Trapping of a quantum particle in a shallow potential well and soliton formation from a cluster of Langmuir waves

V. T. Astrelin, B. N. Breĭzman, V. V. Vasil'ev, Z. Sedláček, and K. Jungwirth

Physics Institute, Czechoslovak Academy of Sciences; Nuclear Physics Institute, USSR Academy of Sciences
(Submitted 14 January 1986)
Zh. Eksp. Teor. Fiz. **91**, 2039–2052 (December 1986)

The dynamics of formation of a soliton in supersonic expansion of a cluster of Langmuir waves is investigated. It is shown that the plasma-density perturbation produced by the expanding cluster can be regarded from the standpoint of quantum mechanics as a shallow potential well. This makes it possible to obtain analytically the dependence of the energy trapped in the well on the width and initial energy of the cluster. Plasmon capture causes the well to become deeper, to emit sound waves, and be transformed ultimately into a soliton. During this stage the number of plasmons in the well is preserved, and this determines the amplitude of the produced soliton. The analytic model constructed is in good agreement with results of corroborative numerical calculations.

1. STATEMENT OF PROBLEM

One of the characteristic features of strong Langmuir turbulence is the tendency of the waves to become self-localized. This tendency is clearly revealed, in particular, by results of one-dimensional numerical calculations that demonstrate the onset of solitons.[1–3] Soliton creation is qualitatively attributed to deformation of the plasma-density profile by the pressure of the Langmuir waves, and to trapping of some fraction of the waves into a region of lower density, with ultimate formation of a self-sustaining bound state. This process is quantitatively described by the following system of equations for the complex amplitude of a high-frequency electric field E and for the perturbation of the plasma density n (Ref. 4):

$$i\frac{\partial}{\partial t}E+\frac{3}{2}\omega_p r_D^2\frac{\partial^2}{\partial x^2}E=\frac{1}{2}\omega_p\frac{n}{n_0}E, \tag{1}$$

$$\frac{\partial^2}{\partial t^2}n-c_s^2\frac{\partial^2}{\partial x^2}n=\frac{1}{16\pi M}\frac{\partial^2}{\partial x^2}|E|^2. \tag{2}$$

Here $\omega_p\equiv(4\pi n_0e^2/m)^{1/2}$ is the electron plasma frequency corresponding to an unperturbed density n_0; $r_D\equiv[T/(4\pi n_0e^2)]^{1/2}$ is the Debye radius; T is the electron temperature; M is the ion mass; c_s is the ion-sound velocity.

If the characteristic group velocity v_g of the Langmuir waves is low compared with c_s, Eq. (2) can be approximately replaced at not too high energy by the relation $n=-|E^2|/(16\pi T)$, which means that the density perturbation adjusts itself instantaneously to the distribution of the high-frequency pressure. In this limiting case we have an exhaustive answer to the question of how the soliton parameters are connected with the initial distribution of the electric field, since the system (1), (2) reduces to a nonlinear Schrödinger equation that can be integrated by the inverse scattering transform method.[5] Things are not as definite when the system (1), (2) cannot be integrated in general form. We attempt here to make this situation clearer.

Let us illustrate how a soliton is formed if the initial values of n and $\partial n/\partial t$ are zero, and the initial electric field distribution is a cluster whose width[1)] λ meets the condition

$$\lambda\ll r_D\left(\frac{M}{m}\right)^{1/2}. \tag{3}$$

It follows from this condition, in particular, that the cluster consists predominantly of supersonic plasmons.

Bearing inequality (3) in mind, it is useful to separate in Eqs. (1), (2) from the very outset the corresponding small parameter

$$g=\frac{2}{3}\frac{\lambda}{r_D}\left(\frac{m}{M}\right)^{1/2}. \tag{4}$$

It is convenient for this purpose to reduce the system (1), (2) to dimensionless form

$$ig\frac{\partial}{\partial t}E+\frac{\partial^2}{\partial x^2}E=nE, \tag{5}$$

$$\frac{\partial^2}{\partial t^2}n-\frac{\partial^2}{\partial x^2}n=\frac{\partial^2}{\partial x^2}|E|^2, \tag{6}$$

by making the substitutions

$$x\to\lambda x,\qquad t\to\frac{\lambda}{c_s}t,\qquad n\to3n_0(r_D/\lambda)^2n,\qquad E\to(48\pi n_0T)^{1/2}(r_D/\lambda)E. \tag{7}$$

Before we proceed to the formal solution of the problem of interest, we demonstrate qualitatively the simplifications brought about by the smallness of g.

At $g\ll1$ the density perturbation produced by the plasmons during their free spreading is relatively small, so that the reaction of this perturbation to the plasmons can be described in terms of the quantum-mechanical problem of a particle in a shallow potential well. Since the density profile is nonstationary, some of the plasmons are trapped by the well and go over into a bound state. The trapping continues until the energy level corresponding to the bound state is not too far from the limit of the continuous spectrum. With increasing depth of the well, this level is gradually lowered and

 0038-5646/86/121210-08$04.00

the trapping ceases, since the time dependence of the well parameters becomes ultimately adiabatically slow. The plasmons trapped in the well comprise, as we shall see, only a small fraction of the initial cluster, whereas the entire cluster participates in the creation of the well. This allows us to distinguish in the trapping problem between two relatively independent parts: description of a well produced by a freely spreading cluster, and determination of the number of plasmons trapped in a well that varies in accordance wtih a known law.

