Brillouin scattering of a noisy polariton wave

N. A. Gippius, L. V. Keldysh, and S. G. Tikhodeev

P. N. Lebedev Physical Institute, Academy of Sciences of the USSR

Institute of General Physics, Academy of Sciences of the USSR

(Submitted 25 June 1986)
Zh. Eksp. Teor. Fiz. **91**, 2263–2275 (December 1986)

We consider the Brillouin scattering of a strong polariton wave, assumed to be narrow-band Gaussian noise, in a direct-gap semiconductor. We show that notwithstanding the incoherence of the pumping the effects caused by the coherence of the scattered polaritons and the phonons which are in resonance with them—the mixing of polariton and phonon states, the renormalization of the spectra, the formation of a gap in the density of states and in the spectral density of scattered anti-Stokes polaritons, and oscillations in the intensity of the anti-Stokes waves after the pumping is switched on—are conserved (although they are appreciably weakened when the intensity and the spectral width of the pump are increased). (These effects were considered for the case of coherent pumping by Ivanov, Keldysh, and Tikhodeev, [Sov. Phys. JETP **57**, 234 (1983); **63**, 1086 (1986); **64**, 45 (1986)].)

§1. INTRODUCTION

Characteristic for the behavior of systems which are removed from equilibrium by a strong external field is the occurrence of correlations which are additional to the ones occurring in a state of thermodynamic equilibrium. As a rule, such correlations are not universal (in contrast to the equilibrium case) and are determined by the external field and the specifics of the system; they turn out to affect appreciably the behavior of the system. Phenomena typical of the situation described here take place in Brillouin scattering of a strong coherent polariton wave in a semiconductor. The coherence between scattered polaritons and resonance phonons which arises in this case leads[1] to the formation of mixed polariton-phonon (phonoriton) modes, the restructuring of the spectrum and of the occupation numbers of both the phonons and the polaritons.[1)] There is experimental evidence in favor of the phonoriton restructuring of the spectrum during anti-Stokes scattering by optical phonons in CdS.[3] Various effects caused by this restructuring (near the threshold of induced scattering when the transient wave is abruptly switched on) are considered theoretically in Refs. 4 and 5. In a formal description of these effects the coherence of a strong polariton wave (pump) was used in an essential way. It is therefore of interest to analyze which of the effects considered "survive" when one uses an incoherent pump.

In the present paper we consider the anti-Stokes[2)] Brillouin scattering of a strong noisy polariton wave. In this paper we describe such an electromagnetic wave with a frequency ε_0 close to the polariton resonance frequency as Gaussian noise with a vanishing average field amplitude, $\langle E \rangle = 0$, and a pair correlation function

$$\langle E(0,t)E^{*}(\mathbf{r},t')\rangle \propto n_0 \exp\{-\delta|t'-t| + i\varepsilon_0(t'-t) - i\mathbf{p}_0\mathbf{r}\}. \quad (1.1)$$

Here $\varepsilon_0 = \varepsilon(\mathbf{p}_0)$, $\mathbf{p}_0$ is the quasimomentum of the wave, $\varepsilon(\mathbf{p})$ the polariton dispersion law, n_0 the spatial polariton density which is connected with the intensity of the passing wave through the relation

$$I = \hbar\varepsilon_0 n_0 c_0, \quad c_0 = |\partial\varepsilon(\mathbf{p})/\partial\mathbf{p}|_{\mathbf{p}=\mathbf{p}_0}, \quad (1.2)$$

and δ is the spectral width of the noise (the reciprocal of the correlation time). In other words, such a wave is a macro-occupied polariton mode (the number of particles $n_0 V$ in it is proportional to the volume) with a quasimomentum $\mathbf{p}_0$ and random phase. The averaging in (1.1) is performed over the appropriate density matrix which from an experimental point of view is equivalent to averaging over an ensemble of realizations. However, when we study stationary phenomena this averaging is equivalent, by virtue of the ergodicity of stationary random processes,[6] to averaging over the observation time $t \gg \delta^{-1}$.

The simplest to describe is the scattering in the case of narrow-band noise $\delta \ll \Gamma$ where $\Gamma = \gamma_{\rm pol} + \gamma_{\rm ph}, \gamma_{\rm pol(ph)}$ is the reciprocal of the polariton (phonon) life time. From a formal point of view one finds the solution for $\delta = 0$. The general prescription for finding any final answer reduces [see §2, Eq. (2.5)] to averaging the appropriate expression evaluated for coherent pumping over a Rayleigh intensity distribution. Such a result is natural for Gaussian noise (the so-called adiabatic approximation, see Ref. 6). Many of the results obtained are therefore intuitively obvious. We analyze in §3 the behavior of the polariton density of states (the imaginary part of the retarded Green function) and the spectral density of the scattered polaritons in the frequency and momentum range close to the anti-Stokes resonance for stationary backward scattering. We show that when the pumping intensity increases there occurs a pseudo-gap (as in disordered systems[7]) in the density of states which is considerably more smeared out than the gap for coherent pumping. In particular, the density of states in the center of the gap decreases with increasing I proportional to $I^{-1}\ln I$ and not to I^{-1} as for coherent pumping.[1]

We consider in §3 also non-stationary scattering when the pumping is switched on suddenly. In the coherent case there occur after the switching on of the pumping oscillations in the intensity of the anti-Stokes line[5] similar to the

0038-5646/86/121344-08$04.00

nutations of a two-level system. In the case of noisy pumping the oscillations of the anti-Stokes line are damped. This result corresponds to the suppression of the nutations of a two-level system in a noisy field.[6]

At the end of §3 we consider Stokes scattering. In the approximation for the given noisy pumping used in the present paper there do not exist stationary solutions for the Stokes lines in contrast to the scattering of a coherent wave when there are stationary solutions right up to the threshold I_c for induced scattering (stochastic instability[6]). The growth in the amplitude of the Stokes waves starts without having a threshold; in the time range $t > \tilde{t}$, where $\tilde{t} \propto I^{-1}$ it proceeds faster than exponentially, proportional to $\exp[t^2\Gamma/\tilde{t}]$.

