Resonant and parametric interaction of ultrashort light pulses in a multilevel nonlinear medium

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The coherent resonant interaction between multifrequency light and a multilevel system is considered together with the N-wave interaction of wave packets in a medium with a quadratic nonlinearity. The Lax representation of the self-consistent set of Maxwell-Bloch equations for this interaction is found. Soliton solutions resulting from the parametric transformation of ultrashort pulses in three- and four-level system ($N = 3$, and $N = 6$, respectively) are constructed. Equations implementing the Bäcklund transformation, and the recurrence relations defining an infinite series of integrals of motion, are given for $N = 3$.

INTRODUCTION

The resonant and coherent interaction of light with a nonlinear medium has been actively investigated in recent times, both experimentally and theoretically.[1–6] The advent of picosecond techniques has necessitated the development and analysis of models describing the evolution and transformation of ultrashort pulses of light in nonlinear media.

The propagation of ultrashort pulses in two-level systems has attracted most attention.[1,2] Application of the inverse scattering method[7] has resulted in a description of the evolution of ultrashort pulses in nondegenerate media, both damping[2] and growing (see the bibliography in Ref. 8). The inverse scattering method has been successfully applied to the Maxwell-Bloch equations for certain degenerate transitions.[9,10] The transformation of ultrashort pulses with one carrier frequency into one or more ultrashort pulses with other carrier frequencies is possible in the course of parametric multiphoton interaction.[2–6,11–13] Resonance Raman scattering[5,6] is a simple scheme for this transformation. Using the inverse scattering method and numerical techniques, the authors of Refs. 5 and 6 show that efficient transformation of ultrashort pulses is possible in a three-level system interacting resonantly with two- and three-frequency fields.

In general, the self-interaction of light fields must be taken into account in addition to the resonant interaction occurring during the propagation of light in a nonlinear nonresonant medium. The quadratic nonlinearity, giving rise to coherent three-wave interaction,[11–13] is the weakest nonlinearity (whenever it is not forbidden by the symmetry of the problem). The characteristic time for nonresonant three-wave interaction is frequently the shortest of the nonlinear interaction times, and this indicates that the nonresonant three-wave interaction must be taken into account in studies of the propagation and transformation of ultrashort pulses. The following conditions must be satisfied if the nonresonant three-wave interaction is to take place:

$$k_{im}+k_{mj}=k_{ij}, \quad \omega_{im}+\omega_{mj}=\omega_{ij}, \tag{1}$$

where k_{im}, ω_{im} are the central wave vector and the frequency of the wave $\mathscr{E}_{im}$, respectively. The resonant interaction occurs when the frequency ω_{im} is close to the frequency of the transition betwen levels i and m. The nonresonant three-wave interaction is used in parametric amplification and transformation of ultrashort waves.[11–14] The resonant interaction may play a significant role in the dynamics of the parametric transformation of ultrashort pulses in a nonlinear medium and, conversely, the nonresonant three-wave interaction may accelerate the transformation of ultrashort pulses in a resonance medium, or it may slow it down.

Theoretical analyses of coherent multiphoton interactions for times much shorter than the relaxation times of the medium encounter considerable difficulties due to the large dimension of the set of equations involved. Accurate trial solutions are required for the correct numerical solution of such models. Asymptotic solutions describing the final stages of the parametric transformation of ultrashort pulses can often be found but, unless a rigorous solution of the Cauchy problem is available, it is practically impossible to determine the initial conditions that takes the system to a given asymptotic state. The inverse scattering method is the most suitable technique for theoretical analysis of multiphoton processes. Unfortunately, the application of this method to the solution of the Cauchy problem for the evolution of ultrashort pulses is restricted by a number of conditions imposed on the parameters of the medium. Roughly speaking, the number of these restrictions increases as the square of the number of interacting fields. However, numerical calculations[6] have shown that, even when the deviations from these conditions are large and of the order of unity, the dynamics of the transformation of ultrashort pulses retains the basic features typical for the integrable case.