After the end of the trapping, the well continues to deepen by inertia for some time, and is then gradually restructured under the influence of the trapped plasmons. In the course of this restructuring the well emits sound and is transformed into a soliton. It is important that since the entire concluding stage of the process is adiabatic, the number of the plasmons in the well is conserved during this stage. This conservation law, together with energy and momentum conservation, enable us to find the amplitude of the produced soliton and the energy of the emitted sound.

Following the indicated procedure, we begin the analysis by determining the potential well produced by the spreading cluster; this is the subject of Sec. 2 of the article. In Sec. 3 we solve the problem of plasmon capture by the well. The conditions under which the results of Secs. 2 and 3 are valid are made more precise in Sec. 4, where the subsequent soliton-formation dynamics is discussed. The last (fifth) section contains the results of corroborative numerical calculations and a comparison with the analytical results.

2. DENSITY PERTURBATION AND EFFECTIVE POTENTIAL

The shallow-well approximation used below requires that the amplitude n and the spatial scale l of the density perturbation satisfy the inequality

$$nl^2 \ll 1. \tag{8}$$

We emphasize that at $g \ll 1$ this requirement is easily met. Indeed, the spreading time of a plasmon cluster with initial width of order unity can be estimated at g, and the growth rate of the perturbation produced by this cluster is tentatively equal to [see Eq. (6)] $\partial n/\partial t \sim g|A|^2$, where A is the amplitude of the electric field in the cluster. The spatial scale of this perturbation is equal to the initial size of the cluster, i.e., we must put $l = 1$ in inequality (8).

After the spreading of the cluster, the density perturbation continues, by inertia, to increase linearly with time up to $t \sim 1$ (by that instant its growth stops on account of spreading at the speed of sound). Thus, the perturbation is bounded from above by $n \sim g|A|^2$. It can be seen from this preliminary estimate that condition (8) allows us to consider quite high plasmon-energy densities. The corresponding bound will be formulated more accurately in Sec. 4.

Inequality (8) means that the characteristic wavelengths of the plasmons trapped in the well are large compared with the dimensions of the well itself. The well can be regarded as pointlike relative to such plasmons, and a corresponding transition can be made from Eq. (5) to an equation with a δ-function potential

$$ig\frac{\partial}{\partial t}E + \frac{\partial^2}{\partial x^2}E = -\eta\delta(x)E. \tag{9}$$

The δ function is taken to be localized here at the maximum of the initial plasmon bunch. Since the well velocity does at any rate not exceed that of sound, and the trapped plasmons are assumed to be supersonic, we have neglected in (9) the possible displacement of the well in the course of the trapping.

The coefficient η preceding the δ function is chosen to satisfy the condition that the first derivative of the electric field with respect to x experience the same jump as in the real well. Since η is equal to $-\int_{-\infty}^{+\infty} n\,dx$, it vanishes in first order in the parameter nl^2 by virtue of the conservation of the total number of ions. In the next (second) order, the standard successive-approximation procedure yields

$$\eta = \int_{-\infty}^{+\infty} dx \left[\int_{-\infty}^{x} n(x')\,dx' \right]^2. \tag{10}$$

The function

$$\xi = -\int_{-\infty}^{x} n(x';t)\,dx'$$

in Eq. (10) satisfies an equation

$$\frac{\partial^2}{\partial t^2}\xi - \frac{\partial^2}{\partial x^2}\xi = -\frac{\partial}{\partial x}|E|^2 \tag{11}$$

that follows directly from (6). In the problem of interest to us, the initial values of $\xi(x;t)$ and $\dot{\xi}(x;t)$ are zero. The corresponding solution of (11) is then

$$\xi(x;t) = \frac{1}{2}\int_0^t [|E(x-t+\tau;\tau)|^2 - |E(x+t-\tau;\tau)|^2]\,d\tau. \tag{12}$$

Assuming the plasmons that produce the well to disperse freely, we put

$$|E(x;t)|^2 = \int \bar{E}_{k_1}\bar{E}_{k_2}^* \exp[i(k_1-k_2)x - it(k_1^2-k_2^2)/g]\,dk_1\,dk_2, \tag{13}$$

where $\bar{E}_k$ are the Fourier components of the initial field. It is convenient to replace k_1 and k_2 in the integral (13) by new variables $q = k_1 - k_2$ and $p = (k_1 + k_2)/g$. Integrating next with respect to time in (12), we can write $(x;t)$ in the form

$$\xi = i\frac{g}{2}\int dp\,dq \frac{e^{iqx}}{q(1-p^2)}(\cos qt - \cos pqt)G_+(gp;q) + \frac{g}{2}\int dp\,dq\frac{e^{iqx}}{q(1-p^2)}(p\sin qt - \sin pqt)G_-(gp;q), \tag{14}$$

where G_+ and G_- are the even and odd (*in* p) parts of the function

$$G(gp;q) \equiv \bar{E}_{(gp+q)/2}\bar{E}^*_{(gp-q)/2}.$$

Since the parameter g is small and the characteristic scales of the variation of the function G with respect to p are $1/g$ and unity, respectively, it is possible to replace, if $t \gg g$, the function $G_+(gp;q)$ in (14) by $G_+(0;q)$. This cannot be done in the case of G_-, since the remaining integral with respect to p diverges as $p \to \pm\infty$. The integral that contains G_- is therefore governed by the large values of p, so that the term $\sin pqt$ can be left out of the integrand, and unity in the denominator can be neglected compared with p^2. In the upshot, $\xi(x;t)$ takes the form

$$\xi = -\frac{g}{2}\int dq \frac{\sin qt}{q} e^{iqx} R(q), \tag{15}$$

where

$$R(q) \equiv \lim_{\varepsilon\to 0} \int \frac{dp}{p - i\varepsilon^2 q/|q|} \bar{E}_{(p+q)/2} \bar{E}^*_{(p-q)/2}. \tag{15a}$$