In the case of Gaussian noise with a finite spectral width, considered in §4, there is no general rule for calculating any quantities such as there is when $\delta = 0$. However, in the framework of the τ-approximation (see Ref. 4) one can for the case of stationary scattering completely sum the perturbation theory series in the external field for the retarded Green function. The summation method (reduction to an infinite continued fraction) was, as far as we know, first applied in Ref. 8 to calculate the linear polarizability of a three-level system. This method was used also in the theory of disordered systems[9] to calculate the electron density of states. It was shown in Ref. 9 that writing the solution as a continued fraction is convenient for a numerical analysis. As to the statistical Green function, even in the τ-approximation and the stationary case one can only carry out an exact summation under the condition that the phonon life time is considerably longer than the polariton life time, or vice versa. (From an experimental point of view this case is, of course, the most common one.) Taking into account that δ is finite leads (as in Ref. 9) to a yet larger smearing out of the pseudo-gap in the density of states; as $\delta/\Gamma \to 0$ the solution goes over into the one obtained for $\delta = 0$ by averaging over the Rayleigh distribution.

On the whole we can conclude that when a noisy polariton wave is scattered in a semiconductor the effects connected with the additional coherence of the scattered polaritons and phonons do not disappear although they are considerably weakened.

Concluding this section we consider how these effects must manifest themselves experimentally. We asusme that we use as a pump a narrow-band noisy source with $\delta \ll \Gamma$. Stationary effects (renormalization of the spectrum and of the populations) can be studied using a single realization under the condition that the observation time $t \gg \delta^{-1}$. Transient processes (oscillations of the anti-Stokes and growth of the Stokes components) develop over times $t \lesssim \Gamma^{-1}$. Noise effects must thus manifest themselves when one averages over a series of pulses (of length $t_{\text{pulse}} \ll \delta^{-1}$) in each of which one observes the scattering of a coherent pump.

§2. PERTURBATION THEORY FOR SCATTERING OF AN INCOHERENT WAVE

We use as in Refs. 4 and 5 a diagram technique for non-equilibrium processes.[10,11] We consider first stationary scattering. The rules for constructing a diagram perturbation theory in terms of the polariton-phonon interaction which was formulated in Ref. 4 remain valid, except for the rule for describing the external field. As the amplitude of the field is zero on average the anomalous vertices[3)] of (I.1.7) for the creation and annihilation of a polariton with $p = p_0$ are also zero. The anomalous Green functions (I.1.8) and (I.1.9) also vanish. All quantities with diagrams which in the case of coherent scattering contain different numbers of anomalous creation and annihilation vertices are, in general, also zero due to averaging over the phase.

The statistical component of the free polariton Green function which is proportional to the correlator (1.1) depends on the intensity of the external field. We isolate it and take it into account separately. After Fourier transforming with respect to the frequencies and momenta it has (in the triangular representation) the form

$$\begin{pmatrix} 0 & 0 \\ 0 & -4in_0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}_0)\,\delta/[(\varepsilon-\varepsilon_0)^2+\delta^2] \end{pmatrix}. \qquad (2.1)$$

As we assume the field to be Gaussian, the higher correlators vanish. The action of an incoherent external field is in the resonance approximation thus completely described by a diagram perturbation theory containing the lines (2.1) besides the free propagators of scattered polaritons and phonons (I.1.2) to (I.1.4) (Fig. 1). It is convenient for what follows to combine the latter with the vertices (I.1.5) of the polariton-phonon interaction and to write them in the form (see Fig. 1)

$$\begin{aligned} \Xi_{i'j'i''j''}(p', p'') = {} & \Phi_{\mathbf{p}'}\Phi_{\mathbf{p}''}(\sigma_x)_{i'j'}(\sigma_x)_{i''j''}(2\pi)^4 \\ & \times \delta(k'+p-p')(2\pi)^4\delta(k''+p-p'') \\ & \times (2\pi)^3\delta(\mathbf{p}-\mathbf{p}_0)\,2\delta/[(\varepsilon-\varepsilon_0)^2+\delta^2], \end{aligned} \qquad (2.2)$$

where

$$\Phi_{\mathbf{p}} = D(n_0|\mathbf{p}-\mathbf{p}_0|/2\hbar\rho u)^{1/2}, \quad k=(\mathbf{k}, \omega), \quad p=(\mathbf{p}, \varepsilon),$$

$i', j', i'', j'' = 1,2$ are time indexes.[4)]

Comparing the perturbation-theory series constructed thus with the series for coherent pumping one can easily formulate the following correspondence rule illustrated by Fig. 2. To obtain all diagrams for any quantity A in the incoherent case one must construct all diagrams for A with the same number of anomalous creation and annihilation vertices in the coherent case. After that one must join by lines (2.2) the creation vertices with annihilation vertices in all possible ways. We note that from a diagram for the coherent case with m anomalous vertices of each kind we obtain $m!$ diagrams for the incoherent pumping. For finite δ these diagrams are, in general not equal to one another. However, in the limiting case of narrow-band noise, $\delta = 0$, the relation

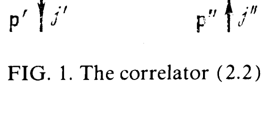

FIG. 1. The correlator (2.2) of the external field, Ξ.

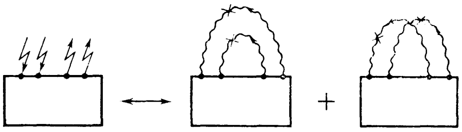

FIG. 2. Rule for the correspondence of diagrams for coherent and noisy polariton waves.

$$\Xi(p', p'')=\Phi(p')\otimes\Phi(p''), \tag{2.3}$$

where

$$\Phi_{ij}(p')=\Phi_{\mathbf{p}'}(2\pi)^4\delta(p-p_0)(2\pi)^4\delta(k'+p-p')(\sigma_x)_{ij} \tag{2.4}$$

is the anomalous vertex (I.1.7) for coherent pumping, is satisfied. Hence, in that case all $m!$ diagrams are equal to one another and to the original diagram for coherent pumping.

