Stimulated Brillouin scattering in forward scattering of light

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Brillouin scattering with the scattered light wave propagating in the same direction as the incident one is possible in an anisotropic medium. The problem of stimulated Brillouin scattering (SBS) for such a scattering geometry has already been solved for the stationary regime. It is shown in this article that the problem of SBS in a nonstationary regime reduces to solution of a nonlinear sine-Gordon equation with a specified initial condition. The known inverse scattering transform method is used to solve this problem and yields in explicit form the amplitudes of the interacting waves. It is shown that an effective-interaction region is produced in the crystal, and in this region the intensity of the incident light wave is transferred to the scattered one. In the case of nonstationary SBS, this region moves in the course of time to the entrance face of the sample.

Stimulated Brillouin scattering (SBS) is usually observed in backscattering geometry, when the scattered light wave emerges from the sample counter to the incident one. In an isotropic medium, the frequency and wave-vector conservation laws forbid forward scattering of light from sound. A crystal, however, has birefringence, differently polarized light waves have different velocities, and diffraction of light by a moving lattice of sound is possible also in forward-scattering geometry. SBS is also observed in this case.[1] The SBS can frequently be observed only for large intensity transfer from the incident to the scattered light wave. A mathematical description of this process follows.

Assuming that the spatial and temporal variations of the interacting-wave amplitudes occur over intervals much larger than the lengths and periods of the waves, we can change from the wave equations to the truncated equations for the electric-field intensity amplitudes $E_0(x,t)$ and $E_1(x,t)$ in the incident and scattered light waves, and the amplitude $u(x,t)$ of the displacement in the sound wave[2]:

$$\partial E_0/\partial x+c^{-1}(\partial E_0/\partial t)=-auE_1, \quad a=\varepsilon^{3/2}p\omega q/4c, \tag{1}$$

$$\partial E_1/\partial x+c^{-1}(\partial E_1/\partial t)=auE_0, \tag{2}$$

$$w^{-1}(\partial u/\partial t)+\partial u/\partial x+\alpha(u-u_0)=bE_0E_1, \quad b=\varepsilon^2 p/32\pi\rho w^2. \tag{3}$$

Here ε is the dielectric constant, p is the photoelastic constant, ρ is the density, c and w are the light and sound velocities, q and α are the sound wave vector and damping coefficient, and ω is the light frequency. For simplicity, we do not distingush between the dielectric constants, velocities, and frequencies of the incident and scattered light waves. We neglect the weak damping of the light compared with the stronger damping of the sound.

The system of equations is completely defined by specifying the boundary and initial conditions

$$E_0(x=0,\ t)=\mathscr{E},\ E_1(x=0,\ t)=0,$$
$$u(x=0,\ t)=u_0,\ u(x,\ t=0)=u_0. \tag{4}$$

The amplitude u_0 simulates the thermal-noise level, and only its inclusion in Eq. (3) is compatible with the natural boundary conditions (4).[2]

The intense light wave that causes SBS is usually produced by a laser pulse, whose time $T_0\sim 10^{-8}$ s is significant in the description of SBS. Assuming that the sample length L is traversed by the light instantaneously, $L\ll cT_0$, we can drop the time derivatives in Eqs. (1) and (2). What matters is whether a stationary sound-wave amplitude can be set up within the time T_0. If

$$\alpha w T_0 \gg 1 \tag{5}$$

this does take place and the SBS takes place in a stationary regime. We can then neglect in the system (1)–(3) all the time derivatives. For coordinates $x>1/\alpha$ we neglect also the space derivative in (3), and then the system can be completely integrated.[2] An analogous solution was indicated for stimulated Raman scattering by Loudon[3,4]:

$$u(x)=u_0\frac{\exp(ab\mathscr{E}^2x/\alpha)}{1+(\alpha u_0/b\mathscr{E}^2)^2\exp(2ab\mathscr{E}^2x/\alpha)}, \tag{6}$$

$$E_1(x)=\mathscr{E}\frac{(\alpha u_0/b\mathscr{E}^2)[\exp(ab\mathscr{E}^2x/\alpha)-1)}{[1+(\alpha u_0/b\mathscr{E}^2)^2\exp(2ab\mathscr{E}^2x/\alpha)]^{1/2}}, \tag{7}$$

$$E_0(x)=\mathscr{E}\frac{1-(\alpha u_0/b\mathscr{E}^2)^2\exp(ab\mathscr{E}^2x/\alpha)}{[1+(\alpha u_0/b\mathscr{E}^2)^2\exp(2ab\mathscr{E}^2x/\alpha)]^{1/2}}. \tag{8}$$

The small terms $(\alpha u_0/b\mathscr{E}^2)^2$ were left out of expressions (6) and (8) whenever they came close to unity. It can be seen that an effective-interaction region is produced in the crystal, with length of the order of $\alpha/ab\mathscr{E}^2$. In this region, the amplitude of the incident light wave decreases exponentially, while the amplitude of the scattered wave grows exponentially to the value $\mathscr{E}$. Only in this region does an intense sound wave exist, with amplitude determined by the intensity of the incident light wave. With increase of this intensity, the effective-interaction region shifts towards the entrance face of the sample.

