Nonlinear reflection of a sound wave in a gas mixture

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Energy transfer from an incident sound wave to a reflected one is considered in a mixture consisting of a light gas with a small amount of heavy gas added. The transfer mechanism is based on a nonlinear interaction between the incident and reflected waves.

1. INTRODUCTION

Barodiffusion plays an important role in a mixture of a light and a heavy gas.[1] Thus, it causes an anomalously strong damping of sound in a binary mixture with nearly equal component mass concentrations. The damping rate is M/m times larger than in a pure gas, where M and m are the molecular masses of the heavy and light components, respectively. Note that equal contributions are made to the damping by the internal friction in the gas and by the friction against the wall (see Sec. 2).

The anomalous dissipation in such a medium is also accompanied by strong nonlinear effects. One of them, separation of the mixture components in a standing sound wave, was investigated theoretically and experimentally in Refs. 2 and 3.

We discuss in the present paper the possibility of total reflection of an incident wave as a result of a nonlinear interaction that gives rise to transfer of the energy from the incident to the reflected wave.

It is convenient to describe an effect of second order in the oscillation amplitude in terms of heavy test particles located in the field of a sound wave excited in the lighter gas. Taking part in the oscillations, the test particle acquires additional energy represented by an effective potential

$$U=M\langle v^2\rangle/2. \tag{1.1}$$

Here M is the mass of the trial particle, v is the gas velocity in the sound oscillations, and the angle brackets denote averaging over the period of the oscillations. The particles are redistributed in the effective potential in accordance with the Boltzmann law

$$\frac{\Delta c}{c}=-\frac{U}{T}=-\frac{M}{m}\frac{\langle v^2\rangle}{s^2}, \tag{1.2}$$

where c is the mass concentration of the heavy particles, T the gas temperature, and s is the sound velocity, which is of the same order as the thermal velocity. The sound wave draws the heavy particles into pressure antinodes, and this makes separation of the components possible.[2,3]

If two traveling waves, incident and reflected, propagate in a gas at close frequencies ω_1 and ω_2 respectively, the effective potential is no longer stationary:

$$U=M\langle v_1v_2\rangle, \tag{1.3}$$

where v_1 and v_2 are the velocities in the incident and reflected waves. The averaging is over the fast sound motions.

The redistribution produces in the gas a quasistationary wave with oscillation frequency $\Delta\omega=\omega_1-\omega_2$ ($\Delta\omega\ll\omega_1,\omega_2$). The modulations of the concentration, temperature, and density (c, T, and ρ, respectively) in this wave are of the same order:

$$\frac{\Delta c}{c}\approx\frac{\Delta T}{T}\approx\frac{\Delta\rho}{\rho}\approx\frac{M}{m}\frac{\langle v_1v_2\rangle}{s^2}. \tag{1.4}$$

Strong nonlinear interaction, described by Eq. (1.4), of the three waves of frequency ω_1, ω_2, and $\Delta\omega$, can cause total transfer of the energy from the incident to the reflected wave. As in the earlier papers, we use the large mass ratio as the factor that enhances the nonlinearity which is usually weak in acoustics.

We proceed to estimate the damping length of wave 1 or the growth length of wave 2. Assuming $v_2\ll v_1$ (the reflected wave in a long tube is weakened by linear damping), we estimate the correction to the derivative $\partial v_2/\partial x$ from the mass-flux conservation condition:

$$\delta\left(\frac{\partial v_2}{\partial x}\right)\sim k\frac{\Delta\rho}{\rho}v_1^*\approx k\frac{M}{m}\left|\frac{v_1}{s}\right|^2 v_2, \tag{1.5}$$

where k is the wave number. Rigorous calculations of $\delta(\partial v_2/\partial x)$ and of the amplitudes in a quasistationary wave, using the equations of weakly linear hydrodynamics, are given below (Secs. 4 and 5). The quantity $\Delta\rho$, and with it $\delta(\partial v_2/\partial x)$, is complex. The imaginary part of the complex factor left out of (1.5) determines, depending on the sign, the damping or the growth of the reflected wave. The sign of the nonlinear growth rate (decrement)

$$\Gamma_{NL}\sim k\frac{M}{m}\left|\frac{v_1}{s}\right|^2 \tag{1.6}$$

is determined by the sign of $\Delta\omega=\omega_1-\omega_2$.