This statement allows us to obtain a general rule for the summation of diagrams for any quantity $A(n_0)$ describing the scattering of a narrow-band incoherent polariton wave with density n_0 [or intensity I of (1.2)] if we know the corresponding function $A_{\mathrm{coh}}(n_0)$ for the coherent case:

$$A(n_0)=\int_0^\infty e^{-\zeta}A_{coh}(\zeta n_0)\,d\zeta=\int_0^\infty P_{n_0}\{n\}A_{coh}(n)\,dn, \tag{2.5}$$

where

$$P_{n0}\{n\}=n_0^{-1}\exp\{-n/n_0\} \tag{2.6}$$

is an exponential distribution corresponding to the Rayleigh distribution of the amplitude.

To prove Eq. (2.5) we must expand $A_{\mathrm{coh}}(n_0)$ in a perturbation theory series in powers of n_0:

$$A_{coh}(n_0)=\sum_{m=0}^{\infty}A_{m,coh}n_0^{\,m}. \tag{2.7}$$

We then get in the incoherent case (when $\delta=0$)

$$A(n_0)=\sum_{m=0}^{\infty}m!A_{m,coh}n_0^{\,m}. \tag{2.8}$$

Using the representation

$$m!=\int_0^\infty \zeta^m e^{-\zeta}\,d\zeta \tag{2.9}$$

and interchanging summation and integration in (2.8) we prove (2.5).

In concluding this section we consider the non-stationary scattering of an incoherent wave when it is suddenly switched on. This problem was solved for coherent pumping in the τ-approximation in Ref. 5. We shall assume that the noisy wave, switched on at time $t=0$, is Gaussian noise with a correlation function differing from (1.1) by additional factors $\theta(t)\theta(t')$, where $\theta(t)$ is the step function:

$$\langle E(0,t)E^*(\mathbf{r},t')\rangle\infty n_0\theta(t)\theta(t')$$
$$\times\exp\{-\delta|t-t'|+i\varepsilon_0(t'-t)-i\mathbf{p}_0\mathbf{r}\}. \tag{2.10}$$

As in the case of stationary scattering the anomalous Green functions [the off-diagonal components of the matrix (II.4)] vanish. One sees easily that the above formulated rule of correspondence between diagrams for the coherent and incoherent cases remains valid. In the limit as $\delta\to 0$ the dependence on t and t' of the corresponding function $\Xi(p't,p''t')$ can be factorized and an equation such as (2.3) is satisfied. The summation rule (2.5) is thus also valid for non-stationary scattering.

§3. SCATTERING OF NARROW-BAND NOISE, $\delta\to 0$

We use Eqs. (I.1.18) and (I.2.5) and calculate in the τ-approximation the polariton density of states $|\mathrm{Im}G^r_{\mathrm{pol}}(p)|$ and the spectral density of the backward scattered polaritons $N_{\mathrm{pol}}(p)$ for coherent pumping:[5]

$$|\,\mathrm{Im}\,G^r_{pol}(p)\,|_{coh}=[\gamma_{pol}\,|\,b(p)\,|^2+\Phi_{\mathbf{p}}^2\gamma_{ph}]\,|\,Z^r(p)\,|^{-2}, \tag{3.1}$$

$$[N_{pol}(p)={}^1\!/_2 i(F_{pol}-G^r_{pol}+G^a_{pol})]_{coh}=\gamma_{ph}N_+\Phi_{\mathbf{p}}^2\,|\,Z^r(p)\,|^{-2}, \tag{3.2}$$

where

$$Z^r(p)=a(p)b(p)-\Phi_{\mathbf{p}}^2,$$
$$a(p)=\varepsilon-\varepsilon(\mathbf{p})+i\gamma_{pol},\quad b(p)=\varepsilon-\varepsilon_0-u|\mathbf{p}-\mathbf{p}_0|+i\gamma_{ph}, \tag{3.3}$$
$$N_+=[\exp(\hbar u|\mathbf{p}_0-\mathbf{p}_+|/k_BT)-1]^{-1}$$

is the equilibrium number of resonance phonons with momentum $\mathbf{p}_0-\mathbf{p}_+$ which is in resonance for the anti-Stokes backward scattering, see (I.1.1); u is the sound speed. In (3.2) and henceforth we neglect the thermal source of polaritons $N_{0,\mathrm{pol}}=0$.

To obtain the corresponding functions $|\mathrm{Im}\,G^r_{\mathrm{pol}}|,N_{pol}$ for the incoherent case as $\delta\to 0$ we use Eq. (2.5). For a qualitative comparison of the behavior in the coherent and the incoherent cases we performed numerical calculations the results of which are given in Figs. 3 and 4 (for $|\mathrm{Im}G^r_{\mathrm{pol}}|$) and Figs. 5 and 6 (for N_{pol}). Figures a refer to the coherent and figures b to the incoherent case. The quantities $|\mathrm{Im}G^r_{\mathrm{pol}}|$, N_{pol}/N_+ are shown as functions of the frequency and of the longitudinal momentum. The central point in all figures [with coordinates $p_+=(\varepsilon_+,\mathbf{p}_+)$, see (I.1.1)] is the region of anti-Stokes resonance for backward scattering. In that point the polariton and absorbed phonon terms intersect. Quantities with the dimensions of frequency $(\varepsilon,(G^r_{\mathrm{pol}})^{-1},\mathrm{N}^{-1}_{\mathrm{pol}})$ are measured in units γ_{pol}, and momenta in units $2\gamma_{\mathrm{pol}}/c_0$. In the calculations we used the following parameter values: $u=1/3$, $c_0=2$, $\gamma_{\mathrm{pol}}=1$, $\gamma_{\mathrm{ph}}=\frac{1}{2}$. The pumping strength in dimensionless units $\Phi^2_+/\gamma^2_{\mathrm{pol}}$ $(\Phi_+\equiv\Phi_{\mathbf{p}^+})$ is equal to 1 (Figs. 3 and 5) and 9 (Figs. 4 and 6).

It is very clear from Figs. 3(a) and 4(a) how the phonon and polariton modes mix when the intensity of the coherent pumping increases, tails occur in the polariton density of states and extend along the phonon term, and a gap is formed in the density of states. In the incoherent case (Figs. 3b and 4b) there also occurs a mixing and a trough in the density of states is formed but much less well pronounced. The spectral density of the backward scattered polariton wave (Figs. 5 and 6) behaves similarly.