This equation for ξ is valid in the entire range of t, with the exception of very short times comparable with time required to double the width of the initial plasmon cluster ($\xi \sim t^2$ for such time intervals). Since such short times can be neglected if the cluster energy is not too high, Eq. (15) can be treated in this problem as exact. Substituting now $\xi(x;t)$ in (10), we get

$$\eta = \frac{\pi}{2} g^2 \int dq \frac{\sin^2 qt}{q^2} |R(q)|^2. \tag{16}$$

We present also approximte expressions for η at $t \ll 1$ and $t \gg 1$ (recall that the time unit here is the travel duration of the sound wave through the localization region of the initial plasma cluster):

$$\eta = \frac{\pi}{2} g^2 t^2 \int |R(q)|^2 dq, \qquad t \ll 1. \tag{17}$$

$$\eta = \frac{\pi^2}{2} g^2 t |R(0)|^2, \qquad t \gg 1. \tag{18}$$

The factor $|R(0)|^2$ in (18) can be understood as the value of $|R(q)|^2$ *at* $|q| \sim 1/t \ll 1$. Although the function $R(q)$ has generally speaking a discontinuity when the sign of q is reversed [see Eq. (15a)], its absolute value is continuous if E is smooth, and it is this which permits the quantity $|R(0)|^2$ to be introduced.

It follows at first glance from (18) that as $t \to \infty$ the function $\eta(t)$ (and with it the bound-state energy) increases without limit. It is clear, on the other hand, that the binding energy cannot exceed the depth of the density well produced after the spreading of the plasmons, and the well depth is certainly finite. This contradiction indicates that the shallow-well model is untenable at sufficiently large values[2)] of t. Indeed, the potential connected with any one of the sound waves of the solution (12), which propagates in one direction, can be shown to be by itself not shallow. The well produced by the two waves traveling in opposite directions is shallow because these waves cancel each other to a considerable degree, so that the resultant density perturbation decreases more rapidly than the perturbation due to each of the waves taken separately. In other words, the well remains shallow only up to a certain limiting distance between them. It is important that the maximum permissible wave separation is large compared with the width of the initial plasmon cluster. This is in fact why there exists a certain range of t in which $\eta(t)$ is given by Eq. (18).

3. PLASMON TRAPPING

3.1. Fundamental relations

We take the number N of trapped plasmons to be equal to $|A_0|^2$, where A_0 is the amplitude of the bound state in the expansion of the electric field over the entire set of those eigenfunctions of Eq. (9) which correspond to the instantaneous value of $\eta(t)$:

$$E(x;t) = A_0(\eta/2)^{1/2} e^{-|x\eta/2|} + \int a_k(t)\psi_k(x;t)\,dk.$$

Here a_k are the amplitudes of the continuum states characterized by the wave functions ψ_k.

After the spreading of the initial cluster and the onset of the adiabatic stage, in which transitions from the continuum to the bound state and back are forbidden, the amplitude A_0 should obviously be equal, accurate to a factor $(2/\eta)^{1/2}$, to the value of the electric field at $x = 0$, since all the free plasmons ultimately leave the well-localization region. Thus, the problem reduces in fact to finding the asymptotic of the function $E(0;t)$ at large values of t.

An equation for $E(0;t)$ can be easily obtained from (9). Note that according to (9) the coefficients of the Fourier-integral expansion of the function $E(x;t)$ satisfy the equation

$$ig\frac{\partial}{\partial t}E_k - k^2 E_k = -\frac{\eta}{2\pi}\varepsilon,$$

where $\varepsilon \equiv E(0;t)$. Integration of this equation yields

$$E_k(t) = \bar{E}_k e^{-ik^2 t/g} + \frac{i}{2\pi g}\int_0^t \eta(\tau)\varepsilon(\tau)e^{ik^2(\tau - t)/g}\,d\tau, \tag{19}$$

where $\bar{E}_k$ are the initial values of the Fourier coefficients. The function $E_k(t)$ is connected with $\varepsilon(t)$ by the relation $\varepsilon = \int E_k\,dk$. Integration of both halves of (19) with respect to k yields therefore the following integral equation for $\varepsilon(t)$:

$$\varepsilon(t) - \frac{1+i}{2(2\pi g)^{1/2}}\int_0^t \frac{\eta(\tau)\varepsilon(\tau)}{(t-\tau)^{1/2}}\,d\tau = \int \bar{E}_k e^{-ik^2 t/g}\,dk. \tag{20}$$