If the incident-wave amplitude exceeds the threshold at which the nonlinear growth rate (1.6) becomes comparable with the linear one (Sec. 3),

$$\Gamma_L\sim k^2lM/m, \tag{1.7}$$

where l is the mean free path in the gas, an instability sets in. The reflected-wave amplitude $|v_2/s|$ begins to increase. We assume that this process leads ultimately to a highly nonlinear regime in which the incident wave does not penetrate into the gas. To verify this prediction, an ultrasonic radiator must be placed at one end of the tube with the gas mixture, and a sensor at the other. When the ultrasound power rises to the threshold value, it should no longer be received. Estimates detailed at the end of the article yield the following parameter values: the emitter threshold power is ~10–30 W at a tube diameter 0.5 cm and at an ultrasound frequency 0.3 MHz.

The plan of the article is the following: In Sec. 2 we derive the fundamental equations and estimate the various

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processes—barodiffusion, thermal diffusion, thermal conductivity, and viscous friction. In Sec. 3 we consider the linear theory and estimate the linear, bulk, and boundary-layer damping. Sections 4 and 5 deal with the nonlinear theory in second and third order of perturbation theory, respectively. The threshold value of the incident-wave amplitude is obtained in Sec. 5. The final Sec. 6 contains the estimates and a discussion of the results.

2. FUNDAMENTAL EQUATIONS

We use the standard equations of the dynamics of a gas mixture, with account taken of the viscosity, heat conduction, and diffusion.[1] We shall show below that thermodiffusion can be neglected if the binary-mixture molecular-mass ratio is small, $m/M \ll 1$, and the concentrations of the two components are comparable, $c \sim 1 - c \sim 1$. This leads to some simplifications. Under these conditions the barodiffusion is anomalously large and plays the principal role in the dissipation mechanism and in the nonlinear phenomena.

The flux **J** of the heavy component (with concentration c) and the heat flux **Q** are proportional to the gradients of the concentration c, of the pressure P, and of the temperature T (Ref. 1):

$$\mathbf{J} = -\rho D\left[\nabla c + k_T \frac{\nabla T}{T} + k_P \frac{\nabla P}{P}\right], \tag{2.1}$$

$$\mathbf{Q} = \left[k_T \left(\frac{\partial \mu}{\partial c}\right)_{P,T} - T\left(\frac{\partial \mu}{\partial T}\right)_{P,c} + \mu\right]\mathbf{J} - \varkappa \nabla T. \tag{2.2}$$

Here ρ is the gas density, μ the chemical potential per unit mass (see Ref. 1, p. 321), and D and $\varkappa$ respectively the diffusion coefficient and the thermal conductivity.. The dimensionless quantities k_T and k_P are called the thermo- and barodiffusion ratios.[1] The barodiffusion ratio is a purely thermodynamic quantity [Eq. (59.10 of Ref. 1), and can be easily calculated for an ideal gas (Ref. 1, problem in §59). In the limit $M/m \gg 1$ we have

$$k_P = -\frac{M}{m} c(1-c)^2. \tag{2.3}$$

The thermodiffusion ratio k_T is a kinetic quantity that depends on the character of the intermolecular interaction. An estimate of k_T in the hard-elastic-sphere model is given in Ref. 4 [Eqs. (2.3), (2.8), (2.9), and (7.2)–(7.5)]. For $m/M \ll 1$ and $c \sim 1 - c \sim 1$ we have

$$k_T = \frac{5}{4 \cdot 2^{1/2}} \frac{m}{M} \frac{c}{1-c} \left(\frac{d_{12}}{d_2}\right)^2. \tag{2.4}$$

Here d_1 and d_2 are the diameters of colliding unlike molecules, and d_{12} is their arithmetic mean.