The speed at which the density of states diminishes at the center of the gap when the pumping strength increases

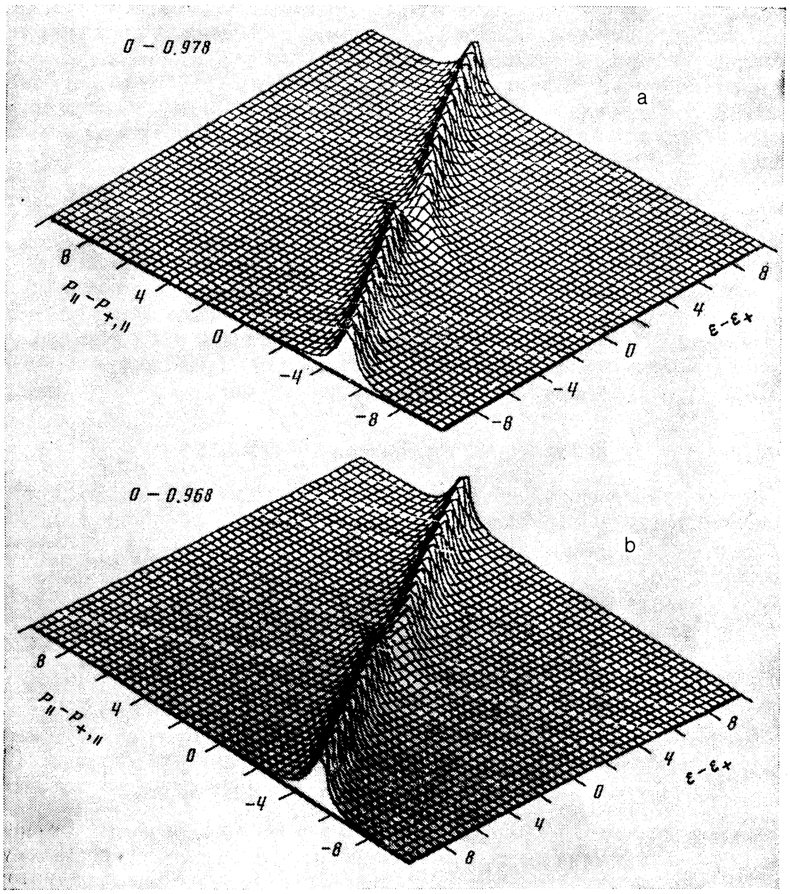

FIG. 3. The polariton density of states $|\mathrm{Im}\, G^{r}_{\mathrm{pol}}|\gamma_{\mathrm{pol}}$ as function of frequency and longitudinal momentum for $\Phi_{+}/\gamma_{\mathrm{pol}}=1$ (explanation in the text): a: coherent case, b: incoherent case (the numbers in the upper left-hand corner of each figure are the minimum and maximum values of the function shown, $p_{\perp}=p_{+,\perp}$).

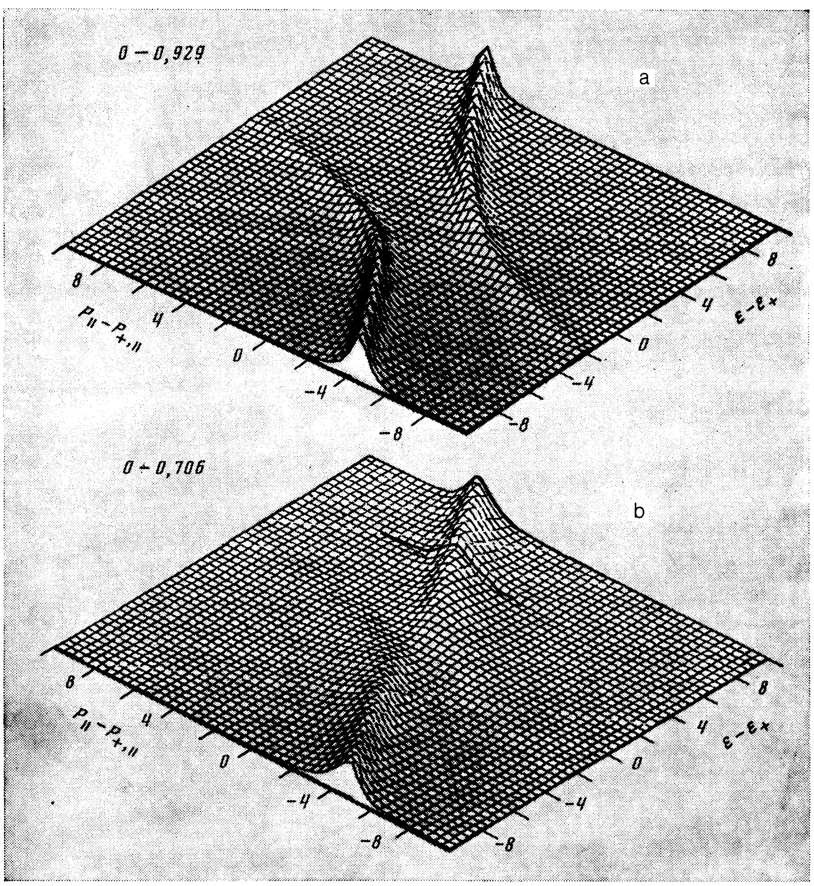

FIG. 4. The same as in Fig. 3, for $\Phi_{+}/\gamma_{\mathrm{pol}}=3$.

