Derivation of a chromoelectric string from non-Abelian gauge field theory

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A consistent derivation is given of the effective action for the topological sector of gauge fields that contains the solution given in Ref. 1 for the relativistic chromoelectric string with quarks at its ends. The lower bound for the Yang-Mills action in this sector is obtained.

1. INTRODUCTION

This is a continuation of our previous paper,[1] in which we described a topologically nontrivial configuration of a gauge field that forms a chromoelectric string with quarks at the ends and a bare tension coefficient k_0. This string differs from the Nambu string (and its subsequent generalizations) by the fact that:

(1) at the classical level, the world surface of the string in Euclidean four-dimensional space has a constant scalar curvature R, where $|R| \sim k_0$, and

(2) the string action is quantized as a consequence of the topological restrictions that are common in quantum soliton (instanton) theory.

These properties suggest that, when quantum fluctuations about the string field are calculated, it will be possible to restrict the analysis to the quasiclassical approximation and thus avoid the well-known pathologies that arise in string quantization.

The essence of the problems that we consider here is as follows. The string field configuration was obtained in Ref. 1 as a solution of the equations of motion, found by taking the variation of the effective action S_{eff} for the hadron field correlators. The form of S_{eff} was determined by observing that, in the expansion in terms of the number N of colors, the principal contribution was provided by planar diagrams. Consequently, when the correlators are evaluated, we need only integrate over a certain subclass of gauge fields that provides the biggest contribution in the leading order in $1/N$. A subclass of this kind, consisting of two-dimensional fields, was determined in Ref. 2 on the basis of intuitive considerations and the hypothesis of dimensional reduction,[3] which states that, in the confinement phase, the four-dimensional Yang-Mills theory reduces effectively to a two-dimensional theory.

This choice of the subclass of fields was subsequently justified by the results reported in Ref. 2, since it led to a string field whose contribution to the functional integral did not vanish as $N \to \infty$ (in contrast to, for example, the contribution of instantons[4]).

In the present paper, we use topological arguments to give a consistent derivation of the above subclass of fields, which, in turn, provides a rigorous justification for the derivation of S_{eff}. This means that the chromoelectric string (with the bare tension coefficient) is obtained from first principles in gauge field theory, using the computational procedure described in Ref. 1.

Our paper is constructed as follows. Section 2 gives a derivation of the effective action for the topological field sector containing the solution describing the string. In Section 3, we consider the stability of the solution against small fluctuations in the gauge field.

2. DERIVATION OF EFFECTIVE ACTION

The hadronic n-particle correlator $K(1, \ldots, n)$ in Euclidean space is given by

$$K(1,\ldots,n)=B^{-1}\int d\mu(A) \times \int D\psi\, D\psi^{+}\{\exp(-S_{\text{Y-M}})\,[M(\Gamma_n)\ldots M(\Gamma_1)]\}, \tag{1}$$

where

$$B=\int d\mu(A)\int D\psi\, D\psi^{+}\exp(-S_{\text{Y-M}}),$$
$$S_{\text{Y-M}}=\int d^4x\left\{\frac{1}{4}G_{\mu\nu}^{a}G_{\mu\nu}^{a}+\psi^{+}i(\gamma_\mu D_\mu+m)\psi\right\},$$
$$G_{\mu\nu}^{a}=\partial_\mu A_\nu^{a}-\partial_\nu A_\mu^{a}-ef^{abc}A_\mu^{b}A_\nu^{c},\quad D_\mu=\partial_\mu+ie\lambda^a A_\mu^{a}/2\equiv\partial_\mu+iA_\mu. \tag{2}$$

The gauge-invariant quantities $M(\Gamma_i)$ $(i = 1, \ldots, n)$ in (1) are given by

$$M(\Gamma_i)\equiv M_\alpha^{\beta}(\Gamma_i)$$
$$=\psi_c^{\beta+}(y_i')\left[P\exp\left(-ie\frac{\lambda^a}{2}\int_{y_i}^{y_i'}dx_\mu A_\mu^{a}\right)\right]_{cc'}\psi_{c',\alpha}(y_i), \tag{3}$$