It follows from the gasdynamic equations at $k_T \equiv 0$ (see below) that $\nabla P/P \gg (m/M)^2 (\nabla T/T)$ (in each order of perturbation theory in the nonlinearity). Thus, the neglect of thermodiffusion in (2.1) is justified. Using the expressions for the chemical potential of an ideal gas [Ref. 5, §§43,93] we readily obtain, to leading order in m/M,

$$\left(\frac{\partial \mu}{\partial c}\right)_{T,P} = \frac{T}{M} \frac{1}{c(1-c)^2},$$
$$T\left(\frac{\partial \mu}{\partial T}\right)_{P,c} - \mu = \frac{c_{P_2}}{m} T + \frac{\varepsilon_2}{m} - \frac{\varepsilon_1}{M}, \tag{2.5}$$

where c_{P_2} is the molecular specific heat of the light gas, and ε_1 and ε_2 are certain constants typical of the given gases.

It can be seen from (2.2) and (2.5) that the term $k_T (\partial\mu/\partial c)_{P,T}$ is a small quantity of order $(m/M)^2$ and its neglect in expression (2.2) for the heat flux is legitimate.

Under our assumptions $(m \ll M, c \sim (1-c) \sim 1)$ the light particles greatly exceed in number the heavy ones. The total number of particles n per unit mass is approximately equal to the number of light particles, and is expressed in terms of the mass concentration c in the form

$$n = m(1-c)^{-1}. \tag{2.6}$$

The average specific heat $\langle c_P \rangle$ per molecule is equal to the molecular specific heat c_{P_2} of the light gas. The adiabatic exponent γ is expressed in terms of c_{P_2} in the usual fashion and is independent of the concentration:

$$\gamma = c_{P_2}(c_{P_2} - 1)^{-1}. \tag{2.7}$$

Finally, the light particles make the overwhelming contribution to the specific heat:

$$c_P = n c_{P_2}. \tag{2.8}$$

In the approximations described, the one-dimensional gas-dynamics is described by the following equations: the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{I}$$

the Navier-Stokes equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho\alpha \frac{\partial v}{\partial x}\right), \tag{II}$$

the diffusion equation

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial x}\left\{\rho D\left[\frac{\partial c}{\partial x} - \frac{M}{m}c(1-c)^2 \frac{1}{P}\frac{\partial P}{\partial x}\right]\right\}, \tag{III}$$

the heat-conduction equation

$$\begin{aligned}\frac{1}{T}\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x}\right) - \frac{(\gamma-1)}{\gamma}\frac{1}{P}\left(\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial x}\right) &= \frac{1}{\rho c_P T}\frac{\partial}{\partial x}\left(\rho c_P \chi \frac{\partial T}{\partial x}\right) \\ &\quad - D\left[\frac{1}{(1-c)}\frac{\partial c}{\partial x} - \frac{M}{m}c(1-c)\frac{1}{P}\frac{\partial P}{\partial x}\right] \\ &\quad \times \left(\frac{1}{T}\frac{\partial T}{\partial x}\right) + \frac{\alpha}{2c_P T}\left(\frac{\partial v}{\partial x}\right)^2,\end{aligned} \tag{IV}$$

and the equation of state

$$P = n\rho T = \rho(1-c)T/m. \tag{V}$$

In Eqs. (I)–(IV) v is the velocity along the x axis, and the coefficient α is the kinematic viscosity:

$$\alpha = (^4/_3\eta + \zeta)/\rho, \tag{2.9}$$

where η and ζ are the first- and second-viscosity coefficients. χ denotes the thermal-diffusivity coefficient

$$\chi = \varkappa/\rho c_P. \tag{2.10}$$

Equation (IV), which we arbitrarily named the heat-conduction equation, is obtained from the heat-transfer equation [Ref. 1, Eq. (58.8)] if the derivatives of the specific entropy are expressed in terms of the derivatives of the pressure, temperature and concentration, and if the diffusion equation (III) is used. Note that all the dissipative coefficients—viscosity, heat-conduction, and diffusion—have the same order of magnitude $v_T l$, where v_T is the thermal velocity of the light particles and l is the mean free path.[2]