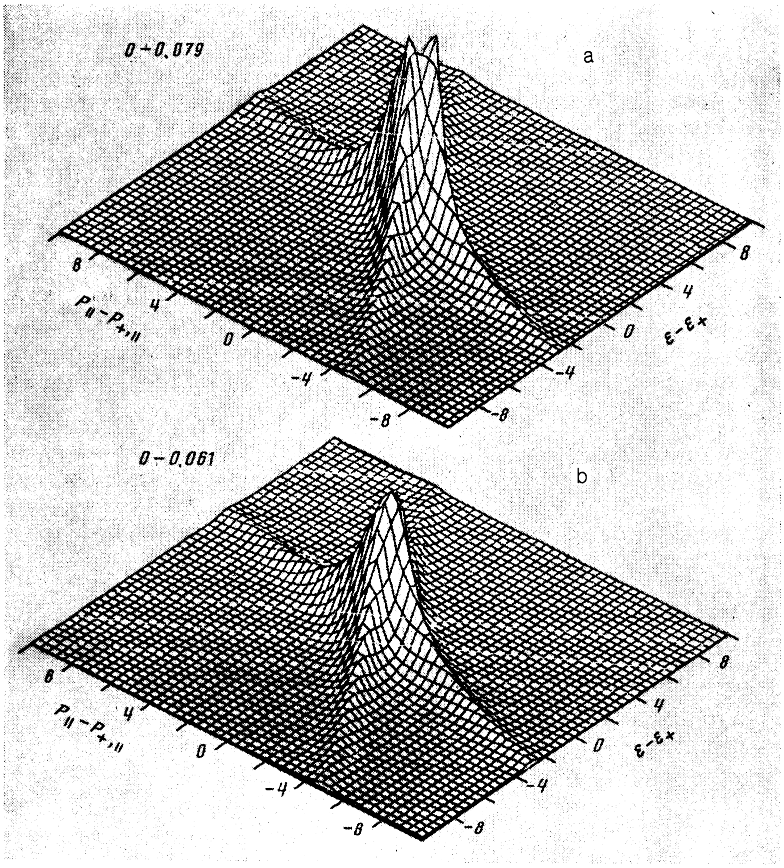

FIG. 5. The spectral density of scattered polaritons $N_{\rm pol}\gamma_{\rm pol}/N_+$ for $\Phi_+/\gamma_{\rm pol} = 1$: a: coherent case, b: incoherent case.

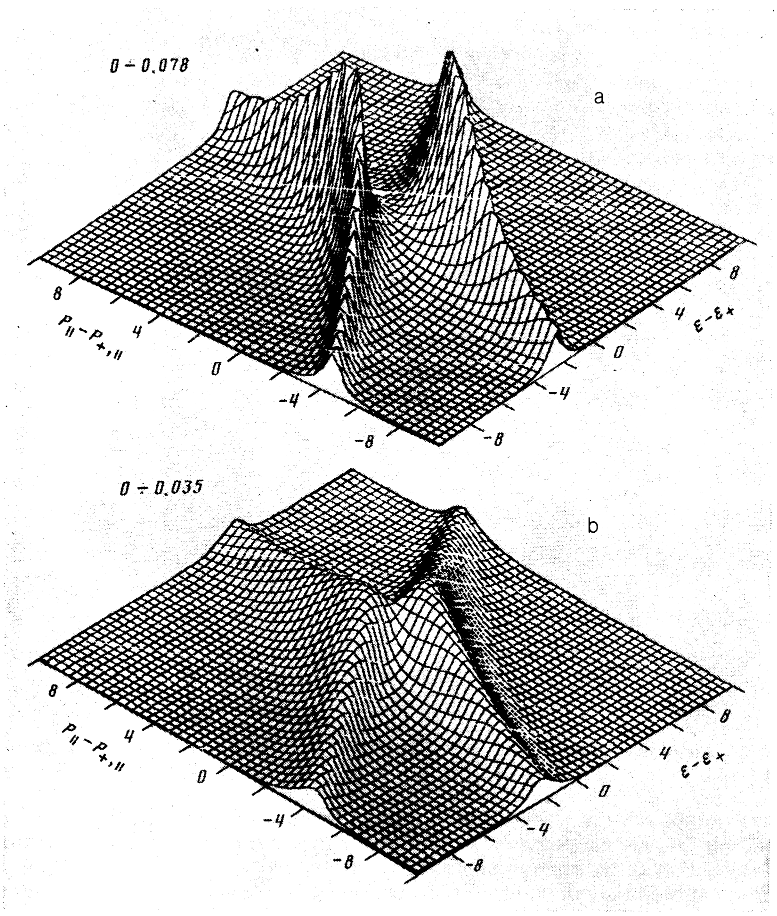

FIG. 6. The same as in Fig. 5 for $\Phi_+/\gamma_{\rm pol} = 3$.

can be estimated quantitatively. It follows from (3.1) that

$$|\operatorname{Im} G^r_{pol}(p_+)|_{coh}=\gamma^{-1}_{pol}\xi/(1+\xi), \tag{3.4}$$

where $\xi = \gamma_{pol}\gamma_{ph}\Phi_+^{-2}$. Using (2.5) we get for incoherent scattering

$$|\operatorname{Im} G^r_{pol}(p_+)|=\gamma^{-1}_{pol}\int_0^\infty e^{-\zeta}(\zeta+\xi)^{-1}\,d\zeta=\gamma^{-1}_{pol}\xi e^{\xi}E_1(\xi). \tag{3.5}$$

In strong fields when $\xi \ll 1$ the exponential integral $E_1(\xi)\sim\exp(-\xi)|\ln\xi|$. Hence, the density of states in the center of the gap decreases in the incoherent case when the pumping strength increases as $\xi\,|\ln\xi|\propto I^{-1}\ln I$ which is appreciably more slowly than in the coherent case (porportional to $\xi\propto I^{-1}$).

The intensity of the scattered polaritons is for non-stationary scattering of a coherent pump described by Eq. (II.5). In the simplest case $\gamma_{pol}=\gamma_{ph}=\Gamma/2$ this formula has for the anti-Stokes component the form

$$[N_{pol}(\mathbf{p},t)]_{coh}=2N_+\left[\frac{\Omega^2}{\Omega^2+\Gamma^2}-\frac{\Omega e^{-\Gamma t}}{(\Omega^2+\Gamma^2)^{1/2}}\sin(\Omega t+\varphi)\right], \tag{3.6}$$

where

$$\Omega^2=\Delta^2+4\Phi_p^2,\quad \Delta=\varepsilon(\mathbf{p})-\varepsilon_0-u|\mathbf{p}-\mathbf{p}_0|,\quad \operatorname{tg}\varphi=\Omega/\Gamma.$$

The time-dependence of (II.15) together with the corresponding dependence in the incoherent case calculated using (2.5) for the same parameter values as before is for $\Phi_+/\gamma_{pol}=3$ shown in Fig. 7. As earlier, *a* refers to the coherent and *b* to the incoherent case. It is clear that the averaging of (2.5) leads to a suppression of the oscillations in the intensity of the scattered polariton wave except for the first period.