and are viewed as the field operators of composite mesons. In these expressions, α, β is the set of indices of the Lorentz group $O(4)$ and the flavor group and $c.\ c'$ are the indices of the color gauge group $SU(N)$. The integral with respect to x_μ in (3) is evaluated along the path Γ_i joining the points y_i, y_i'. The connected part of the correlator (1) within the framework of the $1/N$ expansion can be reduced[5,6] to the expression

$$K(1,\ldots,n)\approx\sum_{perm}\delta_P\prod_{q=1}^{n}\tilde{D}x_q\left\{\exp\left[-\frac{1}{2}\int_{\gamma_1}^{\gamma_2}d\gamma\left(\frac{\dot{x}^2}{\lambda}+\lambda m^2\right)\right] \times\langle O(\Gamma)\rangle_A\right\}, \tag{4}$$

where

$$\tilde{D}x_q=\left[\frac{d\lambda\,\Delta\gamma}{4\pi i}Dx_\mu(\gamma)\right]_q,\quad \dot{x}_\mu=\frac{dx_\mu}{d\gamma},$$
$$O(\Gamma)=\mathrm{Tr}\left[P\exp\left(-ie\oint_\Gamma dx_\mu A_\mu\right)\right], \tag{5}$$

and δ_p is the parity of the Fermi-field permutation. The closed contour Γ in (5) consists of the path Γ_i in the opera-

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tors $M(\Gamma_i)$, whose ends are joined by the quark trajectories $x_\mu(\gamma)$ (see the figure in Ref. 1). The variable γ parametrizes the contour Γ in accordance with $x_\mu = x_\mu(\gamma)$, and varies monotonically from zero to unity with $x_\mu(0) = x_\mu(1)$.

Next, $O(\Gamma)$ can be rewritten in the form of a functional integral over the Grassmann fields $\xi_c(\gamma)$ $(c = 1, ..., N)$ (Refs. 5 and 7) that belongs to the fundamental representation of $SU(N)$ and is scalar under the Lorentz group $O(4)$:

$$O(\Gamma)=\int\prod_{\gamma}[iD\xi_c(\gamma)D\xi_c^*(\gamma)]\exp(-S[\xi])i\xi_c(1)\xi_c^*(0), \tag{6}$$

$$S[\xi]=\oint_\Gamma d\gamma\,\xi_c^*(\gamma)\left[\frac{d}{d\gamma}+ieA_\mu\frac{dx_\mu}{d\gamma}\right]_{cd}\xi_d(\gamma), \tag{7}$$

$$(\xi_1\xi_2)^*=\xi_1^*\xi_2^*. \tag{7'}$$

After that, each term in (4) is written in the form

$$B^{-1}\int d\mu(A)\int\prod_{q=1}^{n}\mathcal{D}x_q\int[iD\xi(\gamma)D\xi^*(\gamma)]$$
$$\times\exp(-S[A,\xi,x])i\xi_c(1)\xi_c^*(0), \tag{8}$$

where

$$S[A,\xi,x]=\sum_{q=1}^{n}\left[\frac{1}{2}\int_{\gamma_1}^{\gamma_2}d\gamma\left(\frac{\dot{x}^2}{\lambda}+\lambda m^2\right)\right]_q+\frac{1}{4}\int d^4x\,G_{\mu\nu}^aG_{\mu\nu}^a$$
$$+\oint_\Gamma d\gamma\,\xi_c^*(\gamma)\left[\frac{d}{d\gamma}+ieA_\mu\dot{x}_\mu\right]_{cd}\xi_d(\gamma). \tag{9}$$

From now on, the functional integral (8) will be examined in the quasiclassical approximation.[1)] This means that we must have classical self-consistent solutions A^{cl}, ξ^{cl}, x^{cl} of the equations of motion, obtained from the condition $\delta S = 0$. In particular, if we take the variation of (9) over the Grassmann field ξ, we obtain the equation

$$\frac{d\xi_c}{d\gamma}+ie\left(\frac{\lambda^a}{2}\right)_{cd}\xi_dA_\mu^a\dot{x}_\mu=0, \tag{10}$$

from which it follows that ξ^{cl} is covariantly constant. The formal solution of (10) can now be expressed in terms of the ordered exponent

$$\xi_c^{cl}(\gamma)=\left[P\exp\left(-ie\frac{\lambda}{2}\int_0^\gamma d\gamma'\frac{dx_\mu}{d\gamma'}A_\mu^a\right)\right]_{cd}\xi_d(0) \tag{11}$$