In this paper, we present the Lax representation for a new, self-consistent set of nonlinear equations that can be integrated by the inverse scattering method. The system includes both special cases of the Maxwell-Bloch equations describing self-induced transparency and resonant parametric transformation of ultrashort pulses in a multilevel medium and the equations for the M-wave interaction.[11–13] Section 1 examines a scheme for the resonant and nonresonant three-wave interaction between three-wave packets and a three-level medium. Soliton solutions describing the transformation of ultrashort pulses are described. In the following Section, the results are generalized to the four-level case and $N = 6$. Some integrable reductions of cascade transitions are discussed. The Bäcklund transformations and the recurrence relations defining an infinite series of integrals of

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motion of the new integrable system are given in the Appendix for $N = 3$.

1. RESONANT INTERACTION WITH A THREE-LEVEL MEDIUM AND NONRESONANT THREE-WAVE INTERACTION

Three wave packets with slowly-varying envelopes

$$E=\sum_{i,j=1}^{3}[E_{ij}(x,t)\exp(i\omega_{ij}t-ik_{ij}x)+\text{c.c.}] \qquad (2)$$

interact resonantly with a three-level medium (see Fig. 1). The carrier frequency ω_{ij} is close to the i-j transition frequency. When the nonresonant three-wave interaction is taken into account, the truncated Maxwell equations assume the form

$$(\partial_t+V_{ij}\partial_x)E_{ij}=i\frac{4\pi n_0\omega_{ij}}{n_{ij}^2}d_{ij}\langle\rho_{ij}\rangle+\frac{2\pi i\omega_{ij}}{n_{ij}^2}\chi E_{im}E_{mj},$$

$$E_{im}=E_{mi}^*,\quad V_{ij}=cn_{ij}^{-1};\quad i,j,m=1-3,\quad i\neq j,\quad i\neq m,\quad j\neq m. \qquad (3)$$

where d_{ij}, ω_{ij}, n_{ij} are, respectively, the dipole moments, frequencies, and nonresonant refractive indices corresponding to the $i\leftrightarrow j$ transition, χ is the nonlinear (second-order) susceptibility of the medium, $\langle\rho_{ij}\rangle$ are the off-diagonal components of the density matrix averaged over the frequency distribution, and n_0 is the density of the resonance medium. The Bloch equations for the density matrix ρ_{ij} are conveniently written in the form:

$$2\partial_t\rho_{ij}+i(\nu_i-\nu_j)\rho_{ij}=i\mathscr{E}_{ij}g_{ij}(\rho_{jj}-\rho_{ii})+i(\mathscr{E}_{ik}g_{ik}\rho_{kj}-\mathscr{E}_{kj}g_{kj}\rho_{ik}),$$

$$\partial_t\rho_{kk}=\operatorname{Im}(\mathscr{E}_{ki}\rho_{ik}g_{ki}+\mathscr{E}_{kj}\rho_{jk}g_{kj}), \qquad (4)$$

where

$$\mathscr{E}_{ij}=E_{ij}\frac{(8\pi\hbar n_0\omega_{ij})^{-1/2}}{n_{ij}},\qquad g_{ij}=d_{ij}\frac{(8\pi\hbar n_0\omega_{ij})^{1/2}}{n_{ij}},$$

$$\rho_{ij}=\rho_{ji}^*,\quad \rho_{11}+\rho_{22}+\rho_{33}=1;\quad i,j,k=1-3,\quad i\neq j,\quad i\neq k,\quad j\neq k.$$

The set of equations defined by (3) can now be written in the form

$$(\partial_t+V_{ij}\partial_x)\mathscr{E}_{ij}=ig_{ij}\langle\rho_{ij}\rangle+i\varkappa\mathscr{E}_{ik}\mathscr{E}_{kj},$$
$$i,j,k=1-3,\quad i\neq j,\quad j\neq k,\quad i\neq k; \qquad (5)$$

where $\varkappa = 16\pi^2 n_0 h_\chi/(\omega_{12}\omega_{13}\omega_{23})^{1/2}(n_{12}n_{13}n_{23})^{-1}$. Under certain specific restrictions on the physical constants g, V and $\varkappa$, the set of equations defined by (4) and (5) can be represented by the compatibility conditions

$$\partial_x L+\partial_t M=i[M,L] \qquad (6)$$

for the two sets of linear equations

$$i\partial_t\psi=L(A,B,\mathscr{E},\rho,\lambda)\psi, \qquad (7)$$

$$-i\partial_x\psi=M(A,B,\mathscr{E},\rho,\lambda)\psi, \qquad (8)$$

where A and B are constant diagonal matrices and λ is the spectral parameter. The matrices L and M that implement the Lax representation of (4) and (5) have the form