This result corresponds to the damping of the nutations of a two-level system in a noise field.[6]

In concluding this section we give some results referring to the Stokes scattering of a narrow-band noisy wave. The considerations given in §2 are formally independent of whether we consider anti-Stokes or Stokes scattering. One can thus expect that Eq. (2.5) remains valid also in the Stokes case. However, in our statement of the problem when the pumping intensity is assumed to be given by an external source there is no stationary solution in arbitrarily weak fields (stochastic instability[6]) in contrast to the scattering of a coherent pump when there are stationary solutions right up to the threshold of induced scattering. This is clear from (2.5) in which the integration is performed over all intensities, among which there are also those which exceed the threshold in the coherent case. We consider therefore the non-stationary problem. We write down that part of (II.5) which gives the exponential increase of the Stokes wave $[N_{pol}]_{coh}$ when $I>I_c$ for $\gamma_{ph}=\gamma_{pol}=\Gamma/2$ and $p=p_-$ [see (I.1.1)]:

$$[N_{pol}(\mathbf{p}_-,t)]_{coh}\sim-\frac{1+N_-}{4}\frac{\Phi_-}{\gamma_2(\mathbf{p}_-)}\exp[-2\gamma_2(\mathbf{p}_-)t], \tag{3.7}$$

where

$$\gamma_2(\mathbf{p}_-)=-\Phi_-+\Gamma/2,\quad \Phi_-\equiv\Phi_{\mathbf{p}_-},$$

$$N_-=[\exp(\hbar u|\mathbf{p}_--\mathbf{p}_0|/k_BT)-1]^{-1}.$$

For a noisy pump we have

$$N_{pol}(\mathbf{p}_-,t)\sim\frac{1+N_-}{2}\int_0^\infty d\zeta\,\frac{\zeta^2\Phi_-}{\zeta\Phi_--\Gamma/2}\exp(-\zeta^2+2\zeta\Phi_-t-\Gamma t). \tag{3.8}$$

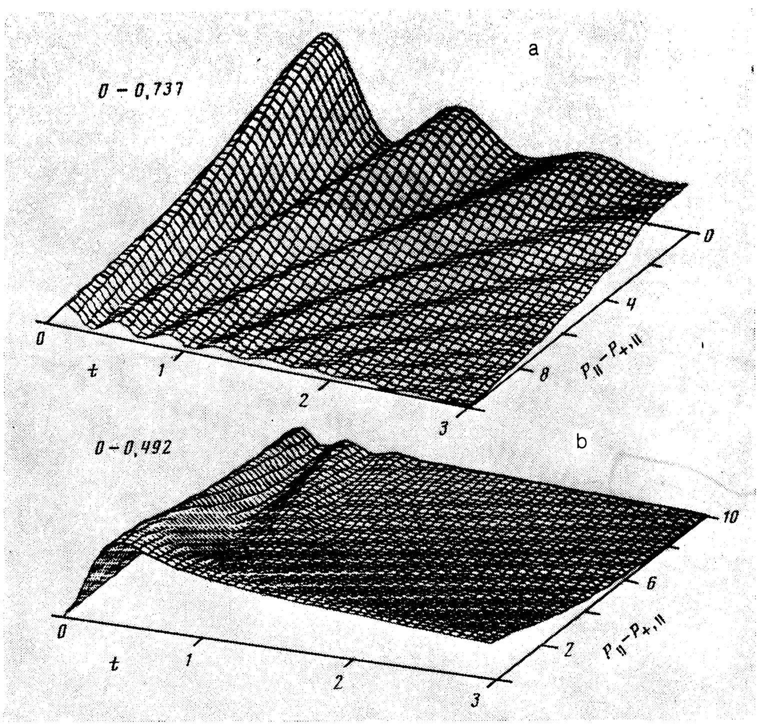

FIG. 7. The function $N_{pol}(t,p_\parallel)/N_+$ for $\Phi_+/\gamma_{pol}=3$. The time is measured in units γ_{pol}^{-1} and the momentum in $2\gamma_{pol}/c_0$.

The integrand in (3.8) contains a simple pole on the integration contour. This non-integrable singularity arises in the threshold region for induced scattering in the framework of the τ-approximation which is, as was shown in Ref. 4, not applicable in that region. We assume that taking the divergent diagrams near the threshold completely into account leads to the singularity in (3.8) becoming integrable. Using the Laplace method to estimate the integrals[12] and noting that the singularity of the factor of the exponent does not fall for large t in the important region of integration we find that for $t \gg \tilde{t} = \Gamma\Phi_-^{-2}$

$$N_{pol}(\mathbf{p}_-, t) \sim {}^1/_2\pi^{1/2}(1+N_-)\Phi_- t \exp[(\Phi_- t)^2 - \Gamma t]. \qquad (3.9)$$

For the scattering of a noisy wave of arbitrary intensity the Stokes waves must thus grow faster than exponentially. Leaving this growth regime occurs with a delay which is inversely proportional to the pumping strength (thanks to this there does not arise a paradox when we are considering pumping with $I \to 0$). For clarity we recall that this effect must occur after averaging over a large number of realizations.

Under actual conditions, of course, the pumping strength in each point of the semiconductor is not fixed by an external source (as in our idealized statement of the problem). The growth process is limited by particles leaving the passing wave and the amplitude of the Stokes waves emerges at a stationary value. It follows from our considerations that the establishment of a stationary picture proceeds completely differently for coherent and for noisy pumps.

§4. SCATTERING OF NOISE WITH A FINITE SPECTRAL WIDTH

If $\delta \neq 0$ the rule (2.3) for factorization is not satisfied and there does not exist a general rule like (2.5) for summing any diagrams. However, in the framework of the τ-approximation one can for the stationary case solve the problem for the retarded and advanced Green functions by a method proposed by Elyutin.[8] This method breaks down already for the statistical Green functions and allows us to solve the problem only when $\gamma_{\rm pol} \ll \gamma_{\rm ph}$ or vice versa (and arbitrary δ).

