Transition from the classical to the quantum Hall effect in a system with a one-dimensional periodic potential

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A degenerate two-dimensional electron gas, in a magnetic band formed from a Landau level in the presence of a one-dimensional periodic potential, is considered. In strong magnetic fields, when the magnetic length λ is smaller than the period A of the potential, the system becomes substantially nonuniform. The dependence of the components σ_{ij} $(i,j=x,y)$ of the conductivity tensor on the occupancy ν of the magnetic band in the presence of scattering by point impurities is calculated for different λ/A. In the case of small λ/A the diagonal components σ_{ii} have a sharp maximum at $\nu=\frac{1}{2}$. The Hall conductivity σ_{xy} does not depend on the parameters of the scatterers. A smooth transition from the classical to the quantum Hall effect as λ/A decreases is traced.

1. INTRODUCTION

The energy spectrum of two-dimensional ($2D$) electrons in a quantizing magnetic field B is a set of Landau levels broadened as a result of scattering by a random impurity potential. The quantum Hall effect (QHE),[1] which is observed in such systems in sufficiently strong fields B and low temperatures T, consists in the fact that in finite intervals of B or of the electron density n_s the Hall conductivity takes quantized values

$$\sigma_{xy}=-\sigma_{yx}=-e^2p/2\pi\hbar, \tag{1}$$

where $p=0,1,...$; here $-e$ is the electron charge, $\hbar$ is Planck's constant, and the dissipative components of the conductivity tensor vanish:

$$\sigma_{xx}=\sigma_{yy}=0. \tag{2}$$

Deviations from (1) are associated with violation of (2). With decreases of B or increase of T a smooth transition to the classical Hall effect is observed:

$$\sigma_{xy}=-\sigma_{yx}=-en_sc/B=-e^2\nu/2\pi\hbar. \tag{3}$$

Here $\nu=2\pi\lambda^2n_s$ is the occupancy factor of the Landau levels, $\lambda=(\hbar c/eB)^{1/2}$ is the magnetic length, and c is the velocity of light. The theory of the QHE in Ref. 2 explains (1) and (2), but is not able at present to give a quantitative explanation of the real deviations from the ideal quantization (1) or of the transition to (3).

The construction of a consistent theory of $\sigma_{ij}(B,T)$ is made difficult by the absence of a small parameter associated with the scattering of electrons by impurities. The reason for this is that the Landau levels with respect to the center of the Larmor orbit are infinitely degenerate in the absence of impurities. This difficulty is removed, at least partially, in $2D$ systems, in which the electron energy depends on the coordinates of the center of the orbit in the absence of impurities. This situation is realized for a $2D$ electron system in a periodic $1D$ potential $V(x)=V(x+A)$, where A is the period of the potential and is much greater than the lattice constant. In the present paper we trace the transition from (3) to (1) in such a system.

The potential $V(x)$ arises naturally in inversion layers on semiconductor surfaces with high Miller indices,[3] or can be created artificially in MIS structures in the form of a periodic striped structure.[4] We shall assume that the amplitude of the potential is small in comparison with the cyclotron energy $\hbar\omega_c$ and shall neglect mixing of the Landau levels. The potential $V(x)$ transforms the Landau levels into narrow (in comparison with $\hbar\omega_c$) $1D$ magnetic bands[5,6] with a dispersion law $\varepsilon_N(k_y)$ that can be calculated in lowest order in $V/\hbar\omega_c$:

$$\varepsilon_N(k_y)=\langle Nk_y|V|Nk_y\rangle. \tag{4}$$

It is assumed that the layer of $2D$ electrons lies in the plane $z=0$, the magnetic field $\mathbf{B}=(0,0,B)$ is perpendicular to the layer, k_y is the wave vector in the y direction, and $|Nk_y\rangle$ are the Landau wavefunctions, which, in this approximation, characterize the states of the electrons in the magnetic band. The band energy (4) is relative to the Nth Landau level. It is obvious that $\varepsilon_N(X)=\varepsilon_N(X+A)$, where $X=-\lambda^2k_y$ is the x coordinate of the center of the orbit and is a good quantum number in the absence of impurity scattering; therefore, the energy in the band is periodic in k_y space, with period A/λ^2. In the following we shall confine ourselves to considering the fundamental magnetic band $N=0$, and shall omit the index N.