$$L=-A^{-1}B\lambda+BA^{-1}QA-QB, \qquad (9)$$

$$M=A^{-1}\lambda+A^{-1/2}\langle\tilde\rho\rangle A^{1/2}+Q-A^{-1}QA. \qquad (10)$$

where $A_{ii}=a_i$, $B_{ii}=b_i$, $A_{ij}=B_{ij}=0$, $i\neq j$,

$$Q_{ii}=0,\quad Q_{ij}=\mathscr{E}_{ij}(a_i-a_j)^{-1/2},\quad i\neq j;$$

$$\langle\tilde\rho_{ij}\rangle=\int_{-\infty}^{\infty}\rho_{ij}(x,t,\zeta)f(\zeta)(\zeta-\lambda+i0)^{-1}\,d\zeta,\quad i,j=1-3, \qquad (11)$$

and $f(\zeta)$ is the frequency distribution of the resonance medium. Substituting (9)–(11) in (6), and comparing with (4) and (5), we find, after some laborious calculations, that the existence conditions for the Lax representation (9)–(11) are

$$K_{12}+K_{23}=K_{13},\quad \Omega_{12}+\Omega_{23}=\Omega_{13}, \qquad (12)$$

$$\varkappa=(V_{12}-V_{13})(K_{12}K_{13}K_{23}^{-1})^{1/2}, \qquad (13)$$

where

$$K_{ij}=d_{ij}^2\omega_{ij}\cdot 8n_0\pi\hbar c^{-2},\quad \Omega_{ij}=K_{ij}V_{ij}.$$

The physical constants V, g, ν and $\varkappa$ can be expressed in terms of the elements of matrices A and B, as follows:

$$V_{ij}=\frac{a_ib_j-b_ja_i}{a_i-a_j},\qquad g_{ij}=\frac{a_ib_j-b_ia_j}{[a_ia_j(a_i-a_j)]^{1/2}},\qquad \nu_i=\frac{\zeta}{a_i},$$

$$\varkappa=(a_1b_2-b_1a_2+a_2b_3-b_2a_3+a_3b_1-b_3a_1)\times[(a_1-a_2)(a_2-a_3)(a_1-a_3)]^{-1/2},$$

where we are assuming that $0<a_3<a_2<a_1$.

Conditions (12) and (13) correspond to the case where all three optical dipole transitions are allowed. Let us now consider the case of two allowed dipole transitions (transitions 1–2 and 1–3 in the Fig. 1). The 2–3 transition is forbidden. The conditions for the existence of the Lax representation, given by (9)–(11), now become much simpler and take the form

$$V_{23}=0,\quad \Omega_{12}=\Omega_{13}, \qquad (14)$$

$$\varkappa=\Omega_{12}(K_{12}^{-1}-K_{13}^{-1})^{1/2}. \qquad (15)$$

In the special case where $V_{12}=V_{13}$, conditions (12) and (14) are found to be identical with the conditions for the validity of the inverse scattering method (given in Refs. 5 and 6). This reduction describes only the resonant interaction with atomic transitions. Moreover, the symmetry of the problem was used in constructing the Lax representation (9)–(11). At exact resonance $[f(\zeta)=\delta(\zeta)]$, the Lax rep-

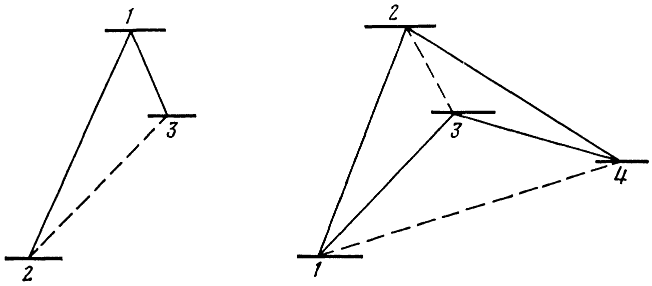

FIG. 1. Transition and resonance interaction schemes for wave packets. Solid lines correspond to the wave packet with frequency close to the dipole-allowed transition frequency, whereas broken lines correspond to the wave packet with carrier frequency close to the dipole-forbidden transition frequency.