We start with the calculation of $G^r_{\rm pol}$. In the framework of the τ-approximation any diagram for $G^r_{\rm pol}$ consists (see Fig. 8) of a "spine" containing a product of alternating functions

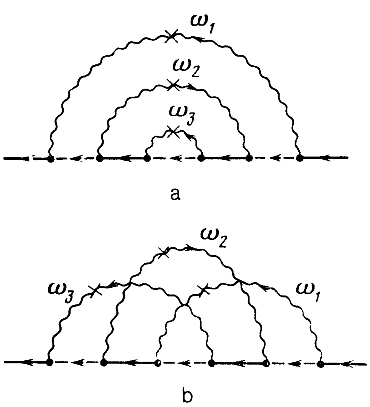

FIG. 8. Diagrams for $G^r_{\rm pol}$.

$$G_{\tau,pol}(\varepsilon-\omega_1+\omega_2-\ldots) = [a(\varepsilon-\omega_1+\omega_2-\ldots)]^{-1}, \qquad (4.1)$$

$$G^r_{\tau,ph}(\varepsilon-\omega_1+\omega_2-\ldots) = [b(\varepsilon-\omega_1+\omega_2-\ldots)]^{-1} \qquad (4.2)$$

and "ribs"—arbitrarily entangled lines Ξ, (2.2), which after integration over momenta are reduced to the form

$$\Xi(\omega_j) = \Phi_{\mathbf{p}}^2 2\delta[(\omega_j-\varepsilon_0)^2+\delta^2]^{-1}. \qquad (4.3)$$

The functions $a(\varepsilon)$ and $b(\varepsilon)$ occurring in (4.1), (4.2) are defined in (3.3). Here and henceforth we shall not write down the momentum arguments: $\mathbf{p}$ for the $G^r_{\tau,\rm pol}$ lines and $\mathbf{p}-\mathbf{p}_0$ for $G^r_{\tau,\rm ph}$.

The method for summing such diagrams[8] is based upon the fact that their magnitude depends only on the number of ribs passing over each line of the spine and does not depend on the entanglement of the ribs. For instance, the diagrams of Fig. 8 have the same magnitude and are equal to

$$(\Phi_{\mathbf{p}}^2)^3 a(\varepsilon)^{-2}[b(\varepsilon)+i\delta]^{-2}[a(\varepsilon)+2i\delta]^{-2}[b(\varepsilon)+3i\delta]^{-1}.$$

This property follows from the analyticity of the functions $G^r_{\tau,\alpha}$ in the upper ε-halfplane. It enables us to calculate $G^r_{\rm pol}$ as a sum of "simple" diagrams (such as Fig. 8a) for which the ribs are not entangled and for which, hence, the vertices are not renormalized. One needs only correctly take into account the number of diagrams which are equal in magnitude to the given simple diagram. As a result (see Refs. 8 and 9) finding the function $G^r_{\rm pol}$ reduces to solving an infinite set of coupled equations

$$\begin{aligned}
G^r_{pol} &= a^{-1}[1+\Phi_{\mathbf{p}}^2 G^r_{1,ph}G^r_{pol}],\\
G^r_{1,ph} &= [b+i\delta]^{-1}[1+\Phi_{\mathbf{p}}^2 G^r_{1,pol}G^r_{1,ph}],\\
G^r_{1,pol} &= [a+2i\delta]^{-1}[1+2\Phi_{\mathbf{p}}^2 G^r_{2,ph}G^r_{1,pol}],\\
&\vdots\\
G^r_{m,pol} &= [b+(2m-1)i\delta]^{-1}[1+m\Phi_{\mathbf{p}}^2 G^r_{m,pol}G^r_{m,ph}],\\
G^r_{m,ph} &= [a+2mi\delta]^{-1}[1+(m+1)\Phi_{\mathbf{p}}^2 G^r_{m+1,ph}G^r_{m,pol}].\\
&\vdots
\end{aligned} \qquad (4.4)$$

We can write the solution of the set (4.4) in the form of an infinite continued fraction

$$G^r_{pol}(\varepsilon) = \frac{1}{a(\varepsilon)-}\,\frac{\Phi_{\mathbf{p}}^2}{b(\varepsilon)+i\delta-}\,\frac{\Phi_{\mathbf{p}}^2}{a(\varepsilon)+2i\delta-}\,\frac{2\Phi_{\mathbf{p}}^2}{b(\varepsilon)+3i\delta-}\,. \qquad (4.5)$$

To find $G^r_{\rm ph}$ one must in (4.5) interchange the functions a and b.

We now turn to the calculation of $F_{\rm pol}(p)$. In the τ-approximation any diagram for $F_{\rm pol}$ also consists of ribs and vertebra which in contrast to the diagrams for $G^r_{\rm pol}$ contain (at an arbitrary place) one function such as

$$F_{\tau,pol} = 2i\,\mathrm{Im}\,G^r_{\tau,pol} \quad (N_{0,pol}=0) \qquad (4.6)$$

or

$$F_{\tau,ph}=2i(1+2N_+)\,\mathrm{Im}\,G^r_{\tau,ph}, \tag{4.7}$$

to the left of which stand the $G^r_{\tau,\alpha}$ and to the right the $G^a_{\tau,\alpha}$, $\alpha=$ pol, ph. As the functions (4.6), (4.7) are not analytical either in the vertex or in the lower ε-half-plane diagrams with entangled ribs are not equal to the corresponding simple diagram. We were not able to obtain a general rule of summation similar to (4.5) for any relation between δ, $\gamma_{\rm ph}$, and $\gamma_{\rm pol}$. However, in the case when one of the dampings is appreciably less than the other ($\gamma_{\rm pol}\ll\gamma_{\rm ph}$ or vice versa) one easily finds a solution. For instance, when $\gamma_{\rm pol}\ll\gamma_{\rm ph}$

$$N_{pol}(p)=2N_+|\,\mathrm{Im}\,G^r_{pol}|, \tag{4.8}$$

where $G^r_{\rm pol}$ is the infinite continued fraction (4.5).