The halfwidth of the band (4) increases exponentially with increase of B in relatively weak magnetic fields,[6] and, as $\lambda/A\to 0$, reaches a maximum value equal to the amplitude of the potential $V(x)$. In this case,[7] $\varepsilon(X)=V(X)$. We shall assume for simplicity that the equation $\varepsilon(X)=\text{const}$ has no more than two real roots on a period of the function $\varepsilon(X)$. In other respects, the explicit form of $\varepsilon(X)$ is not specified. We assume that the band (4) is not smeared out by the scattering. The presence of a small parameter, equal to the ratio of the collisional broadening of the Landau levels to the width of the magnetic band, makes it possible to calculate σ_{ij} correctly by treating the scattering by impurities in the lowest Born approximation.

The electrons in the magnetic band tend to occupy the minima of the potential $V(x)$. The properties of the system depend strongly on the magnitude of the parameter λ/A. In relatively weak fields B ($\lambda\gtrsim A$), the wavefunctions of electrons in neighboring minima of $V(x)$ overlap strongly, and

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the system can be regarded as almost uniform. The diagonal components of the conductivity tensor in this case have been calculated in Ref. 6 by means of the Kubo formula. It was found that the conductivity is strongly anisotropic ($\sigma_{xx} \ll \sigma_{yy}$) and is determined by two different mechanisms. The conductivity σ_{yy} has a band character and is inversely proportional to the scattering. The conductivity σ_{xx} is due to migration of the centers of the orbits along the x axis as a result of the scattering, and is directly proportional to the scattering. Qualitatively, these two mechanisms are present for all values of λ/A.

In strong fields B ($\lambda < A$) the overlap of wavefunctions from neighboring minima of the potential $V(x)$ is exponentially small, and the system becomes highly nonuniform. It is this case that will be considered in this paper. The electron concentration is a periodic function of x with period A. The potential $V(x)$ in this case must be calculated self-consistently. This causes the width of the magnetic band to depend on ν.

More fundamental is another consequence of the nonuniformity of the system. An external electric field is distributed in the system nonuniformly. This casts doubt on the applicability of the usual Kubo formula for the conductivity. Therefore, in this paper we use for the calculation of the current the physically intuitive quasiclassical ideas first introduced by Titeica.[8]

The article is arranged as follows. In Sec. 2 we calculate the distribution function of electrons over the band in weak electric fields in the presence of scattering by point impurities. It is assumed that, because of fast electron-phonon relaxation, it is a local-equilibrium distribution function. In Sec. 3 we find general expressions for the components of the conductivity tensor. In Sec. 4 we give an analysis of them. Special attention is paid to an investigation of the behavior of the Hall components of the conductivity. In Sec. 5 we discuss the limits of applicability of the results. Preliminary results of this work were published in Ref. 9.

2. SOLUTION OF THE KINETIC EQUATION

We shall consider a degenerate electron gas in the lowest magnetic band $\varepsilon(X)$ (Fig. 1a). In equilibrium the electrons occupy states in the minima (wells) of $\varepsilon(X)$, up to the Fermi energy E_F. By virtue of the one-dimensional nature of the magnetic band (4), the Fermi surface $\varepsilon(X_F) = E_F$ is the set of points $X_F = X_F^{(1,2)} + nA$, where $n = 0, \pm 1, \ldots$, and $X_F^{(1,2)}$ are chosen as shown in Fig. 1a. The density of states at the Fermi level has the form

$$N(E_F) = \frac{1}{2\pi\hbar A}\left(\frac{1}{v_F^{(1)}} + \frac{1}{v_F^{(2)}}\right), \tag{5}$$

where $v_F^{(1,2)}$ are the Fermi velocities of the electrons, $v_F^{(1,2)} = |\partial\varepsilon/\partial X|\lambda^2/\hbar$ at $X = X_F^{(1,2)}$. The number of occupied states is equal to $|X_F^{(1)} - X_F^{(2)}|S/2\pi\lambda^2 A$, where S is the area of the system, whence it follows that $|X_F^{(1)} - X_F^{(2)}| = A\nu$. Thus, in the space of the orbit centers the electrons occupy bands of width $A\nu$, separated by intervals $A(1-\nu)$; see Fig. 1b.