resentation (9)–(11) can be found by a reduction of the Lax representation for the set of equations that corresponds to the six-wave interaction.[14]

The Lax representation given above enables us to use the inverse scattering method to investigate the transformation of ultrashort pulses. Some of the results obtained with this representation are given in the Appendix. The simplest single-pulse solution can be found by the Zakharov-Shabat method.[15] The soliton solution is defined by the projector

$$P_{ij}=m_i m_j\Big(\sum_{k=1}^{3}|m_k|^2\Big)^{-1},$$

where m_i are vectors forming the basis of the space $M=\mathrm{Im}\,P$:

$$R_i=\iint_{-\infty}^{\infty}\rho_{ii}(x,0,\zeta)f(\zeta)(\zeta-\lambda+i0)^{-1}\,dx\,d\zeta,$$

$$m_i=\alpha_i\exp\left[-i\lambda(xa_i^{-1}+tb_ia_i^{-1})-iR_i\right].$$

Following Ref. 15, we find that

$$\mathscr{E}_{ik}=(\lambda-\lambda^*)(|a_i-a_k|)^{1/2}b_i^{-1}P_{ik}=\beta_{ik}P_{ik}. \tag{16}$$

In the interaction scheme with two allowed dipole transitions 1–2 and 1–3, we find from (16) that

$$\begin{aligned}|\mathscr{E}_{12}|&=\beta_{12}[\mathrm{ch}\,(T_{12})+|\alpha_3/\alpha_2|^2\exp\,(T_{12}-q_{23})]^{-1},\\ |\mathscr{E}_{13}|&=\beta_{13}[\mathrm{ch}(T_{13})+|\alpha_2/\alpha_3|^2\exp(T_{13}+q_{23})]^{-1},\end{aligned} \tag{17}$$

where

$$T_{ij}=\eta(K_{ij}x+\Omega_{ij}t)-\tilde{R}_i+\tilde{R}_j,$$

$$q_{23}=2\left[\left(\frac{\Omega_{12}}{V_{12}}-\frac{\Omega_{12}}{V_{13}}\right)\eta+\tilde{R}_2-\tilde{R}_3\right],$$

$$\eta=\mathrm{Im}\,\lambda,\quad \tilde{R}_i=\mathrm{Im}\,R_i.$$

Solution (17) can be obtained with the aid of a Bäcklund transformation (see Appendix). The asymptotic behavior of (17) is determined by the sign of q_{23}. When $q_{23}>0$, we have

$$\begin{aligned}&x\to-\infty,\quad |\mathscr{E}_{12}|\to\beta_{12}\,\mathrm{sech}\,(T_{12}),\quad \mathscr{E}_{13}\to0,\\ &x\to\infty,\quad \mathscr{E}_{12}\to0,\quad |\mathscr{E}_{13}|\to\beta_{13}\,\mathrm{sech}\,(T_{13}).\end{aligned} \tag{18}$$

The solution given by (17) and (18) describes resonant Raman scattering in a three-level system with nonresonant three-wave interaction taken into account for $(V_{12}-V_{13})\times(\tilde{R}_2-\tilde{R}_3)<0$.

The transfer of ultrashort-pulse pump energy $\mathscr{E}_{12}$ to the Stokes ultrashort pulse is determined by the amplitude of the pump pulse. Thus, for $\eta>\eta_k$, where η_k is given by the condition $q_{23}(\eta_k)=0$, the transfer of energy to the Stokes ultrashort pulses does not occur. Unstable simultaneous propagation of the ultrashort pulses over allowed dipole transitions occurs when $q_{23}=0$. The Lax representation given above can be generalized to a larger number of resonance levels and fields.