By virtue of the linearity of (20), the function ε can be written as a superposition of the functions ε_k satisfying Eq. (20) with a right-hand side in the form $\exp(-ik^2 t/g$. In the adiabatic stage, where N is independent of time, we have

$$N = \frac{2}{\eta}\left|\int \bar{E}_k \varepsilon_k\,dk\right|^2. \tag{21}$$

If the well depth has a power-law variation

$$\eta = Qt^m \tag{22}$$

the equation for ε_k reduces, by the natural substitutions

$$t \to (g/Q^2)^{1/(2m+1)}\tau, \qquad k \to (g^m Q)^{1/(2m+1)}\varkappa \tag{23}$$

to the universal form

$$\varepsilon_\varkappa(\tau)-\frac{1+i}{2(2\pi)^{1/2}}\int_0^\tau\frac{\tau_1^m\varepsilon_\varkappa(\tau_1)\,d\tau_1}{(\tau-\tau_1)^{1/2}}=e^{-i\varkappa^2\tau}. \qquad (24)$$

This universal property enables us to determine directly the dependence of N, defined by (21), on the parameters Q and g. Recognizing that at $g \ll 1$ the well traps mainly long-wave plasmons, we replace the function E_k in (21) by its value at $k = 0$. Using in addition relations (22)–(24) we get

$$N=|\bar{E}_0|^2(g^mQ)^{1/(2m+1)}C_m, \qquad (25)$$

where

$$C_m=2\lim_{\tau\to\infty}\left\{\tau^{-m}\left|\int\varepsilon_\varkappa(\tau)\,d\varkappa\right|^2\right\}. \qquad (26)$$

Equation (25) gives the sought dependence of N on Q and g accurate to an as yet unknown numerical factor C_m determined by the asymptotic solution of Eq. (24). For $m = 1$ and $m = 2$ [it can be seen from (17) and (18) that it is just these two cases which are of primary interest] the factor C_m can be obtained analytically. The result of the corresponding calculations (which are given in the second half of the section) is

$$C_2=\frac{8\pi^4(4/125)^{1/5}}{\Gamma^2(1/5)\Gamma^2(3/5)} \qquad (27)$$

$$C_1=\pi6^{1/3}\Gamma^2(2/3), \qquad (28)$$

where Γ is the Euler gamma function. Combining (17), (18), (22), and (25)–(28) we find ultimately that

$$N=\frac{8\pi^4(2\pi/125)^{1/5}}{\Gamma^2(1/5)\Gamma^2(3/5)}g^{4/5}|\bar{E}_0|^2\left(\int|R(q)|^2dq\right)^{1/5} \qquad (29)$$

for the well's quadratic growth described by (17), and

$$N=3^{1/3}\pi^{5/3}\Gamma^2(2/3)g|\bar{E}_0|^2|R(0)|^{2/3} \qquad (30)$$

for linear growth [see Eq. (18)].

3.2. Calculation of capture coefficient

Proceeding to solve Eq. (24), we start out from the case $m = 1$. Taking the Laplace transform of the function $\varepsilon_\varkappa(\tau)$

$$\varepsilon(\omega)=\int_0^\infty\varepsilon_\varkappa(\tau)e^{i\omega\tau}\,d\tau \qquad (31)$$

and applying Eq. (24) to this transform, we find that $\varepsilon(\omega)$ satisfies the equation

$$\varepsilon-\frac{1}{2\omega^{1/2}}\frac{\partial}{\partial\omega}\varepsilon=\frac{i}{\omega-\varkappa^2}. \qquad (32)$$

All the functions are defined here in the upper half-plane of the complex variable ω $(0<\arg\omega<\pi)$. Therefore, in particular, $\operatorname{Re}\omega^{1/2}>0$. The solution of (32) is

$$\varepsilon=\frac{i}{\omega-\varkappa^2}-\exp\left(\frac{4\omega^{3/2}}{3}\right)\int_\omega^\infty\frac{i\exp(-4\omega_1^{3/2}/3)}{(\omega_1-\varkappa^2)^2}d\omega_1, \qquad (33)$$

where the integration is over any contour located in the upper half-plane and going off to infinity in the sector $0<\arg\omega_1<\pi/3$, thereby ensuring a decrease of the integrand as $\omega_1\to\infty$. This choice of the integration constant is governed by the initial condition for the function $\varepsilon_\varkappa(\tau)$:

$$\varepsilon_\varkappa(\tau)=1,\ \tau=+0,$$
$$\varepsilon_\varkappa(\tau)=0,\ \tau=-0. \qquad (34)$$

We shall find it convenient to assume that the integration contour in (33) goes off to infinity along the real axis.

Equation (33) allows us to represent the solution of (24) in the form

$$\varepsilon_\varkappa(\tau)=\exp(-i\varkappa^2\tau)-\frac{i}{2\pi}\int_{-\infty+i0}^{+\infty+i0}\exp\left(-i\omega\tau+\frac{4\omega^{3/2}}{3}\right)\times d\omega\int_\omega^{+\infty+i0}\frac{\exp(-4\omega_1^{3/2}/3)}{(\omega_1-\varkappa^2)^2}d\omega_1. \qquad (35)$$