The orbit centers move in the y direction with velocity $v = -\lambda^2\hbar^{-1}\partial\varepsilon/\partial X$, thereby creating currents in the system. These currents are associated with the Hall drift of electrons in the field $V(x)$, have a nondissipative character, and

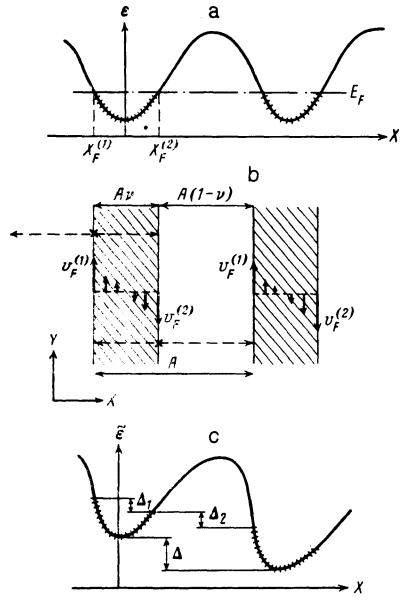

FIG. 1. Energy and distribution of electrons in a magnetic band at $T = 0$ K, as a function of the x coordinate X of the center of the orbit, in equilibrium (a) and in an external electric field (c); b) trajectories of the centers of electron orbits; the occupied states are shaded; the solid arrows represent the velocities of orbit centers; the dashed arrows show transitions between trajectories on account of scattering by impurities, which are denoted by crosses; $\Delta = \Delta_1 + \Delta_2$ is the change of the Fermi level in a period of the function $\varepsilon(X)$.

do not destroy the equilibrium in the system. When scattered by impurities, electrons can pass over from one trajectory to another and, as a result of this, be displaced in the x direction. Obviously, in equilibrium the average current is equal to zero.

An external electric field $\mathbf{F} = (F_x, F_y)$ renormalizes the spectrum [$\varepsilon(X) \to \tilde{\varepsilon}(X)$] and destroys the equilibrium in the system. The field component F_x tilts the curve $\epsilon(X)$; see Fig. 1c. As the Fermi-level electrons are displaced along the x axis on account of scattering by impurities, they accumulate at one of the edges of each well because of the difference in the intrawell and interwell overlap integrals. The same accumulation effect can be produced by a field F_y. In this case the electrons are displaced along the x axis with the Hall velocity $\dot{X} = cF_y/B$, and impurity scattering limits the accumulation process. As a result, the external electric field is supplemented by an induced field, and the total electric field acting on an electron becomes dependent on x. Model estimates[6] show that, as a rule, typical velocities of electrons in the band exceed the sound velocity. This makes it possible to assume that in a weak electric field the rapid electron-phonon relaxation at each well edge on account of Čerenkov emission of phonons will lead to the establishment of local equilibrium described by the Fermi distribution function $f(X)$ with a Fermi energy depending on X (Ref. 10).

The current in the system depends on the differences Δ_1 and Δ_2 of the Fermi levels inside each well and in neighboring wells, respectively (see Fig. 1c). We introduce the quantity $\Delta = \Delta_1 + \Delta_2$. In this nonuniform system the quantity $-\Delta/eA$ plays the role of the average (over a period A) electric field F_x^*.

To determine Δ_1 and Δ_2 it is necessary to solve the kinetic equation for f in the X-representation:

$$\frac{\partial f}{\partial X}\dot{X}+\frac{2\pi}{\hbar}\sum_{X'}\overline{|\langle X'|U|X\rangle|^2}\delta(\tilde{\varepsilon}(X)-\tilde{\varepsilon}(X'))\times[f(X)-f(X')]=0, \quad (6)$$

where $U=u_0\Sigma_i\delta(\mathbf{r}-\mathbf{r}_i)$ is the potential of the point impurities; the bar denotes the standard averaging over the impurity configurations; the perturbed spectrum $\tilde{\varepsilon}(X)$ differs from the unperturbed spectrum $\varepsilon(X)$ to the extent that Δ_1 and Δ_2 differ from zero. The average value of the square of the matrix element in (6),

$$\overline{|\langle X|U|X'\rangle|^2}=\frac{n_iu_0^2}{(2\pi)^{1/2}\lambda L_y}\exp\left[-\frac{(X-X')^2}{2\lambda^2}\right],$$

depends exponentially on B and on the hopping length $|X-X'|$, on account of the overlap integral; n_i is the concentration of impurities, and L_y is the normalization length of the sample. Assuming that the deviations of the local-equilibrium distribution function $f(X)$ from the equilibrium distribution function $f_0(X)$ are small, we obtain, after linearization of (6), an equation for $E_F(X)$:

$$\frac{\partial f_0}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial X}\frac{cF_y}{B}+\frac{n_iu_0^2}{(2\pi)^{1/2}\lambda^3\hbar}\int dX'\exp\left[-\frac{(X-X')^2}{2\lambda^2}\right]\times\delta(\varepsilon(X)-\varepsilon(X'))\left(-\frac{\partial f_0}{\partial\varepsilon}\right)[E_F(X)-E_F(X')]=0. \quad (7)$$