2. RESONANT INTERACTION WITH A FOUR-LEVEL MEDIUM AND THE SIX-WAVE INTERACTION

The Maxwell-Bloch equations for the resonant interaction of six-wave packets with a four-level medium and the simultaneous six-wave interaction in a medium with a quadratic nonlinearity will now be written in the form

$$\begin{aligned}&\partial_t\rho_{\alpha\alpha}=\mathrm{Im}\,(U_{\alpha,\alpha+1}\rho_{\alpha+1,\alpha}+U_{\alpha,\alpha-1}\rho_{\alpha-1,\alpha}+U_{\alpha,\alpha+2}\rho_{\alpha+2,\alpha}),\\ &2\partial_t\rho_{\alpha,\alpha+1}+i(\nu_{\alpha+1}-\nu_\alpha)=i[U_{\alpha,\alpha+1}(\rho_{\alpha+1,\alpha+1}-\rho_{\alpha\alpha})\\ &+U_{\alpha,\alpha-1}\rho_{\alpha-1,\alpha+1}-U_{\alpha+2,\alpha+1}\rho_{\alpha,\alpha+2}\\ &\qquad+U_{\alpha,\alpha+2}\rho_{\alpha+2,\alpha+1}-U_{\alpha-1,\alpha+1}\rho_{\alpha,\alpha-1}],\\ &2\partial_t\rho_{\alpha-1,\alpha+1}+i(\nu_{\alpha+1}-\nu_{\alpha-1})=U_{\alpha-1,\alpha}\rho_{\alpha,\alpha+1}+U_{\alpha-1,\alpha-2}\\ &\times\rho_{\alpha-2,\alpha+1}-U_{\alpha+2,\alpha+1}\rho_{\alpha-1,\alpha+2}-U_{\alpha,\alpha+1}\rho_{\alpha-1,\alpha}\\ &\qquad+U_{\alpha-1,\alpha+1}(\rho_{\alpha+1,\alpha+1}-\rho_{\alpha-1,\alpha-1}),\\ &(\partial_t+V_{ij})\mathscr{E}_{ij}=\varkappa_{ijk}\mathscr{E}_{ik}\mathscr{E}_{kj}+\varkappa_{iqj}\mathscr{E}_{iq}\mathscr{E}_{qj}+g_{ij}\langle\rho_{ij}\rangle,\\ &\qquad U_{\alpha\beta}=\mathscr{E}_{\alpha\beta}g_{\alpha\beta},\end{aligned} \tag{19}$$

where $i, j, k, q=1$–4 and are not equal in pairs, and α, $\beta=1$–4 (mod 4). Apart from the indices, the notation is the same as in the last Section. The coupling constant $\varkappa_{imj}$ corresponds to the nonresonant three-wave interaction between fields for which (1) is satisfied. The figure shows the interaction scheme. The Lax representation for (19) is given by (9) and (10) with 4×4 matrices:

$$\begin{aligned}&A_{ik}=\delta_{ik}a_i,\quad B_{ik}=\delta_{ik}b_i,\quad Q_{ij}=\mathscr{E}_{ij}(a_i-a_j)^{-1/2},\\ &Q_{ii}=0,\quad \langle\tilde{\rho}_{ij}\rangle=\int_{-\infty}^{\infty}\rho_{ij}(x,t,\zeta)f(\zeta)(\zeta-\lambda+i0)^{-1}\,d\zeta,\\ &i,j=1\text{–}4.\end{aligned} \tag{20}$$

The conditions for the validity of the Lax representation (9), (10), and (20) will, in general, consist (all dipole transitions are allowed) of a set of conditions for each triad of fields for which (1) is satisfied:

$$\begin{aligned}&K_{ik}+K_{kj}=K_{ij},\quad \Omega_{ik}+\Omega_{kj}=\Omega_{ij},\\ &\varkappa_{ijk}=(V_{ij}-V_{kj})(K_{ij}K_{kj}K_{ik}^{-1})^{1/2},\\ &i,\ j,\ k=1\text{–}4,\quad i\neq j,\quad j\neq k,\quad i\neq k.\end{aligned} \tag{21}$$

When the 2–3 and 1–4 transitions are forbidden, the conditions for the validity of the inverse scattering method assume the form

$$\begin{aligned}&V_{23}=V_{14}=0,\quad \Omega_{ij}=\Omega,\quad i,j=12,\ 24,\ 34,\ 13;\\ &\frac{1}{V_{12}}+\frac{1}{V_{24}}+\frac{1}{V_{34}}=\frac{1}{V_{13}},\quad \varkappa_{ijk}=(\Omega\,|V_{ij}-V_{kj}|)^{1/2},\\ &i,\ j,\ k=1\text{–}4,\quad i\neq j,\quad j\neq k,\quad k\neq i.\end{aligned} \tag{22}$$