3. LINEAR APPROXIMATION

Bulk and surface damping

Linearization of the system (I)–(V) entails no difficulty. We present, however, a system of 3D linear equations, bearing in mind an application to the boundary-layer damping problem:

$$\frac{\partial\rho}{\partial t}+\rho_0 \operatorname{div}\mathbf{v}=0, \tag{3.1}$$

$$\frac{\partial\mathbf{v}}{\partial t}=-\frac{1}{\rho_0}\nabla P+\lambda_0\nabla(\operatorname{div}\mathbf{v})+\nu_0\Delta\mathbf{v}, \tag{3.2}$$

$$\frac{\partial}{\partial t}\left(\frac{c}{1-c_0}\right)=D_0\Delta\left[\frac{c}{(1-c_0)}-\frac{M}{m}c_0(1-c_0)\frac{P}{P_0}\right], \tag{3.3}$$

$$\frac{\partial}{\partial t}\left[\frac{T}{T_0}-\frac{(\gamma-1)}{\gamma}\frac{P}{P_0}\right]=\chi_0\Delta\left(\frac{T}{T_0}\right), \tag{3.4}$$

$$\frac{P}{P_0}=\frac{\rho}{\rho_0}+\frac{T}{T_0}-\frac{c}{1-c_0}. \tag{3.5}$$

The subscript zero labels unperturbed equilibrium quantities. Symbols without subscripts denote deviations. The quantity λ_0 in (3.2) is a linear combination of the kinematic viscosities:

$$\lambda_0=({}^1/_3\eta_0+\zeta_0)/\rho_0, \quad \nu_0=\eta_0/\rho_0. \tag{3.6}$$

If the solution takes the form of a plane sound wave $\exp(-i\omega t+ikx)$, the velocity v is directed along the wave vector (the x axis). The dispersion equation relating the frequency and wave number can be easily solved if the quantities k^2D_0/ω, $k^2\lambda_0/\omega$, $k^2\nu_0/\omega$, $k^2\chi_0/\omega$ are assumed small. It is necessary to impose the stronger constraint

$$\frac{M}{m}\frac{k^2D_0}{\omega}\ll 1. \tag{3.7}$$

To first order in this small parameter, the dispersion equation takes the form

$$\frac{ks}{\omega}=\pm\left[1\pm i\frac{\Gamma(k)}{k}\right], \quad \frac{\Gamma(k)}{k}=\frac{\gamma}{2}\frac{M}{m}c_0(1-c_0)\frac{|k|D_0}{s}, \tag{3.8}$$

where s is the speed of sound and is determined by the relation

$$\rho_0 s^2=\gamma P_0. \tag{3.9}$$

The $\pm$ signs in (3.8) correspond to different wave-propagation directions. $\Gamma(k)$ stands for the linear bulk damping rate. All the relative amplitudes in the wave can be expressed in terms of one of them, say ρ/ρ_0:

$$\frac{v}{s}=\pm\frac{\rho}{\rho_0}\left[1-i\frac{\Gamma(k)}{k}\right], \quad \frac{P}{P_0}=\gamma\frac{\rho}{\rho_0}\left[1-2i\frac{\Gamma(k)}{k}\right], \tag{3.10}$$

$$\frac{T}{T_0}=(\gamma-1)\frac{\rho}{\rho_0}\left[1-2i\frac{\Gamma(k)}{k}\right], \quad \frac{c}{1-c_0}=\pm 2i\frac{\Gamma(k)}{k}\frac{\rho}{\rho_0}.$$

The $\pm$ signs are in one-to-one correspondence in expression (3.10) for the velocity and in the dispersion relation (3.8).

The condition (3.7) means that the linear damping of the sound in the gas mixture is small. Note that in a pure gas the damping is determined by the viscosity and by the thermal conductivity, which make contributions of order kl to $\Gamma(k)/k$. The contribution (3.8) due to barodiffusion, is M/m times larger.

We proceed to calculate the damping in the boundary layer. When the gas moves along a boundary of a solid having a high thermal conductivity, the following boundary conditions must be met: the velocity $\mathbf{v}$ at the wall and the diffusion-flux component normal to the wall are zero, while the temperature at the wall is equal to the specified T_0. These boundary conditions can be satisfied by using the solutions of the system (3.1)–(3.5); these solutions fall off rapidly with increasing distance from the wall (see, e.g., Ref. 1, problem 2 of §79). Assuming the wall to be locally planar, i.e., $kR\gg 1$ where R is the radius of the tube, we seek a solution in the form

$$\exp\left[ik(x-st)-\varkappa z\right], \quad \varkappa\gg k.$$

Here x is the coordinate along the sound-wave propagation direction and z is the coordinate perpendicular to the wall. The system (3.1)–(3.5) has three such solutions: for the viscosity, thermal conductivity, and diffusion modes. The dispersion relations for a given k determine the values of $\varkappa$:

$$\varkappa_\nu=\left(i\frac{ks}{\nu}\right)^{1/2}, \quad \varkappa_\chi=\left(i\frac{ks}{\chi}\right)^{1/2}, \quad \varkappa_D=\left(i\frac{ks}{D}\right)^{1/2}. \tag{3.11}$$