In the degenerate case, from (7) there follows a simple algebraic relation between Δ_1 and Δ:

$$\Delta_1=\Delta\left\{\sum_{n=-\infty}^{\infty}n\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}\times\left\{\sum_{n=-\infty}^{\infty}\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}^{-1}-(2\pi)^{1/6}e\hbar^2v_F^{(1)}v_F^{(2)}\lambda F_y\left\{n_iu_0^2\sum_{n=-\infty}^{\infty}\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}^{-1}. \quad (8)$$

The infinite sums in (8) have arisen because in the integral in (7) we have taken into account hopping of electrons between different points of the Fermi surface upon scattering by impurities. We note that for $F_y=0$ the expression (8) contains no parameters associated with the scatterers.

3. CALCULATION OF THE CURRENT

We shall consider first the y component of the current. It arises as a result of the skewness of the Fermi level within each well. The average current density j_y is found in the usual way:

$$j_y=-\frac{e}{\hbar S}\sum_{k_y}\frac{\partial\tilde{\varepsilon}}{\partial k_y}f(k_y)=\frac{e}{2\pi\hbar A}\int_{-A/2}^{A/2}dX\frac{\partial\tilde{\varepsilon}}{\partial X}f(X). \quad (9)$$

Linearizing (9) with respect to the perturbation Δ_1, we obtain

$$j_y=-e\Delta_1/2\pi\hbar A. \quad (10)$$

It follows from (8) and (10) that

$$j_y=\sigma_{yy}F_y+\sigma_{yx}(-\Delta/eA), \quad (11)$$

where

$$\sigma_{yy}=e^2v_F^{(1)}v_F^{(2)}\hbar\lambda\left\{(2\pi)^{1/2}n_iu_0^2A\sum_{n=-\infty}^{\infty}\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}^{-1} \quad (12)$$

formally coincides, for $v_F^{(1)}=v_F^{(2)}$, with the corresponding expression in Ref. 6, and describes the classical conductivity of a degenerate electron gas in a $1D$ band, and

$$\sigma_{yx}=\frac{e^2}{2\pi\hbar}\left\{\sum_{n=-\infty}^{\infty}n\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}\times\left\{\sum_{n=-\infty}^{\infty}\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}^{-1} \quad (13)$$

is the Hall conductivity, since it is the coefficient of proportionality between j_y and the effective electric field $F_x^*=-\Delta/eA$.

The current density $j_x=j_x^{(1)}+j_x^{(2)}$ along the x axis contains two contributions. The first contribution

$$j_x^{(1)}=-\frac{en_sc}{B}F_y=-\frac{e^2\nu}{2\pi\hbar}F_y\equiv\sigma_{xy}^{(1)}F_y \quad (14)$$

is related to the usual Hall drift of electrons in a field F_y to zeroth order in the scattering, while the second contribution is due to the migration of the orbit centers X on account of impurity scattering and can be calculated by the method proposed for the $3D$ case in Ref. 8:

$$j_x^{(2)}=-\frac{e}{S}\frac{2\pi}{\hbar}\sum_{X'>X}(X'-X)\overline{|\langle X'|U|X\rangle|^2}\delta(\tilde{\varepsilon}(X')-\tilde{\varepsilon}(X))\times[f(X)-f(X')]. \quad (15)$$

The subsequent calculations are analogous to those which were performed in the solution of the kinetic equation. Linearizing (15), we obtain

$$j_x^{(2)}=-\frac{en_iu_0^2}{2(2\pi)^{1/2}\hbar A\lambda^5}\int_{-A/2}^{A/2}dX\int_{-\infty}^{\infty}dX'\,(X'-X)\times\exp\left[-\frac{(X-X)^2}{2\lambda^2}\right]\times\delta(\varepsilon(X)-\varepsilon(X'))\left(-\frac{\partial f_0}{\partial\varepsilon}\right)[E_F(X)-E_F(X')],$$

whence, for degenerate statistics, it follows that

$$j_x^{(2)}=-\frac{en_iu_0^2}{(2\pi)^{1/2}\hbar^3\lambda v_F^{(1)}v_F^{(2)}}\times\left\{\left[\sum_{n=-\infty}^{\infty}(n-\nu)n\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]+\frac{(v_F^{(1)})^2+(v_F^{(2)})^2}{2v_F^{(1)}v_F^{(2)}}\sum_{n=-\infty}^{\infty}n^2\exp\left[-\frac{A^2n^2}{2\lambda^2}\right]\right]\Delta-\Delta_1\sum_{n=-\infty}^{\infty}(n-\nu)\times\exp\left[-\frac{A^2(n-\nu)^2}{2\lambda^2}\right]\right\}. \quad (16)$$