As in the last Section, the solution of (19) can be performed with the aid of the Riemann problem. The final expressions for the field moduli in the case of the single-pole solution are:

$$\begin{aligned}\mathscr{E}_{12}=\beta_{12}\Big[\mathrm{ch}(T_{12})&+\left|\frac{\alpha_3}{\alpha_1}\right|^2\exp(-T_{12}+q_{32})\\ &+\left|\frac{\alpha_4}{\alpha_2}\right|^2\exp(T_{12}+q_{41})\Big]^{-1},\\ \mathscr{E}_{13}=\beta_{13}\Big[\mathrm{ch}(T_{13})&+\left|\frac{\alpha_2}{\alpha_3}\right|^2\exp(-T_{13}-q_{32})\\ &+\left|\frac{\alpha_4}{\alpha_2}\right|^2\exp(T_{12}+q_{41})\Big]^{-1},\end{aligned}$$

$$\mathscr{E}_{24}=\beta_{24}\Big[\operatorname{ch}(T_{24})+\Big|\frac{\alpha_3}{\alpha_2}\Big|^2\exp(T_{24}+q_{32})+\Big|\frac{\alpha_1}{\alpha_4}\Big|^2\exp(-T_{12}-q_{41})\Big]^{-1},$$

$$\mathscr{E}_{34}=\beta_{34}\Big[\operatorname{ch}(T_{34})+\Big|\frac{\alpha_1}{\alpha_3}\Big|^2\exp(-T_{34}-q_{41})+\Big|\frac{\alpha_2}{\alpha_4}\Big|^2\exp(T_{34}-q_{32})\Big]^{-1},$$

$$T_{ij}=\eta(K_{ij}x-\Omega_{ij}t)+\ln\Big|\frac{\alpha_i}{\alpha_j}\Big|-\tilde{R}_i+\tilde{R}_j,$$

$$q_{ij}=2[K_{ij}\eta x-(\tilde{R}_i-\tilde{R}_j)],$$

$$R_i=\operatorname{Im}\iint_{-\infty}^{+\infty}\rho_{ii}(x,0,\zeta)f(\zeta)(\zeta-\lambda+i0)^{-1}\,dx\,d\zeta,$$

$$\eta=\operatorname{Im}\lambda. \qquad (23)$$

This solution was obtained for the case of two forbidden dipole transitions 2–3 and 1–4 (see Fig. 1). The asymptotic behavior of (23) is determined by the signs of q_{32} and q_{41}. When $q_{32}<0$ and $q_{41}>0$, we find from (23) that

$$x\to-\infty,\quad \mathscr{E}_{13}=\beta_{13}\operatorname{sech}(T_{13});\quad \mathscr{E}_{12},\mathscr{E}_{24},\mathscr{E}_{34}\to0,$$
$$x\to\infty,\quad \mathscr{E}_{24}=\beta_{24}\operatorname{sech}(T_{24});\quad \mathscr{E}_{12},\mathscr{E}_{13},\mathscr{E}_{34}\to0. \qquad (24)$$

As can be seen from (24), the single-pole solution (23) describes the parametric transformation of ultrashort pulses with a carrier frequency that is in resonance with the 1–3 transition into ultrashort pulses with a carrier frequency close to the 2–4 transition frequency. The many-pole solutions of (19) correspond to the decay of the pump soliton into a number of solitons with different frequencies.

We note that the Lax representation cannot be constructed for the three dipole-forbidden transitions 2–3, 1–4, and 3–4. Numerical analysis of this case is of interest in this connection. The above system, which can be integrated by the inverse scattering method, can be generalized to a large number of resonance transitions and fields. The Lax representation for this system is given by (9) and (10) with matrices A, B, and Q of large dimension.