The square root is taken here with a positive real part. These are approximate expressions. They are the leading terms of an expansion in the small parameter kl. We write down with the same accuracy the oscillation amplitudes, which we label with the same subscripts as $\varkappa$, but use no subscripts for the amplitudes of the planar sound wave far from the wall:

$$\begin{gathered} v_\nu/s=\rho/\rho_0, \\ \rho_\chi/\rho_0=-T_\chi/T_0=(\gamma-1)\rho/\rho_0, \\ \rho_D/\rho_0=c_D/(1-c_0)=\gamma(M/m)c_0(1-c_0)\rho/\rho_0. \end{gathered} \tag{3.12}$$

The remaining amplitudes, including the transverse velocity component, are zero to the indicated accuracy. Note that the amplitudes in the boundary-layer diffusion mode are M/m times larger than the amplitudes in the viscosity and thermal conductivity modes. The main contribution to the losses in the boundary layer is made by the diffusion mode, which does not exist in the pure gas:

$$\frac{dE}{dt}=-\frac{1}{\rho_0 D_0}\left(\frac{\partial\mu_0}{\partial c_0}\right)_{P,T}\int J^2\,dV. \tag{3.13}$$

Substituting $\mathbf{J}$ from (2.1) and $(\partial\mu/\partial c)_{P,T}$ from (2.5) in (3.13), we obtain for the boundary-layer damping decrement the expression

$$\Gamma_S=\frac{1}{R}\gamma\frac{M}{m}c_0(1-c_0)\left(\frac{kD_0}{2s}\right)^{1/2}, \tag{3.14}$$

where R is the tube radius. Note that, notwithstanding the smallness of $(\partial\mu/\partial c)_{P,T}\sim T/M$ the contribution of the diffusion mode is M/m times larger than the contributions of the two other modes (Ref. 1, problem 2 of §79).

Equations (3.8) and (3.14) show that when a relatively small fraction of heavy admixture is added to a light gas both the bulk and the surface damping of the sound increase by a factor M/m.

The contribution of the diffusion mode to the surface damping is not taken into account in Ref. 3. The Q of the acoustic system was therefore overestimated by not less than 10 times the experimental value. Allowance for the diffusion contribution may eliminate this discrepancy.

4. NONLINEAR THEORY

Quasistatic beats

Consider nonlinear interaction of two sound waves, one incident along the axis from left to right, and the other reflected in the opposite direction. All the quantities are proportional to $\exp[ik_1(x-st)]$ in the incident wave and to $\exp[ik_2(x+st)]$ in the reflected. We assume that the wave numbers, and hence the frequencies, are close to one another:

$$k_1=k+{}^1/_2\Delta k,\quad k_2=k+{}^1/_2\Delta k,\quad \Delta k\ll k. \tag{4.1}$$

The nonlinear interaction gives rise to a third wave of frequency $\Delta\omega=\omega_1-\omega_2$ and wave number $2k=k_1+k_2$. Since the frequency difference is small, the resultant wave is quasistatic. The amplitudes of the various quantities in this wave can be calculated by expanding in the small parameter $\Delta k/k$. We use in addition, as before, the large parameter M/m. All the kinetic coefficients [of the (Kinematic) viscosity, of the thermal diffusivity, and of the diffusion) are of the same order. This allows us to neglect in the right hand sides of Eqs. (I)–(V) all the nonlinear terms containing these coefficients, compared with the barodiffusion terms. We label all the amplitudes in the incident wave by the index 1, in the reflected by 2, and in the quasistatic by 3.