To determine the coefficient C_1 it suffices here to retain only the second term. Its asymptotic form at $\tau\to\infty$ is determined by a saddle point located at large negative values of ω $(\omega = -\tau^2/4)$. This allows us to replace in (35) the lower limit of the integration with respect to ω_1 by $-\infty$, after which evaluation of the integral by the saddle-point method yields

$$\varepsilon_\varkappa(\tau)\approx-\frac{1+i}{2(2\pi)^{1/2}}\tau^{1/2}\exp\left(\frac{i\tau^3}{12}\right)\int_{-\infty+i0}^{+\infty+i0}\frac{\exp(-4\omega_1^{3/2}/3)}{(\omega_1-\varkappa^2)^2}d\omega_1.$$

It follows hence that

$$\int\varepsilon_\varkappa(\tau)\,d\varkappa\approx(1-i)\left(\frac{\pi\tau}{2}\right)^{1/2}\exp\left(\frac{i\tau^3}{12}\right)\int_{-\infty+i0}^{+\infty+i0}\exp\left(-\frac{4\omega_1^{3/2}}{3}\right)d\omega_1.$$

Integrating here with respect to ω and substituting the result in (25), we obtain ultimately Eq. (28) for C_1.

In the case $m = 2$, a Laplace transformation converts Eq. (24) into the second-order differential equation

$$+\frac{i}{2\omega^{1/2}}\frac{\partial^2}{\partial\omega^2}\varepsilon=\frac{i}{\omega-\varkappa^2}. \qquad (36)$$

Replacements of the independent variable

$$z=4(1-i)\omega^{5/4}/5 \qquad (37)$$

and of the unknown function $f=\varepsilon/\omega^{1/2}$ reduces (36) to an inhomogeneous Bessel equation whose solution is best expressed in terms of the Hankel functions $H^{(1)}_{2/5}$ and $H^{(2)}_{2/5}$. The resultant expression for ε is

$$\varepsilon=\frac{i}{\omega-\varkappa^2}+\frac{2\pi}{5}\omega^{1/2}H^{(2)}_{2/5}(z)\int_{-\infty+i0}^{\omega}\frac{\omega_1^{1/2}H^{(1)}_{2/5}(z_1)}{(\omega_1-\varkappa^2)^3}d\omega_1+\frac{2\pi}{5}\omega^{1/2}H^{(1)}_{2/5}(z)\int_\omega^{+\infty+i0}\frac{\omega_1^{1/2}H^{(2)}_{2/5}(z_1)}{(\omega_1-\varkappa^2)^3}d\omega_1. \qquad (38)$$

Just as in (33), the integration here is along contours located in the upper ω_1 half-plane and going off to inifinity along the real axis. This leads to satisfaction of the initial condition

(34). It is easy to verify that the solution (38) is unique: addition to it of any solution of the homogeneous equation (36) leads inevitably to violation of the condition (34).

An examination of (38) shows that to find the asymptotic form of the function $\varepsilon_\varkappa(\tau)$ it suffices to retain in (38) only the last term, for just as in the case $m = 1$ the asymptotic $\varepsilon_\varkappa(\tau)$ is determined by the behavior of ε at large negative values of ω. Using the asymptotic expression for the function $H^{(1)}_{2/5}(z)$

$$H^{(1)}_{2/5}(z) \approx (2/\pi)^{1/2}(1/z)^{1/2} e^{i(z-9\pi/20)}$$

(see, e.g., Ref. 6) and replacing the lower limit of integration with respect to ω_1 by $-\infty$, we find that for the frequencies of interest to us we have

$$\varepsilon(\omega) \approx \frac{(\pi/5)^{1/2} 2^{1/4}}{\omega^{7/8}} \exp\left[\frac{4}{5}(1+i)\omega^{5/4} - \frac{13}{40}\pi i\right] \times \int_{-\infty+i0}^{+\infty+i0} \frac{\omega_1^{1/2} H^{(2)}_{2/5}(z_1)}{(\omega_1-\varkappa^2)^3} d\omega_1. \qquad (39)$$

On going from $\varepsilon(\omega)$ to $\varepsilon_\varkappa(\tau)$ the integral with respect to ω can be evaluated by the saddle-point method (the saddle point is located at $\omega = -\tau^4/4$). The result takes the form

$$\varepsilon_\varkappa(\tau) \approx \frac{\tau}{5^{1/2}} \exp\left[i\left(\frac{\tau^5}{20} - \frac{7\pi}{10}\right)\right] \int_{-\infty+i0}^{+\infty+i0} \frac{\omega_1^{1/2} H^{(2)}_{2/5}(z_1)}{(\omega_1-\varkappa^2)^3} d\omega_1.$$

Integration of this expression with respect to $\varkappa$ yields

$$\int \varepsilon_\varkappa(\tau)\, d\varkappa = \frac{\pi\tau}{5^{1/2}} \exp\left[i\left(\frac{\tau^5}{20} - \frac{7\pi}{10}\right)\right] \int_{-\infty+i0}^{+\infty+i0} \omega_1^{1/2} H^{(2)}_{2/5}(z_1)\, d\omega_1.$$

Evaluation of the integral with respect to ω_1 (for which it is expedient to change from ω_1 to the variable z_1) and substitution of the result in (26) yields ultimately Eq. (27) for C_2.

4. DYNAMICS OF SOLITON FORMATION

We consider, on the basis of the contents of Secs. 2 and 3, various soliton-formation situations and indicate the conditions for realizing each.