We now consider an exmaple of integrable Maxwell-Bloch equations of another type. Consider a model consisting of a set of cascade transition. We assume that the dipole moment $\mu_{k-1,k}=\mu(k)$ of a cascade transition is a slowly-varying function of the transtion number k. Coherent excitation of the levels is described by the following equations for the state amplitudes:

$$\partial_t a_k=i[\mu(k+1)\mathscr{E}_{k+1}a_{k+1}+\mathscr{E}_k^* a_{k-1}\mu(k)]. \qquad (25)$$

The truncated Maxwell-Bloch equations for the field amplitudes $\mathscr{E}_k$ that are in resonance with the $k-1\to k$ transition have the following form in the case of high radiative losses:

$$\gamma\mathscr{E}_k=\mu(k)\omega_k n_0(2\hbar)^{-1}a_k a_{k-1}^*, \qquad (26)$$

where γ is the radiative damping coefficient and ω_k is the $k-1\to k$ transition frequency. For large k, the set of equations given by (25)–(26) reduces to the differential-difference equation

$$\partial_\tau N_k=N_k(N_{k+1}-N_{k-1}),\quad \tau=t\gamma^{-1}\mu^2(1)\omega_1 n_0,$$
$$N_k\sim a_k a_k^*/\mu^2(k)\omega_k. \qquad (27)$$

This equation originally appeared in the connection with the study of Langmuir oscillations and was integrated by the inverse scattering method by Manakov.[7] Soliton solutions of (27) describe the coherent transfer of excitation over a chain of energy levels.

3. CONCLUSION

We have investigated a new integrable model. The Maxwell-Bloch equations describing self-induced transparency and parametric transformation of solitons in a multilevel resonant medium, and the decay type N-wave interaction, are special cases of this model. The Lax representation of this integrable system enables us to use the well-known exact methods employed in the solution of the Cauchy problem.[7] We have not examined the influence of the resonance interaction of the parametric transformation of solitons in the course of the nonresonance three-wave interaction. The two-pole solution, which demonstrates the decay of the pump soliton into two stable solitons, can readily be constructed as in Ref. 13. The influence of the resonant medium on the dynamics of the nonresonant three-wave interaction is determined by the initial level populations.

We note that the integrability conditions given above contain the parameters of both nonresonant and resonant media. This may facilitate the choice of media with parameters close to the "exact" values. This is an important consideration because radiative losses by the ultrashort pulses decrease as we approach the integrable case.

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APPENDIX

We shall now find the Bäcklund transformation for $N=3$ and exact resonance $[f(\zeta)=\delta(\zeta)]$. We shall represent the set of linear equations (7) and (8) by the second-order Riccati equations:

$$\partial_x R_1=-i\lambda\left(\frac{1}{a_1}-\frac{1}{a_2}+\frac{\rho_{11}-\rho_{33}}{\lambda^2}\right)R_1+W_{12}R_2+W_{13}-W_{31}R_1^2-W_{32}R_1R_2, \qquad (\text{A.1})$$

$$\partial_x R_2=-i\lambda\left(\frac{1}{a_2}-\frac{1}{a_3}+\frac{\rho_{22}-\rho_{33}}{\lambda^2}\right)R_1+W_{21}R_1+W_{23}-W_{31}R_1R_2-W_{32}R_1^2, \qquad (\text{A.2})$$

$$\partial_t R_1=i\lambda\left(\frac{b_1}{a_1}-\frac{b_3}{a_3}\right)R_1-U_{12}R_2-U_{13}+U_{31}R_1^2+U_{32}R_1R_2, \qquad (\text{A.3})$$

$$\partial_t R_2=i\lambda\left(\frac{b_2}{a_2}-\frac{b_3}{a_3}\right)R_2-U_{21}R_1-U_{23}+U_{31}R_1R_2+U_{32}R_1^2, \qquad (\text{A.4})$$

where

$$W_{ij}=i\left[\frac{\mathscr{E}_{ij}(a_i-a_j)}{(|a_i-a_j|)^{1/2}a_i}-\frac{1}{\lambda}\left(\frac{a_j}{a_i}\right)^{1/2}\rho_{ij}\right],$$

$$U_{ij}=i[\mathscr{E}_{ij}(b_ia_j-a_ib_j)|a_i-a_j|^{-1/2}a_i^{-1}].$$

Since the problem is invariant under the operation of complex conjugation, we find that

$$\mathscr{E}_{23}=\mathscr{E}_{23}'-2i\operatorname{Im}\lambda R_2(a_2-a_3)^{1/2}a_2^{-1}D^{-1}, \qquad (\text{A.5})$$

$$\mathscr{E}_{13}=\mathscr{E}_{13}'-2i\operatorname{Im}\lambda R_1(a_1-a_3)^{1/2}a_1^{-1}D^{-1}, \qquad (\text{A.6})$$

$$\mathscr{E}_{12}=\mathscr{E}_{12}'-2i\,\mathrm{Im}\,\lambda R_1R_2^*(a_1-a_2)^{1/2}a_1^{-1}D^{-1}, \qquad (A.7)$$

where

$$D=1+|R_1|^2+|R_2|^2.$$

The Bäcklund transformation (A.1)–(A.7) relates to two solutions $\mathscr{E}_{ij}'$ and $\mathscr{E}_{ij}$ of (4) and (5).