It follows from the continuity equation (I) that v_3/s is smaller by a factor $\Delta k/k$ than ρ_3/ρ_0 and by a factor $\Gamma(k)/k$ than $\rho_1\rho_2/\rho_0^2$. Neglecting in the Navier-Stokes equation (II) the quantity v_3/s and also the small terms due to viscosity, we express P_3/P_0 in terms of

$$P_3/P_0=2\gamma\rho_1\rho_2/\rho_0^2. \tag{4.2}$$

We retain in the heat-conduction equation (IV) all the terms linear in T_3/T_0 and in P_3/P_0. The drift nonlinear terms (i.e., those containing $v\partial/\partial x$) cancel out in the left-hand side by virtue of the linear-theory relations between the temperature and the pressure on the adiabat. Nonlinearities in the left-hand side of (IV) stem only from the logarithmic derivatives of the pressure and temperature with respect to time. The main nonlinearity in the right-hand side of (IV) is due to the barodiffusion term that contains the large ratio M/m. Substituting P_3/P_0 from (4.2), we obtain an expression for T_3/T_0:

$$\frac{T_3}{T_0}=G_T\frac{\rho_1\rho_2}{\rho_0^2},\quad G_T=\frac{1-4i\Gamma(k)/\Delta k}{1+4ik^2\chi/s\,\Delta k}. \tag{4.3}$$

It is necessary to take into account in the diffusion equation (III) all the terms linear in c_3. The nonlinear drift terms $v\partial c/\partial x$ in the left-hand side are of the same order as the nonlinear barodiffusion terms in the right-hand side. Allowance must also be made for the dependence of the diffusion coefficient on temperature and pressure. For diffusion of heavy particles in a gas this dependence is relatively simple (Ref. 6, §12):

$$D=\frac{3T^3}{mP\langle\sigma_t v^3\rangle}=\frac{4}{3}\left(\frac{2}{\pi}\right)^{3/2}\frac{T^{3/2}}{mPd_{12}^2}, \tag{4.4}$$

where σ_1 is the transport cross section, and the angle brackets denote averaging over a Maxwell distribution. The second equal sign in the equation pertains to the model of hard elastic spheres. In this model we have

$$D/D_0={}^3/_2T/T_0-P/P_0. \tag{4.5}$$

Using (4.5), we can express $c_3/(1-c_0)$ with the aid of the diffusion equation in terms of $\rho_1\rho_2/\rho_0^2$:

$$\frac{c_3}{1-c_0}=G_c\frac{\rho_1\rho_2}{\rho_0^2},\quad G_\rho=-\frac{4i(\gamma-1)\Gamma(k)/\Delta k}{1+4ik^2D/s\Delta k}. \tag{4.6}$$

Finally, taking the logarithm of the equation of state (V), we can obtain ρ_3/ρ_0:

$$\frac{\rho_3}{\rho_0}=G_\rho\frac{\rho_1\rho_2}{\rho_0^2},\quad G_\rho=2-G_T+G_c. \tag{4.7}$$

5. INTERACTION OF QUASISTATIC AND REFLECTED WAVES

The nonlinear interaction of a reflected wave with a quasistatic one can enhance as well as attenuate the latter. We separate the slow and fast oscillating parts of the dependence of the reflected-wave density on the coordinate x:

$$\frac{\rho_2(x)}{\rho_0}=\frac{\tilde\rho_2(x)}{\rho_0}\exp[ik(x+st)], \tag{5.1}$$

where $\tilde\rho_2(x)$ is a slowly varying function of the coordinates. We assume the quantities

$$\frac{1}{k}\frac{\partial}{\partial x}\left(\frac{\tilde\rho_2}{\rho_0}\right),\quad \frac{\Gamma(k)}{k}\frac{\tilde\rho_2(x)}{\rho_0}\ \text{и}\ \frac{\Gamma(k)}{\Delta k}\left|\frac{\rho_1}{\rho_0}\right|^2\frac{\rho_2}{\rho_0}$$

to be of the same order of smallness. We assume the second derivative of $\tilde\rho_2/\rho_0$ with respect to x to be a quantity of higher order. The other quantities must also be similarly separated into slow and fast parts.

Our aim is to derive equations that describe the evolution of the slowly varying amplitudes. The above ordering of the quantities enables us to discard in the equations the nonlinear terms that contain the dissipative coefficients α, χ, and D. The only significant linear dissipative term, just as in the linear approximation, is the barrodiffusion term containing the factor M/m in the diffusion equation.