Expressing Δ_1 in terms of Δ by means of (8), we obtain

$$j_x^{(2)}=\sigma_{xx}\left(-\frac{\Delta}{eA}\right)+\sigma_{xy}^{(2)}F_y, \quad (17)$$

where

$$\sigma_{xx}=\frac{e^2 n_i u_0^2 A}{(2\pi)^{3/2}\hbar^3\lambda v_F^{(1)}v_F^{(2)}}\Big\{\sum_{n=-\infty}^{\infty}(n-\nu)^2\exp\Big[-\frac{A^2(n-\nu)^2}{2\lambda^2}\Big]$$
$$+\frac{(v_F^{(1)})^2+(v_F^{(2)})^2}{2v_F^{(1)}v_F^{(2)}}\sum_{n=-\infty}^{\infty}n^2\exp\Big[-\frac{A^2n^2}{2\lambda^2}\Big]\qquad(18)$$
$$-\Big\{\sum_{n=-\infty}^{\infty}(n-\nu)\exp\Big[-\frac{A^2(n-\nu)^2}{2\lambda^2}\Big]\Big\}^2$$
$$\times\Big\{\sum_{n=-\infty}^{\infty}\exp\Big[-\frac{A^2(n-\nu)^2}{2\lambda^2}\Big]\Big\}^{-1}\Big\},$$

$$\sigma_{xy}^{(2)}=-\frac{e^2}{2\pi\hbar}\Big\{\sum_{n=-\infty}^{\infty}(n-\nu)\exp\Big[-\frac{A^2(n-\nu)^2}{2\lambda^2}\Big]\Big\}$$
$$\times\Big\{\sum_{n=-\infty}^{\infty}\exp\Big[-\frac{A^2(n-\nu)^2}{2\lambda^2}\Big]\Big\}^{-1}.\qquad(19)$$

It follows from (14), (17), and (19) that

$$\sigma_{xy}\equiv\sigma_{xy}^{(1)}+\sigma_{xy}^{(2)}=-\sigma_{yx},\qquad(20)$$

with σ_{yx} determined by the expression (13). The generalization of the results obtained to the case of finite T can be carried out using the formula of Mott.[11]

4. ANALYSIS OF THE RESULTS

The expressions obtained above for σ_{ij} can be simplified in the limiting cases of almost uniform and highly nonuniform systems.

In relatively weak magnetic fields the system is almost uniform, and for $2\pi^2\lambda^2/A^2\gg 1$ the infinite sums in the expressions for the tensor components σ_{ij} can be replaced, with an exponentially small error, by integrals.[6] The equations (12) and (18) for σ_{ii} go over into the corresponding expressions obtained in Ref. 6, and the Hall conductivity is described by the classical expression (3).

We shall consider the opposite limiting case of very strong magnetic fields ($2\lambda^2/A^2\ll 1$), when the system becomes highly nonuniform. In all the infinite sums we may retain only the first few terms, corresponding to hops between neighboring Fermi points. The "bottleneck" for the scattering is the least probable hop; the hopping length for the latter along the x axis is equal to $A(1-\nu)$ for $\nu<\frac{1}{2}$, and $A\nu$ for $\nu>\frac{1}{2}$ (see Fig. 1b). The bottleneck disappears for $\nu=\frac{1}{2}$, and with this occupancy of the band we should expect singularities in the conductivity. Actually, this implies that at $\nu=\frac{1}{2}$ there is a transition from electronic conduction ($\nu<\frac{1}{2}$) to hole conduction ($\nu>\frac{1}{2}$).