It can be seen from (29) and (30) that two different regimes of plasmon trapping in a potential well are possible. The first is realized at $g < t < 1$ and corresponds to deepening of the well by inertia, while the second (called acoustic below) is realized at $t > 1$, when the well width becomes larger than the width of the initial plasmon cluster.

We turn first to the inertial trapping regime. The time t_N during which plasmons are trapped on a discrete level characterized by an eigenfrequency

$$\Omega \sim \eta^2/g, \qquad (40)$$

can be estimated from the uncertainty relation

$$\Omega t_N \sim 1, \qquad (41)$$

or, which is the same, from the condition $\tau \sim 1$, where τ is defined in (23). Recognizing that in the inertial regime we have $\eta \sim g^2 N_0^2 t^2$, where N_0 is the number of plasmons in the initial cluster, we get

$$t_N \sim g^{-3/5} N_0^{-4/5}. \qquad (42)$$

The estimate (42) implies that the adiabaticity condition begins to be satisfied prior to the transition from the inertial to the acoustic regime, i.e., $t_N < 1$. This imposes a lower bound on N_0:

$$N_0 > g^{-3/4}. \qquad (43)$$

For the well to remain shallow at $t = t_N$, its size, equal in the inertial regime to the initial size of the plasmon cluster, should be smaller than the spatial scale of the bound-state wave function $\eta^{-1}(t_N)$. This leads to an upper bound on N_0:

$$N_0 < g^{-2}. \qquad (44)$$

This inequality is equivalent to the condition $t_N > g$ that ensures smallness, compared with N_0, of the number of plasmons trapped in the well. Note that since the parameter g is small there exists a large interval of N_0 in which the inequalities (43) and (44) can be satisfied simultaneously.

We estimate next the influence of the trapped plasmons on the shape of the well. Let $\delta n(x;t)$ be the density perturbation produced by these plasmons. It follows from (6) that at the instant of trapping we have

$$|\delta n| \sim N\eta^3 t_N^2. \qquad (45)$$

It is taken into account here that the spatial scale of the function $\delta n(x;t)$ is equal to the width η^{-1} of the bound-state wave function. The excess pressure of the trapped plasmons on the well is significant in the case when the bound-state-energy correction necessitated by the perturbation δn turns out to be comparable with the energy itself. The latter takes place at

$$\delta n \sim \eta^2. \qquad (46)$$

If, however,

$$\delta n \ll \eta^2, \qquad (47)$$

the pressure of the trapped plasmons can be neglected. The condition (47) means also that the perturbation leaves the potential well shallow. Using expression (29) for N, we can easily verify that the restriction imposed on N_0 by relations (45) and (47) coincides with the inequality (44).

In the situation considered by us, when the plasmon pressure and the gas kinetic pressure are negligibly small at the instant of trapping, the well continues to deepen by inertia for some time after trapping the plasmons. The trapped plasmons are then adiabatically compressed and their pressure increases. The well deformation δn increases corresponding. The time t_D starting with which the deformation becomes substantial is estimated with the aid of relations (17), (29), (45), and (46):

$$t_D \sim g^{-14/20} N_0^{-17/20}. \qquad (48)$$

If the inequality

$$N_0 > g^{-14/17} \qquad (49)$$

is satisfied, this time precedes the transition from the inertial to the acoustic regime.

In the region $t > t_D$, the main process is supersonic compression of the well by the plasmons it contains. The compression dynamics is described by the self-similar law

$$|E|^2 \infty (t_0-t)^{-2}, \quad \delta n \infty (t_0-t)^{-4} \tag{50}$$

(see, e.g., Ref. 3). In the course of this compression the gas-kinetic pressure increases more rapidly than the plasmon pressure, so that the compression ultimately stops, the well is transformed into a soliton, and the matter forced out of it is carried away by the sound waves that go off to infinity.

We return now to relation (43). If it is violated, plasmon trapping is determined not by the inertial but by the acoustic stage of the well evolution, during which

$$\eta \sim t g^2 N_0^2 \tag{51}$$

[see Eq. (18)]. It can be seen from (40) and (51) that trapping terminates at

$$t = t_N \sim g^{-1} N_0^{-4/5} > 1. \tag{52}$$

In the acoustic regime, the well dimension is estimated at t. For the well to become shallow at the instant of trapping, the condition $\eta t < 1$ must be met at $t = t_N$; with allowance for the estimates (51) and (52), this condition reduces to the inequality $N_0 > 1$. It can be easily shown that this inequality permits at the same time to describe the trapping process without allowance for the reverse influence of the plasmons on the well shape. The motion of the trapped plasmons becomes significant only after they are additionally compressed adiabatically. The corresponding time t_D is estimated at $g^{-1} N_0^{-1/2}$, and exceeds t_N. From the instant of time $t = t_D$ the behavior of the energy level in the potential well is governed not by the restructuring of the plasma-density profile but by the density perturbation δn produced by the trapped plasmons. The perturbation δn itself and the plasmon energy density $|E|^2$ evolve in accordance with the self-similar law (5), followed by decay into a soliton and sound.