Integrable nonlinear sets of equations have an infinite number of integrals of motion.[7] The recurrence relation defining this set for (4) and (5) in exact resonance will now be reproduced. Following Ref. 13, we write

$$\psi_{ij}^+(x,\lambda)=\delta_{ij}\exp\left(-i\frac{\lambda}{a_j}x+\int_x^\infty \chi_j(s,\lambda)\,ds\right)$$
$$+(1-\delta_{ij})A_{ij}(x,\lambda)\exp\left(-i\frac{\lambda}{a_j}x+\int_x^\infty \chi_j(s,\lambda)\,ds\right). \qquad (A.8)$$

Substituting this in (8), we obtain

$$ia_ia_j\partial_xA_{ij}-a_iA_{ij}\sum_k\left[(a_j-a_k)Q_{jk}-\frac{(a_ja_k)^{1/2}}{\lambda}\rho_{jk}\right]A_{kj}$$
$$+a_j\left[(a_i-a_j)Q_{ij}-\frac{(a_ia_j)^{1/2}}{\lambda}\rho_{ij}\right]$$
$$+a_j\sum_{k\neq j}\left[(a_i-a_k)Q_{ik}-\frac{(a_ia_j)^{1/2}}{\lambda}\rho_{ik}\right]A_{kj}$$
$$=\lambda A_{ij}\left[a_j-a_i+\frac{(\rho_{ii}-\rho_{jj})}{\lambda^2}a_ia_j\right].$$

The coefficients of the expansion in powers of $1/\lambda$

$$A_{ij}(x,\lambda)=\sum_{n=1}^\infty A_{ij}^{(n)}(x)\lambda^{-n}$$

are given by

$$A_{ij}^{(n+1)}(a_j-a_i)=(\rho_{jj}-\rho_{ii})a_ia_jA_{ij}^{(n-1)}+ia_ia_j\partial_xA_{ij}^{(n)}$$
$$-a_i\sum_{n_1+n_2=n}A_{ij}^{(n_1)}\sum_k(a_j-a_k)Q_{jk}A_{kj}^{(n_2)}$$
$$+a_i\sum_{n_1+n_2=n-1}(a_ia_k)^{1/2}A_{ij}^{(n_1)}\rho_{jk}A_{kj}^{(n_2)}$$
$$+a_j\sum_{k\neq j}[(a_i-a_k)Q_{jk}A_{kj}^{(n)}-(a_ia_j)^{1/2}\rho_{ik}A_{kj}^{(n-1)}].$$

For the density of the integral of motion

$$I_j^{(n)}=i\int_{-\infty}^{\infty}\chi_j^{(n)}(x)\,dx,$$

we have the relation

$$\chi_j^{(n)}=-\frac{i}{a_j}\sum_k[(a_j-a_k)Q_{jk}(x)A_{kj}^{(n)}-(a_ja_k)^{1/2}\rho_{jk}A_{kj}^{(n-1)}].$$

In particular, $\chi_j^{(1)}$ is identical with the integral $I_j^{(1)}$, found in Ref. 13. The next integral has the density

$$\chi_j^{(2)}=-\frac{i}{a_j}\sum_k\Big\{(a_j-a_k)Q_{jk}\Big[\rho_{kj}a_j(a_ka_j)^{1/2}+(\rho_{kk}-\rho_{jj})a_ka_j$$
$$+ia_k^2a_j\partial_xQ_{kj}$$
$$-a_k^2\sum_{k'}(a_k-a_{k'})Q_{kk'}Q_{k'j}\Big]-(a_ja_k)^{1/2}\rho_{jk}a_jQ_{kj}\Big\}.$$

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