In analyzing the expressions obtained we shall start from the case $\Delta=0$. It is realized when the Hall contacts along the x axis are short-circuited. On the energy diagram shown in Fig. 1c, we must set $\Delta_1=-\Delta_2$. Under the action of the field F_y a dissipative current j_y and a Hall current j_x flow in the system. The collisionless component $j_x^{(1)}$ of the Hall current [see (14)] tends to increase Δ_1, while scattering decreases Δ_1. Therefore, Δ_1 is directly proportional to F_y and inversely proportional to the scattering probability, and this provides a qualitative explanation of the expression (8). Since the current satisfies $j_y\sim\Delta_1$, σ_{yy} is inversely proportional to the scattering probability. For $2\lambda^2/A^2\ll 1$ the principal contribution to the scattering probability is given by hops within wells for $\nu<\frac{1}{2}$ and by hops between wells for $\nu>\frac{1}{2}$ (see Fig. 1c). The overlap integrals corresponding to these hops increase as the Fermi level approaches the edges of the band, and so the dependence $\sigma_{yy}(\nu)$ has a maximum at $\nu=\frac{1}{2}$. As B increases the overlap integrals decrease and σ_{yy} increases. The dependence $\sigma_{yy}(\nu)$ for different values of B is shown in Fig. 2a.

The main result of the paper is described by Eqs. (13) and (20). In the calculation of σ_{xy} we made essential use of the scattering of electrons by impurities, but the answer turned out to be independent of the parameters n_i and u_0 of the scatterers.[9] This result, which is surprising at first sight, can be explained as follows. The collisional component $\sigma_{xy}^{(2)}F_y$ of the Hall current is proportional to the number of electrons in an energy interval of width Δ_1, and to the probability of scattering of these electrons. The quantity Δ_1, as has been pointed out, is inversely proportional to the scattering probability. Therefore, in the expressions for $\sigma_{xy}^{(2)}$ and σ_{xy} the parameters of the scatterers cancel.

The dependence $\sigma_{xy}(\nu)$ for different values of B is shown in Fig. 2b. In the uniform case ($2\pi^2\lambda^2/A^2\gg 1$) the expressions (13) and (20) go over into (3) (the classical Hall effect). With decrease of λ/A the dependence $\sigma_{xy}(\nu)$ deviates more and more strongly from the classical straight line, when $\nu\neq\frac{1}{2}$, and, as $B\to\infty$, takes the form of an ideal step with discontinuity $e^2/2\pi\hbar$ at $\nu=\frac{1}{2}$ (the ideal quantum Hall effect). For $2\lambda^2/A^2\ll 1$ the deviations from the ideal step are exponentially small:

$$\sigma_{xy}\approx-\frac{e^2}{2\pi\hbar}\Big\{1+\exp\Big[\frac{A^2(1/2-\nu)}{\lambda^2}\Big]\Big\}^{-1}.\qquad(21)$$

This behavior of σ_{xy} has a simple physical explanation. In the lower half of the band, as B increases the probability that an electron hops between neighboring wells upon scattering decreases faster than the probability of hopping with-

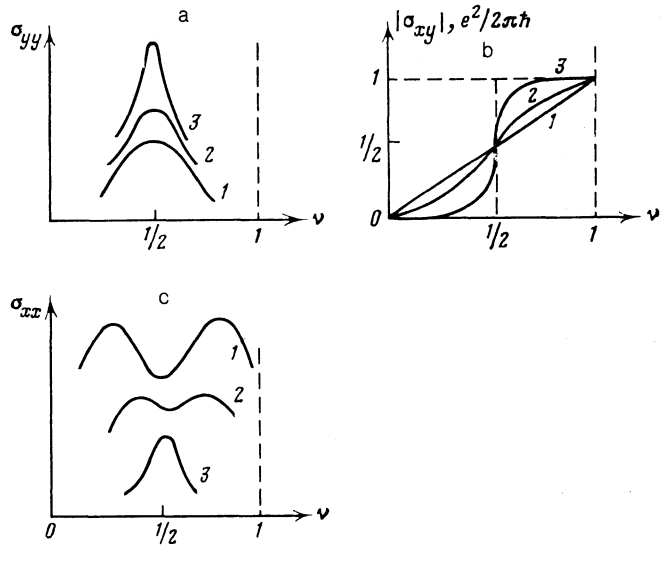

FIG. 2. Dependence of the diagonal components σ_{yy} (a) and σ_{xx} (c) and the Hall component σ_{xy} (b) of the conductivity tensor on the occupancy ν of the magnetic band in weak ($2\pi^2\lambda_1^2\gg A^2$, curves 1), moderate ($1/2\pi^2<\lambda_2^2/A^2<\frac{1}{4}$, curves 2), and strong ($2\lambda_3^2\ll A^2$, curves 3) magnetic fields B: $B_1<B_2<B_3$; $\lambda_i=(\hbar c/eB_i)^{1/2}$, $i=1,2,3$.

in the well (see Fig. 1c) and for $\lambda/A \to 0$ the individual wells become isolated and we have $\sigma_{xy} \to 0$. In the upper half of the band, analogous arguments are applicable to the holes, and for $\lambda/A \to 0$ the quantity σ_{xy} tends to the Hall conductivity $e^2/2\pi\hbar$ of a completely filled Landau level.