To determine the possible parameters of the produced solitons, we use the conservation laws for the number of the plasmons N, the energy H, and the momentum P, excluding from them the contribution of the freely spreading plasmons. The considered integrals of motion of Eqs. (5) and (6) are of the form

$$N = \int |E|^2 \, dx,$$

$$H = \int \left(\frac{n^2+u^2}{2} + n|E|^2 + \left| \frac{\partial}{\partial x} E \right|^2 \right) dx,$$

$$P = \int \left[nu + \frac{ig}{2} \left(E \frac{\partial}{\partial x} E^* - E^* \frac{\partial}{\partial x} E \right) \right] dx,$$

where u is the hydrodynamic velocity of the plasma, defined by the equation

$$\frac{\partial}{\partial t} u = -\frac{\partial}{\partial x} (n + |E|^2).$$

In this case N is the number of plasmons trapped in the shallow well. Since the processes that follow the trapping are adiabatic, the number of plasmons in the soliton is also N. The energy and momentum of a soliton moving with velocity V satisfy then the expressions (see, e.g., Ref. 7)

$$H(N; V) = g^2 V^2 N/4 + \frac{N^3}{48} \frac{(5V^2-1)}{(1-V^2)^3} \approx \frac{N^3}{48} \frac{5V^2-1}{(1-V^2)^3},$$

$$P(N; V) = g^2 V N/2 + \frac{N^3}{12} \frac{V}{(1-V^2)^3} \approx \frac{N^3}{12} \frac{V}{(1-V^2)^3}.$$

Simple estimates show that since the parameter g is small, the values of H and P directly after the plasmon capture are negligibly small compared with $H(N;V)$ and $P(N;V)$. This allows us to write the conservation laws in the form

$$H(N; V) + \varepsilon_+ + \varepsilon_- = 0, \quad P(N; V) + \varepsilon_+ - \varepsilon_- = 0,$$

where ε_+ and ε_- are the energies of the sound waves that go off to the right (+) and to the left (−) from the soliton. Since ε_+ and ε_- are positive, it follows that $|H(N;V)| \geqslant |P(N;V)|$, meaning that in the regimes considered by us the soliton velocity is bounded from above by the inequality $V \leqslant 1/5$. The numerical smallness of V allows us to find, with high accuracy, the amplitude of the electric field in the soliton, $E_0 = N\sqrt{8}$, the soliton width $\Delta = 4/N$, and the total energy of the emitted sound waves, $\varepsilon_+ + \varepsilon_- = N^3/48$. The quantity N in these equations is given by relations (29) and (30). Determination of the soliton velocity calls for a more thorough investigation of the process of its formation, which is outside the scope of the present paper.

Our estimates allow us to demarcate, on the plane of the parameters g and N_0 (see Fig. 1), three regions (A, B, C) with different soliton-formation dynamics. These regions correspond to the following situations:

Region A $(g^{-14/17} < N_0 < g^{-2})$.

1. Plasma trapping in the inertial regime $(N \sim N_0^{7/5} g^{4/5})$.
2. Adibatic compression of the plasmons in the inertial regime.
3. Self-similar well compression by the trapped plasmons.

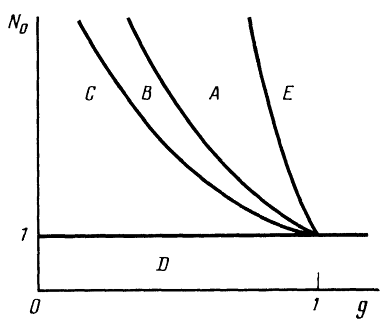

FIG. 1. Regions with different soliton-formation situations.

4. End of compression, emission of sound, and formation of soliton.

Region $B(g^{-3/4} < N_0 < g^{-14/17})$.
1. The same as in region A.
2. Adiabatic compression of the plasmons, first in the inertial and next in the acoustic regime.

3,4. The same as in region A.

Region $C(1 < N_0 < g^{-3/4})$.

0. Inertial deepening of the well without substantial plasmon trapping.
1. Plasmon trapping in the acoustic regime $(N \sim N_0^{5/3} g)$.
2. Adiabatic compression of plasmon in acoustic regime.

3,4. The same as in regions A and B.

Besides the regions A, B, and C, Fig. 1 shows also regions D and E, to which our analysis is inapplicable. These regions are partially described by a nonlinear Schrödinger equation. Typical of region E is an almost complete transition of the plasmons into a bound state, and the possibility of formation of not only one but also several solitons. On the contrary, region D is characterized by almost free dispersal of the plasmons and a low probability of their combining to form a soliton.

5. CORROBORATIVE NUMERICAL CALCULATIONS

To illustrate the results and assess the accuracy of the considered analytic model of soliton formation, we have numerically integrated Eqs. (5) and (6) with the aid of the program used earlier in Ref. 8. The parameter g in Eq. (5) was set equal to 0.125, the initial values of n and $\partial n/\partial t$ were assumed equal to zero, and the initial distribution of the electric field was taken to be Gaussian:

$$E(x;\ 0) = Ae^{-x^2}. \tag{53}$$

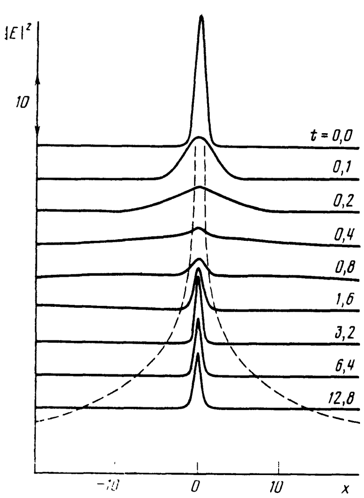

FIG. 2. Spreading, and contraction into a soliton, of plasmons having the Gaussian initial distribution (53), at $g = 0.125$ and $A = 5$. The dashed lines show for comparison the trajectories corresponding to motion at the speed of sound.