The resultant system of equations is relatively simple and leads to the following equation for $\tilde\rho_2/\rho_0$:

$$\frac{\partial}{\partial x}\left(\frac{\tilde\rho_2}{\rho_0}\right)=\frac{\tilde\rho_2}{\rho_0}\left\{\Gamma(k)+ik\left[2+(\gamma-1)(\gamma-2)-\frac{1}{2}G_\rho\right]\left|\frac{\rho_1}{\rho_0}\right|^2\right\}. \tag{5.2}$$

Equation (5.2) contains information on the evolution of both the modulus and the phase of the amplitude $\tilde\rho_2/\rho_0$. It yields readily an equation for the squared modulus of the amplitude

$$\begin{aligned}\frac{\partial}{\partial x}\left|\frac{\tilde\rho_2}{\rho_0}\right|^2&=\left[2\Gamma(k)+\operatorname{Im}(G_\rho)\left|\frac{\rho_1}{\rho_0}\right|^2\right]\left|\frac{\tilde\rho_2}{\rho_0}\right|^2\\&=2\Gamma(k)\left|\frac{\tilde\rho_2}{\rho_0}\right|^2\left\{1-2(\gamma-1)f\left(\frac{\Delta k}{k}\right)\left|\frac{\rho_1}{\rho_0}\right|^2\right\},\end{aligned} \tag{5.3}$$

where the function $f(\Delta k/k)$ in (5.3) is of the form

$$f(x)=\frac{x}{x^2+a^2}-\frac{x}{x^2+b^2},\qquad a=4\frac{kD}{s},\qquad b=4\frac{k\chi}{s}. \tag{5.4}$$

The function $f(x)$ is odd, has a single zero at $x=0$, and tends to zero as $x\to\pm\infty$. The extrema of $f(x)$ (one maximum and one minimum) are located at the points

$$x_\pm=\pm(ab)^{1/2}\psi\left(\frac{a}{b}\right),$$

$$\psi(t)=\frac{1}{6}\left\{\left[\left(t+\frac{1}{t}\right)^2+12\right]^{1/2}-\left(t+\frac{1}{t}\right)\right\}. \tag{5.5}$$

The extremal values of $f(x)$ are given by the equation

$$|f_{max}|=\left[\frac{\psi(a/b)}{ab}\right]^{1/2}\frac{(a/b-b/a)}{[\psi(a/b)+a/b][\psi(a/b)+b/a]}. \tag{5.6}$$

In the region of $\Delta k/k$ in which $f(\Delta k/k)>0$ (as determined by the sign of Δk), the decrease of the reflected-wave amplitude (from left to right) gives way to an increase when the amplitude $|\rho_1/\rho_0|$ exceeds its threshold value $[2(\gamma-1)f(\Delta k/k)]^{-1/2}$. This means onset of instability.

6. ESTIMATES AND DISCUSSION

Equation (5.3) for the amplitude is valid if the radius of the tube in which the sound is excited is many times larger than the wavelength λ. In the experiment, the tube radius is bounded by the radiator power. Account must be taken therefore of the contribution made to the linear decrement by the boundary-layer damping. Equation (5.3) is modified in this case as follows [see (3.14)]:

$$\frac{\partial}{\partial x}\left|\frac{\tilde{\rho}_2}{\rho_0}\right|^2=2\Gamma(k)\left|\frac{\tilde{\rho}_2}{\rho_0}\right|^2\left\{1+\frac{1}{kR}\left(\frac{s}{2kD}\right)^{1/2}-2(\gamma-1)f(\Delta k/k)\left|\frac{\rho_1}{\rho_0}\right|^2\right\}. \tag{6.1}$$

It is easy to obtain from (6.1) the threshold value of the amplitude in a tube of large but finite radius for the incident wave:

$$\left|\frac{\rho_1}{\rho_0}\right|^2_{th}=\left[1+\frac{1}{kR}\left(\frac{s}{2kD}\right)^{1/2}\right]\Big/\,2(\gamma-1)f_{max}. \tag{6.2}$$

Note that the contributions of the bulk and surface damping become equal at $kR\sim(kl)^{-1/2}$.