We now consider the case when the electric field is applied along the x axis, and $F_y = 0$ (the Hall contacts along the y axis are closed). A dissipative current j_x and a Hall current j_y flow in the system. In the case $2\lambda^2/A^2 \ll 1$, the bottleneck determining the quantity j_x is an intrawell hop for $\nu > \frac{1}{2}$ and an interwell hop for $\nu < \frac{1}{2}$. Therefore, the voltage drops mainly inside the wells for $\nu > \frac{1}{2}$ ($\Delta_1 \to \Delta$, $\Delta_2 \to 0$ as $B \to \infty$) and between the wells for $\nu < \frac{1}{2}$ ($\Delta_1 \to 0$, $\Delta_2 \to \Delta$ as $B \to \infty$). The electron hops within wells (between wells) form a current j_x proportional to Δ_1 (Δ_2) and to the probability of these hops. It follows from the continuity of the current that Δ_1/Δ_2 does not depend on the parameters of the scatterers, but is determined entirely by the ratio of the overlap integrals [see (8)].

The Hall current j_y is due to the skewness Δ_1 of E_F within the wells, and in calculating it we can regard the sample as a set of independent conducting strips (Fig. 1b), each of which is in the regime of the ideal QHE, with occupancy equal to unity. The total Hall current $j_y A$ across a period is equal to $\Delta_1 e^2/2\pi\hbar$, whence follows the expression (10). The dependence $\sigma_{yx}(\nu)$ [see (13) and Fig. 2b] coincides, to within a constant factor, with the dependence $\Delta_1(\nu)$.

The fact that the usual relation (20) holds in this strongly nonuniform and anisotropic system is not trivial. It does not follow, generally speaking, from the Onsager symmetry relations.[10]

The dissipative current j_x is due to hops of electrons between Fermi points. The formula (18) for σ_{xx} for $2\lambda^2/A^2 \ll 1$ is greatly simplified,[1)] and for $v_F^{(1)} = v_F^{(2)} \equiv v_F$ can be represented in the form of the Einstein relation:

$$\sigma_{xx} \approx e^2 N(E_F)[A^2/2(\tau_1+\tau_2)]. \tag{22}$$

Here $N(E_F)$ is determined by means of (5), and the expression in the square brackets, which has the meaning of the diffusion coefficient D_{xx}, is written in terms of the lifetimes of an electron in respect of transitions within wells (τ_1) and between wells (τ_2) upon scattering by impurities, with

$$\tau_1^{-1} = \frac{n_i u_0^2}{(2\pi)^{1/2}\hbar^2\lambda v_F} \exp\left[-\frac{A^2\nu^2}{2\lambda^2}\right],$$

$$\tau_2^{-1} = \frac{n_i u_0^2}{(2\pi)^{1/2}\hbar^2\lambda v_F} \exp\left[-\frac{A^2(1-\nu)^2}{2\lambda^2}\right].$$

The dependence of $(\tau_1+\tau_2)^{-1}$ on ν [and, consequently, the dependence $\sigma_{xx}(\nu)$] has a maximum at $\nu = \frac{1}{2}$; the quantity $\sigma_{xx}^{\max}$ decreases with increase of B (see Fig. 2c). In the uniform case, when $2\pi^2\lambda^2/A^2 \gg 1$, σ_{xx} qualitatively repeats the behavior of the density of states $N(E_F)$ and has maxima at the edges of the band [the divergences of $N(E_F)$ are removed due to the scattering].