To prove (4.8) we evaluate the imaginary part of any diagram for $G^r_{\rm pol}$. It is proportional to the imaginary part of the vertebra

$$\mathrm{Im}(G^r_{\tau,pol}G^r_{\tau,ph}G^r_{\tau,pol}\ldots).$$

To evaluate the imaginary part we use an identity which is valid for any complex numbers $\alpha_1, \alpha_2,\ldots,\alpha_n$:

$$\mathrm{Im}\,(\alpha_1\alpha_2\ldots\alpha_n)=\mathrm{Im}\,\alpha_1\,\mathrm{Re}\,(\alpha_2{}^*\alpha_3{}^*\ldots\alpha_n{}^*)$$
$$+\mathrm{Im}\,\alpha_2\,\mathrm{Re}(\alpha_1\alpha_3{}^*\ldots\alpha_n{}^*)+\ldots\mathrm{Im}\,\alpha_n\,\mathrm{Re}(\alpha_1\alpha_2\ldots\alpha_{n-1}),$$

and also Eqs. (4.6), (4.7). We get

$$\mathrm{Im}(G^r_{\tau,pol}G^r_{\tau,ph}G^r_{\tau,pol}\ldots)=\frac{1}{2i}F_{\tau,pol}\,\mathrm{Re}\,(G^a_{\tau,ph}G^a_{\tau,pol}\ldots)$$
$$+\frac{F_{\tau,ph}}{2i(1+2N_+)}\mathrm{Re}\,(G^r_{\tau,pol}G^a_{\tau,pol}\ldots)+\ldots=\frac{1}{2i(1+2N_+)}$$
$$+\ [F_{\tau,pol}G^a_{\tau,ph}G^a_{\tau,pol}\ldots+G^r_{\tau,pol}F_{\tau,ph}G^a_{\tau,pol}\ldots+\ldots]$$
$$+\frac{iN_+}{1+2N_+}[F_{\tau,pol}G^a_{\tau,ph}G^a_{\tau,pol}\ldots+G^r_{\tau,pol}G^r_{\tau,ph}F_{\tau,pol}\ldots+\ldots]. \tag{4.9}$$

The first term on the right-hand side of (4.9) is the sum of all vertebra diagrams for $F_{\rm pol}$ with the topological structure considered.[6)] The second term (proportional, as should be the case, to the number of thermal phonons) is small provided the polariton damping is small, as it is proportional to $\gamma_{\rm pol}$. When $\gamma_{\rm pol}\ll\gamma_{\rm ph}$ and for arbitrary δ for each diagram of a given topological structure therefore the relation

$$F_{pol}=2i(1+2N_+)\,\mathrm{Im}\,G^r_{pol}. \tag{4.10}$$

holds. This proves Eq. (4.8). It follows from the proof that a similar relation holds for $F_{\rm ph}$:

$$F_{ph}=2i(1+2N_+)\,\mathrm{Im}\,G_{ph}{}^r. \tag{4.11}$$

Using (4.5), (4.8) and the results of the numerical analysis of similar expressions given in Ref. 9 we may conclude that taking into account a finite δ leads to an even larger (as compared to the case when $\delta=0$) smearing out of the pseudo-gap and of the singularities of the spectral density of the scattered polaritons. One can also verify that the small parameter which leads, when it tends to zero, to the solutions obtained here going over into the solution (2.5) for $\delta=0$ is, indeed, $\delta/\Gamma\ll 1$.

[1)]The effect of the exciton-photon interaction on the phonoriton restructuring of the spectrum was analyzed in Ref. 2.

[2)]Some results regarding Stokes scattering are given at the end of §3.

[3)]Here and henceforth formulae from Ref. 4 are indicated by the Roman number I and those from Ref. 5 by a II.

[4)]In contrast to the representation chosen in Ref. 4 we shall in the present paper use the positive frequency part of the phonon Green function for the anti-Stokes scattering. The directions of the lines of the scattered polaritons and of the phonons in Fig. 1 are thus the same in contrast to the directions chosen in Ref. 4.

[5)]An error slipped into Eq. (I.2.2). The off-diagonal components of the matrix $\Sigma_{\rm ph}$ must change place.

[6)]We have here not written the Re sign in the right-hand side of (4.9) as either it is real (for mirror-symmetric diagrams) or it becomes real when we add to (4.9) the mirror-image diagram.

[1]A. L. Ivanov and L. V. Keldysh, Zh. Eksp. Teor. Fiz. **84**, 404 (1983) [Sov. Phys. JETP **57**, 234 (1983)].

[2]A. L. Ivanov, Zh. Eksp. Teor. Fiz. **90**, 158 (1986) [Sov. Phys. JETP **63**, 90 (1986)].

[3]G. S. Vygovskiĭ, G. P. Golubev, E. A. Zhykov, *et al.* Pis'ma Zh. Eksp. Teor. Fiz. **42**, 134 (1985) [JETP Lett. **42**, 164 (1985)].

[4]L. V. Keldysh and S. G. Tikhodeev, Zh. Eksp. Teor. Fiz. **90**, 1852 (1986) [Sov. Phys. JETP **63**, 1086 (1986)].

[5]L. V. Keldysh and S. G. Tikhodeev, Zh. Eksp. Teor. Fiz. **91**, 78 (1986) [Sov. Phys. JETP **64**, 45 (1986)].

[6]S. A. Akhmanov, Yu. E. D'yakov, and A. S. Chirkin, Vvedenie v Statisticheskuyu Radiofiziku i Optiku [in Russian] Nauka, Moscow, 1981.

[7]M. A. Sadovskiĭ, Thesis, Physical Institute Acad. Sc. USSR, Moscow, 1974.

[8]P. V. Elyutin, Opt. Spektrosk. **43**, 542 (1977) [Opt. Spectrosc. (USSR) **43**, 318 (1977)].

[9]M. A. Sadovskiĭ, Zh. Eksp. Teor. Fiz. **77**, 2070 (1979) [Sov. Phys. JETP **50**, 989 (1979)].

[10]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[11]E. M. Lifshitz and L. P. Pitaevskiĭ, (Physical Kinetics Nauka, Moscow, 1979, Ch. 10 [English translation published by Pergamon Press, Oxford].

[12]N. G. de Bruijn, Asymptotic Methods in Analysis North-Holland, 1958.

Translated by D. ter Haar