[$\xi^{cl}(0)$ is arbitrary], which is an element of the group $SU(N)$. The field $\xi(\gamma)$ is defined on the contour Γ and is a mapping of this contour into the group $SU(N)$. Since $SU(N)$ is singly-connected for $N \geqslant 2$, the mapping (11) is topologically trivial, i.e., $\pi_1[SU(N)] = 0$. The only exception is the spontaneous breaking of $SU(N)$ to the local subgroup $U(1)$, since $\pi_1[U(1)] = \pi_1(S^1) = \mathbf{Z}$, where $\mathbf{Z}$ is the group of integers and π_1 is the first homotopic group of mappings. We know that topologically nontrivial solutions of the classical equation are stable field configurations. We shall therefore confine our attention to evaluating the contribution of precisely these solutions to the integral (8). If the gauge field A^{cl} implements the spontaneous breaking of $SU(N)$ to $U(1)$, we can choose a gauge in which the ordered exponent in (11) reduces to the usual exponent. (Special cases of fields for which this cannot be done will not be examined here).

Moreover, since $\xi^{cl}(\gamma)$ is a solution of the differential equation (10), we demand that it be single-valued. All in all, this leads to the requirement that the argument of the exponential in (11) be determined by the integral of the Abelian field a_μ, where, for a closed contour Γ, this integral must assume the discrete values given by

$$\oint_\Gamma dx_\mu a_\mu=\int_0^1 d\gamma\frac{dx_\mu}{d\gamma}a_\mu=\frac{2\pi Q}{F_N},\qquad Q=0,\pm1,\pm2\ldots, \tag{12}$$

where F_N is a number that depends on the dimension N of the group $SU(N)$ and Q is the number of "windings" in the mapping $\Gamma \to U(1)$. A specific example is given in Ref. 1, where the solution (11) assumes the form $\xi_c(\gamma) = \exp\{-i\varphi(\gamma)\}\xi_c(0)$. The phase

$$\varphi(\gamma)=F_N\int_0^\gamma d\gamma'\frac{dx_\mu}{d\gamma'}a_\mu$$

does not depend on the color index c $(c = 1, ..., N)$ and F_N is identical with the quadratic Casimir operator $F_N = (N^2 - 1)/2N$. The phase $\varphi(\gamma)$ generates the mappings $S^1 \to S^1$, which split into homotopic classes characterized by the number Q.

Mappings in a given class cannot be continuously deformed to the mappings of another class without violating the single-valuedness of $\xi^{cl}(\gamma)$.

Consider an arbitrary surface Σ with the boundary $\delta\Sigma = \Gamma$. This surface is specified by the equation $x_\mu = z_\mu(\eta_i)$; $\eta^0 = \tau$, $\eta^1 = \sigma$. Condition (12) leads to the quantization of the flux of the Abelian field corresponding to the contour Γ:

$$\Phi=\oint_{\partial\Sigma=\Gamma}dx_\mu a_\mu=\frac{1}{2}\int_\Sigma d^2\eta\,\sigma_{\mu\nu}F_{\mu\nu}=2\pi Q/F_N, \tag{13}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, and the quantity

$$\sigma_{\mu\nu}=\frac{\partial z_\mu}{\partial\eta^0}\frac{\partial z_\nu}{\partial\eta^1}-\frac{\partial z_\mu}{\partial\eta^1}\frac{\partial z_\nu}{\partial\eta^0}$$

has only components that are tangential to Σ. The gauge field that is proportional to a_μ is topologically nontrivial, so that it is stable with respect to small fluctuations only when (13) defines the quantization of the total field flux and not of some random part of it. It is only when this condition is satisfied that the number Q determines the lower limit for the action of the gauge field. Since the contour Γ forms a one-dimensional boundary, this requirement can be satisfied only by two-dimensional fields that "live" on the surface Σ with the boundary $\partial\Sigma = \Gamma$. (Known examples of topological solutions do not, of course, contradict this statement. For example, for the 't Hooft-Polyakov monopole, we have a three-dimensional field and a two-dimensional boundary; for instantons in the Yang-Mills theory, we have a four-dimensional field and a three-dimensional boundary. There are analogous propositons for the vortex solutions, such as the Abrikosov filaments; Ref. 9.)