An interesting consequence of the results obtained is a relationship between the small corrections $\delta\sigma_{xy}$ to the quantized Hall-conductivity values (1) and the dissipative components σ_{xx} and σ_{yy} of the conductivity tensor. For $\lambda^2 \ll A^2|\nu - \frac{1}{2}|$, using (12) and (22) we obtain

$$\sigma_{xx}\sigma_{yy} \approx \left(\frac{e^2}{2\pi\hbar}\right)^2 \exp\left[-\frac{A^2}{\lambda^2}|\nu - {}^1/_2|\right]. \tag{c}$$

From this and from (21) it follows that

$$|\delta\sigma_{xy}| \approx (2\pi\hbar/e^2)\sigma_{xx}\sigma_{yy}. \tag{23}$$

Relations of this type have been investigated for an isotropic $2D$ electron–impurity system [$V(x) = 0$, $\sigma_{xx} = \sigma_{yy}$], both experimentally[12] and theoretically.[13,14] In Ref. 13 a linear relationship ($\delta\sigma_{xy} \sim \sigma_{xx}$) was obtained by the renormalization-group method, while in Ref. 14 a quadratic relationship between $\delta\sigma_{xy}$ and σ_{xx} ($\delta\sigma_{xy} \sim \sigma_{xx}^2$) was obtained by numerical methods. For an anisotropic system such a relationship has not been investigated previously.

Finally, we note that in experiments on the quantum Hall effect one usually measures not the conductivity σ_{ij} but the resistivity ρ_{ij}. Using the relation betwen these tensors ($\hat{\rho} = \hat{\sigma}^{-1}$), we shall analyze the behavior of $\rho_{ij}(\nu)$ for $0 < \nu < 1$ as $\Lambda/A \to 0$. In this limit, $\sigma_{xx} \to 0$ and $\sigma_{yy} \to \infty$, and so $\rho_{xx} \to \infty$ and $\rho_{yy} \to 0$ for $\nu = \text{const}$. The behavior of the Hall resistivity turns out to be unusual: Despite the fact that σ_{xy} experiences a discontinuity at $\nu = \frac{1}{2}$ (see Fig. 2b), $\rho_{xy} \to 2\pi\hbar/e^2$ for all ν. Usually, σ_{xy} and ρ_{xy} experience discontinuities at the same values of ν. In the given case the absence of a discontinuity of ρ_{xy} inside the band is explained by the fact that in this limit the sample, for all values of ν, is divided into almost independent conducting strips (Fig. 1b), each of which is in the regime of the quantum Hall effect with a local occupancy equal to unity. The important point is that we are concerned here with the lower magnetic band. Only under this condition are the conducting strips separated by nonconducting regions and not connected electrically. When the higher magnetic bands are occupied a division into independent strips is found to be impossible, and the discontinuities of σ_{xy} and ρ_{xy} occur at the same (half-integer) values of ν.

5. CONCLUSION

The formula (13) describes analytically a smooth transition from the classical to the quantum Hall effect with increasing B. In deriving it we made essential use of a specific feature of this (see Sec. 1). The smallness of the collisional broadening of the magnetic band in comparison with its width (more precisely, with E_F) made it possible to invoke intuitive ideas about the motion of the electrons, and transparent mathematical apparatus. We shall discuss the limits of applicability of the results obtained.

Impurity scattering was taken into account in lowest order of perturbation theory; localization effects were thereby neglected. This is certainly incorrect near the band edges, when the parameter associated with the scattering ceases to be small.[6] A rigorous solution of the problem of localization in this system, including a determination of the edges of the mobility gap, goes beyond the scope of the present work. Nevertheless, the behavior of σ_{ij} for $\nu \cong \frac{1}{2}$ (the peaks of σ_{xx} and σ_{yy}, and the discontinuity of σ_{xy}), is qualitatively similar to the behavior of σ_{ij} in the regime of the quantum Hall effect in ordinary systems and gives grounds to suppose that near the center of the magnetic band there exists a band of delocalized states, within which the ideas used in the paper are applicable.

Certain results in the paper can be obtained by means of the standard Kubo formula. This pertains to σ_{yy} (Ref. 6) and σ_{xy} (Ref. 9), and is connected with the fact that in the y

direction the system is uniform. For σ_{xx} the result obtained by means of the Kubo formula[6] differs from the correct result (18) in that the last term in the curly brackets is absent (it disappears for $2\pi^2\lambda^2/A^2 \gg 1$, when the system becomes uniform). The role of this term increases in strong magnetic fields (with decrease of λ/A), when the nonuniformity of the system in the direction of the x axis becomes more and more important. This conclusion has a general character.

The theory of uniform linear response and, in particular, the usual (local) form of the Kubo formula are very widely used for the calculation of σ_{ij} in the regime of the quantum Hall effect in ordinary (isotropic) $2D$ systems, which are often nonuniform. The present result casts doubt on the admissibility of such an approach.

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[1)]The expression for σ_{xx} in this case was obtained independently by Yu. B. Grebenshchikov, F. R. Ulinich, and N. A. Usov.

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