If our perturbation theory is to be valid all the way to the threshold, it is required that the third-order corrections to the amplitude ρ_2 be small compared with its initial value. This criterion can be written in the form

$$|G_\rho|\,|\rho_1/\rho_0|^2\ll 1. \tag{6.3}$$

On the other hand, it follows from (5.6) and (6.2) that the order of magnitude of the threshold $|\rho_1/\rho_0|^2$ is not less than kl. An estimate of $|G_\rho|$ near the threshold, i.e., at $\Delta k/k\sim kl$, yields according to (4.3), (4.6) and (4.7) a value on the order of M/m. Thus, the condition (6.3) for applicability of the perturbation theory reduces to the following relation between the parameters:

$$\frac{M}{m}(kl)\left[1+\frac{1}{kR(kl)^{1/2}}\right]\ll 1. \tag{6.4}$$

Note that the condition (6.4) guarantees smallness of the linear damping.

Let us estimate the threshold power of the radiator, using the well-known equation

$$W={}^1/_2\rho_0 s^3|\rho_1/\rho_0|^2\pi R^2. \tag{6.5}$$

We assume for the estimates $f=300$ kHz, $\lambda=10^{-3}$ m, $l=10^6$ m, $R=5\cdot10^{-3}$ m, $s=3\cdot10^2$ m/s, and $P=10^5$ Pa. Then we obtain the threshold power ~ 10 W.

The criterion (6.3) is met at the given values of the parameters and at $M/m\lesssim 150$. Note that the nonlinearity level $\sim kl$ at which the instability appears in this system is small. Wavebreaking, second-harmonic generation, and other nonlinear processes set in at a considerably higher nonlinearity level $(M/m)kl$. This suggests that nonlinear reflection in pure form is experimentally observable.

It must be stipulated, however, that the approach a stationary nonlinear regime and its stability are problems beyond the scope of perturbation theory and call for further research.

APPENDIX

We estimate the threshold amplitude level and the threshold power for a mixture containing of hydrogen H_2 and sulfur hexafluoride SF_6. We assume a mass concentration

$$c={}^1/_3\approx 0.33. \tag{A.1}$$

The density, specific heat, adiabatic exponent, and speed of sound for pure hydrogen, and also the diffusion coefficient in the mixture and the ratio of the molecular mass components are[7]

$$\begin{aligned}&\rho_0=0.8\cdot10^{-4}\ \text{g/cm}^3,\quad c_P=2.5\ \text{cal/g}\cdot\text{deg},\\&\gamma=1.7,\quad \varkappa=4.0\cdot10^{-4}\ \text{cal/cm}\cdot\text{s}\cdot\text{deg},\\&s=1.3\cdot10^5\,\text{cm/s},\quad D=0.5\,\text{cm}^2/\text{s},\quad m/M=1.4\cdot10^{-2}.\end{aligned} \tag{A.2}$$

The mixture density is $(1-c)^2$ times larger than that of pure hydrogen. Accordingly the speed of sound in the mixture is smaller than in pure hydrogen at the same pressure by a factor $(1-c)^{1/2}$. To calculate the thermal-diffusivity coefficient in the mixture it is also necessary to introduce the correction factor $1-c$. We have thus in the mixture

$$\rho_0=1.2\cdot10^{-4}\text{g/cm}^3,\quad s\approx1.1\cdot10^5\ \text{cm/s},\quad \chi=1.3\,\text{cm}^2/\text{s}. \tag{A.3}$$

The remaining quantities in (A.2) remain unchanged.

Substituting the given values of the parameters in (5.6), we get

$$f_{max}=0.7s/k\ \text{s/cm}^2, \tag{A.4}$$

where the wave number is in reciprocal centimeters. To estimate the threshold amplitude and power, we choose the following values of the frequency and of the tube radius

$$\nu=1\,\text{MHz},\quad k=60\,\text{cm}^{-1},\quad R=0.5\,\text{cm}. \tag{A.5}$$

Then

$$\left|\frac{\rho_1}{\rho_0}\right|^2_{th}=1.3\cdot10^{-3},\qquad W_{th}=7\ \text{W}. \tag{A.6}$$

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