Consequently, if we wish to isolate the contribution of field configurations with topological index Q [see (13)], we must integrate over the subclass of two-dimensional gauge fields that are related to the four-dimensional fields in accordance with $A_i^a(\eta) = (\partial z_\mu/\partial\eta^i)A_\mu^a[z(\eta)]$, $i=0,1$, $\mu=0,1,2,3$. The fields $A_i^a(\eta)$ are defined on the surface Σ with the boundary $\partial\Sigma = \Gamma$. When this is taken into account, the approximate evaluation of the expectation value $\langle O(\Gamma)\rangle_A$ (5) now reduces to integration over $A_i^a(\eta) \equiv A_i^a(\Sigma)$ for a fixed surface Σ, followed by summation over the surfaces Σ, i.e.,

$$\langle O(\Gamma)\rangle_A = B^{-1}\int d\mu(\Sigma)\int d\mu[A_i^a(\Sigma)]\exp\{-S[A_i^a(\Sigma)\}O(\Gamma), \tag{14}$$

where

$$S[A_i^a(\Sigma)] = \frac{1}{4}\int_\Sigma d^2\eta\, g^{1/2} g^{il} g^{kn} G_{ik}^a G_{ln}^a, \tag{15}$$

$$G_{ik}^a(\eta) = \frac{\partial \tilde{A}_k^a}{\partial\eta^i} - \frac{\partial \tilde{A}_i^a}{\partial\eta^k} - \varepsilon f^{abc}\tilde{A}_i^b \tilde{A}_k^c, \tag{16}$$

and $g = \det g_{ik}, g_{ik}(\eta) = (\partial z_\mu/\partial\eta^i)(\partial z/\partial\eta^k)$ is the metric tensor on the surface Σ. Since the action given by (15) msut be dimensionless, we have changed the normalization of the field $A \to \tilde{A}$ in (16) in accordance with the equation

$$eA_i^a = (e/d)(dA_i^a) \equiv \varepsilon\tilde{A}_i^a,\quad \varepsilon = e/d, \tag{17}$$

where d is an arbitrary constant with dimensions $[1/m]$, and ε is a dimensional bare charge (the symbol $\sim$ will be omitted henceforth).[2)]

The contribution of the topological sector of fields with index Q to the integral (8) can now be written in the form

$$B^{-1}\int\prod_{q=1}^{n} Dx_q \int d\mu(\Sigma)\int d\mu[A_i^a(\Sigma)] \times\int [iD\xi(\gamma)D\xi^*(\gamma)]i\xi_c(1)\xi_e^*(0)\exp(-S_{eff}), \tag{18}$$

where

$$S_{eff} = S_0[x] + S_{Y\text{-}M}[A] + S[\xi] = \frac{1}{2}\sum_{q=1}^{n}\Big[\int_{\tau_1}^{\tau_2} d\gamma\Big(\frac{\dot{x}^2}{\lambda} + \lambda m^2\Big)\Big]_q + \frac{1}{4}\int_\Sigma d^2\eta\, g^{1/2} g^{il} g^{kn} G_{ik}^a G_{ln}^a + \oint_\Gamma d\gamma\,\xi_c^*(\gamma)\Big[\frac{d}{d\gamma} + i\varepsilon A_i\dot{\eta}^i\Big]_{cd}\xi_d(\gamma). \tag{19}$$

Since Γ is the boundary of the surface Σ, in the last term we have put

$$A_\mu^a(z(\eta))\frac{dz_\mu(\eta(\gamma))}{d\gamma} = A_\mu^a(z(\gamma))\frac{dz_\mu}{d\eta^i}\frac{d\eta^i}{d\gamma} = A_i^a\dot{\eta}^i$$

and have changed the normalization of A in accordance with (17). The action is invariant under arbitrary transformations of the coordinates η^i and γ. The Euclidean operation of conjugation defined by (7') ensures that $S[\xi]$ is real. The action given by (19) is, indeed, the result we require. Its variation leads[1] to the equations of motion with boundary condition indicating the colored quark current flowing over the contour $\Gamma = \partial\Sigma$. The two-dimensional field $A_i^a(\eta)$ satisfying these equations forms a relativistic chromoelectric string.[1]

We have therefore established that the action $S_{\rm eff}$ given by (19) arises consistently from the correlator (1) when we evaluate the contribution of the topological sector Q determined by this action. We have thus shown that the relativistic string examined in Ref. 1 must be looked upon not as yet another phenomenological model in hadron physics, but as a nontrivial solution of the field equations in non-Abelian gauge theory.