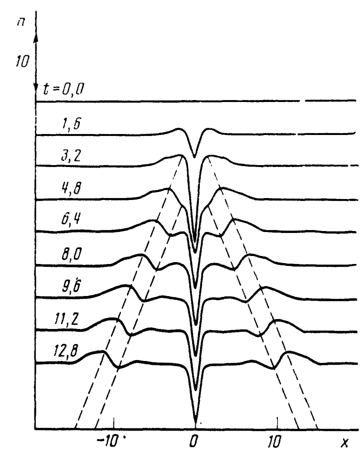

FIG. 3. Evolution of plasma density profile in the regime corresponding to Fig. 2. The dashed lines mark the sound waves going off from the soliton.

The amplitude of the field A was varied in the range from 2.5 to 7.5.

Figures 2 and 3 show the profiles of the squared modulus of the electric field and of the perturbation of the plasma density at various instants of time in the case $A = 5$. To demonstrate the initial dispersal of the plasmons,[3)] the initial time intervals in Fig. 2 are closely spaced, whereas the time spacing in Fig. 3 is uniform. It can be seen from Fig. 2 that after the dispersal of the free plasmons the remainder of the cluster is compressed and assumes ultimately the soliton form. The propagation of the sound waves emitted by the soliton can be distinctly tracked in Fig. 3.

In the course of the calculations we obtained also the dependence of the number N of plasmons in the soliton on the number $N_0 = (\pi/2)^{1/2} A^2$ of plasmons in the initial cluster. This dependence is shown in Fig. 4 by points through which a dashed line is drawn. The lower solid line in the same figure shows the analytically obtained plot for the inertial trapping regime [see Eq. (29)], while the upper solid line is the plot for the acoustic regime [Eq. (30)]. Note that the upper curve is actually incorrectly drawn in Fig. 4, since the same values of the parameters with which the calculations were made correspond, according to the estimates, to

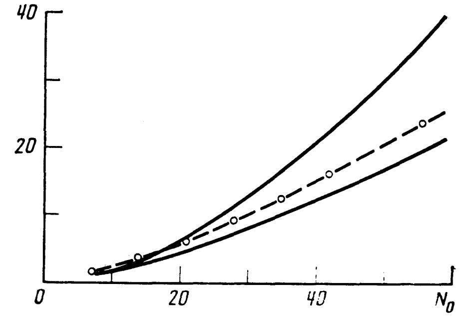

FIG. 4. Comparison of the results of numerical and analytic calculations of plasmon capture into a soliton ($g = 0.125$).

the inertial regime. Nonetheless, even this curve does not deviate too strongly from the calculated points. As expected, the disparity between them increases with N_0. As for the line describing the inertial trapping, it agrees in the entire considered range with the numerical results with accuracy not worse than 20%. This agreement can be regarded as quite satisfactory, since at the chosen calculation parameters the soliton traps already a rather large (up to 40%) fraction of the energy of the inital cluster, i.e., the comparison is made here in fact at the borderline of the validity of the theory.

We note in conclusion that the elementary soliton-creation process discussed by us constitutes a substantial part of a more complicated picture observed in numerical simulation of the evolution of intense supersonic Langmuir waves with random initial phases.[8] A number of laws revealed by this simulation agree qualitatively with the dependences obtained in the present paper.

[1)] In these and subsequent estimates it is implied for simplicity that the cluster has no additional internal scale, i.e., the characteristic wavelength of the plasmons is estimated to be equal to λ.

[2)] In the limit as $t \to \infty$ the assumption that the well is produced predominantly by supersonic plasmons is also violated.

[3)] The presence of a stage of free dispersal of the plasmons makes Fig. 2 qualitatively different from the corresponding figure shown in Ref. 1. The cause of the difference is that the calculations of Ref. 1 were carried out at large values of the parameter g.

[1] L. M. Degtyarev, V. G. Makahn'kov, and L. I. Rudakov, Zh. Eksp. Teor. Fiz. **67**, 533 (1974) [Sov. Phys. JETP **40**, 264 (1975)]

[2] G. J. Morales, Y. C. Lee, and R. B. White, Phys. Rev. Lett. **32**, 457 (1974).

[3] L. M. Degtyarev, R. Z. Sagdeev, G. I. Solov'ev, V. D. Shapiro, and V. I. Shevchenko, Fiz. Plazmy **6**, 485 (1980) [Sov. J. Plasma Phys. **6**, 263 (1980).

[4] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].

[5] V. E. Zakharov and A. B. Shabat, *ibid*. **61**, 118 (1971) [**34**, 62 (1972)].

[6] A. Erdelyi, ed., Higher Transcendental Functions, McGraw, 1953, Vol. 2.

[7] J. Gibbons, S. G. Thornhill, M. J. Wardrop, and D. Ter Haar, Plasma Phys. **17**, 153 (1977).

[8] Z. Sedláček, K. Jungwirth, P. Stavinoha, and B. N. Breizman, Proc. 12th Europ. Conf. on Contr. Fusion and Plasma Physics, Part II, Budapest, 1985, p. 338.

Translated by J. G. Adashko