3. LOWER LIMIT FOR THE ACTION $S_{Y\text{-}M}[A]$ IN THE TOPOLOGICAL SECTOR Q

We shall now show that the gauge part of the action (19), $S_{Y\text{-}M}[A]$, has the lower bound (for fields $A = A^{cl} + \delta A$, where δA is a small fluctuation):

$$S_{Y\text{-}M}[A] \geqslant \pi|Q|. \tag{20}$$

The field A belongs to the homotopic class with the number Q and, conseqently, according to (13), it is identical with A^{cl} on the boundary $\partial\Sigma$ (to within the gauge). The equal sign in (20) corresponds to the solutions of the equations of motion for A, i.e.,

$$\delta S_{eff}/\delta A = 0. \tag{21}$$

It follows from the results reported in Ref. 1 that, for $A = A^{cl}$,

$$G_{ik}^a[A^{cl}] = \varepsilon e_{ik}(g(\eta))^{1/2} I^a(\eta), \tag{22}$$

$$S_{Y\text{-}M}[A^{cl}] = \frac{\varepsilon^2 I^2}{2}\int_\Sigma d^2\eta\, g^{1/2} = \pi|Q|, \tag{23}$$

where $I^a(\eta)$ is a covariantly constant function of the coordinates η^i that belongs to the associated representation of $SU(N)$, where $I^a I^a = (N^2-1)/2N = F_N$, and e_{ik} is the antisymmetric unit tensor. In an arbitrary gauge, the number Q is given by[1]

$$Q = \frac{\varepsilon}{4\pi}\int_\Sigma d^2\eta\, e^{ik} I^a(\eta) G_{ik}^a(\eta). \tag{24}$$

In the Abelian gauge in which $I^a = \text{const}$,

$$A^{cl}(\eta) = I^a a_i(\eta)/\varepsilon,\quad G_{ik}^a[A^{cl}] = I^a(\partial_i a_k - \partial_k a_i)/\varepsilon \tag{25}$$

Q assumes a form corresponding to (13):

$$Q = \frac{F_N}{2\pi}\int_{\Gamma=\partial\Sigma} d\eta^i\, a_i. \tag{26}$$

We now begin with the identity

$$\frac{1}{4}\int_\Sigma d^2\eta\, g^{1/2}\{(G_{ik}^a[A] \mp G_{ik}^a[A^{cl}])^2\} \geqslant 0, \tag{27}$$

where the signs $\mp$ corresponds to $Q = \pm|Q|$, respectively. Using (22) and (23) together with (27), we obtain

$$S_{Y\text{-}M}[A] \mp \frac{\varepsilon}{2}\int_\Sigma d^2\eta\, e^{ik} I^a(\eta) G_{ik}^a[A] + \pi|Q| \geqslant 0. \tag{28}$$

Consider the second term in (28). The field A is not necessarily a solution of (21) but, by definition, belongs to a class with the same Q as A^{cl} (small fluctuations δA do not affect the discrete number Q). On an open surface Σ, we can always transform to the Abelian gauge for the two-dimensional field A_i^a. The integral over the surface Σ then reduces to the integral over the boundary, as in (26). Since the field A is identical with A^{cl} on the boundary $\delta\Sigma$, the second term in (28)

$$\frac{\varepsilon}{2}\int_{\Sigma} d^2\eta\, e^{ik} I^a G_{ik}{}^a$$

is equal to $2\pi Q$, and this takes us directly to (20).

Thus, the small fluctuations δA produce an increase in the action $S_{\text{Y-M}}$, which signifies the stability of the stringlike configuration of the field A^{cl} against δA (other variables being fixed).

[1)] This method is examined in the presence of Grassmann fields in, for example, Ramond's book.[8]

[2)] It is noted in Ref. 1 that the final result for $S[A^{cl}]$ is independent of d.

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