If the inequality (5) is reversed, the SBS regime is nonstationary. We describe it by the same system of equations, but now we simplify Eq. (3) by leaving out the last term of the left-hand side. The second term in the left-hand side,

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which is necessary for the description of the small region $x < wT_0$, can also be left out, since we are interested in large coordinates. Even the simplified system, however, consists of partial differential equations that describe the variation of the amplitudes both in space and in time.

In the linear theory of nonstationary SBS the amplitude $\mathscr{E}$ of the incident wave is assumed constant and Eq. (1) is disregarded. The system (2)–(4) can then be easily solved:

$$u(x,\ t)=u_0 I_0[2(xtabw\mathscr{E}^2)^{1/2}], \tag{9}$$

$$E_1(x,\ t)=u_0(ax/bwt)^{1/2} I_1[2(xtabw\mathscr{E}^2)^{1/2}]. \tag{10}$$

Replacing the Bessel functions of imaginary argument in these equations by their asymptotics $I_\nu(z) = (2\pi z)^{-1/3} e^z$, we obtain an exponential growth that allows the scattered-wave amplitude reach a value of order $\mathscr{E}$, notwithstanding the small prefactor of the exponential. At those values of the argument at which this occurs, however, the approximation with a specified incident-wave amplitude is not valid.

It is easy to show that the system (1)–(4) has a first integral

$$E_0^2(x,\ t)+E_1^2(x,\ t)=\mathscr{E}^2. \tag{11}$$

The value of the constant was chosen from the boundary conditions (4). Using (11), we introduce a new function $\varphi(x,t)$, choosing

$$E_0(x,\ t)=\mathscr{E}\cos[\varphi(x,\ t)/2],\quad E_1(x,\ t)=\mathscr{E}\sin[\varphi(x,\ t)/2]. \tag{12}$$

Equations (1) and (2) together with conditions (4) are transformed into one expression

$$\varphi(x,t)=2a\int_0^x u(x',t)\,dx', \tag{13}$$

substitution of which into (3) reduces the solution of the system to a solution of one nonlinear partial differential (sine-Gordon) equation

$$\frac{1}{wab\mathscr{E}^2}\frac{\partial^2\varphi}{\partial x\,\partial t}=\sin\varphi \tag{14}$$

with initial condition

$$\varphi(x,\ t=0)=2au_0x. \tag{15}$$

The method developed in modern mathematical physics to solve such a Cauchy problem is the inverse scattering transform method. It is described in the known book by Zakharov *et al.*[5] It is shown in the Appendix how to use this method to find a solution of (14) at larger values of the argument $xtabw\mathscr{E}^2$. This solution is

$$\varphi(x,\ t)=4\ \mathrm{arctg}\{(16\pi)^{-1/2}[au_0x^{1/4}(tabw\mathscr{E}^2)^{-3/4}] \times\exp(4xtabw\mathscr{E}^2)^{1/2}\}. \tag{16}$$

Expressions are accordingly obtained for the amplitudes of the interacting waves

$$u(x,t)=\frac{(4\pi)^{-1/2}u_0(xtabw\mathscr{E}^2)^{-1/4}\exp(4xtabw\mathscr{E}^2)^{1/2}}{1+(a^2u_0^2/16\pi)x^{1/2}(tabw\mathscr{E}^2)^{-3/2}\exp(16xtabw\mathscr{E}^2)^{1/2}} \tag{17}$$

$$E_1(x,t)=\mathscr{E}\frac{(4\pi)^{-1/2}au_0x^{1/4}(tabw\mathscr{E}^2)^{-3/4}\exp(4xtabw\mathscr{E}^2)^{1/2}}{1+(a^2u_0^2/16\pi)x^{1/2}(tabw\mathscr{E}^2)^{-3/2}\exp(16xtabw\mathscr{E}^2)^{1/2}}, \tag{18}$$

$$E_0(x,t)=\mathscr{E}\frac{1-(a^2u_0^2/16\pi)x^{1/2}(tabw\mathscr{E}^2)^{-3/2}\exp(16xtabw\mathscr{E}^2)^{1/2}}{1+(a^2u_0^2/16\pi)x^{1/2}(tabw\mathscr{E}^2)^{-3/2}\exp(16xtabw\mathscr{E}^2)^{1/2}}. \tag{19}$$

Expressions (17) and (18) are naturally matched to the asymptotic expressions (9) and (10). These equations describe an exponential decrease of the incident-light-wave amplitude with simultaneous growth of the scattered-wave amplitude. The region in which this takes place depends on the time and shifts in the course of time towards the entrance edge of the sample.

The solution encounters a difficulty, however, viz., reversal of the sign in (19). A similar difficulty can be observed in (8) at very large values of the coordinate. Since the wave amplitude is by definition positive, the thought occurs that the sign reversal corresponds simply to a change of phase, and in Eqs. (8) and (19) one should simply use absolute values.

The set of equations for three interacting waves must take into account the spatial and temporal variations of not only the amplitudes but also the phases $\delta_0(x,t)$, $\delta_1(x,t)$, $\delta_u(x,t)$. This adds to the right-hand sides of (1)–(3) an additional factor $\cos(\sigma_0-\sigma_1-\sigma_u)$. Furthermore, three coupled equations are added:

$$E_0\left(\frac{\partial\delta_0}{\partial x}+\frac{1}{c}\frac{\partial\delta_0}{\partial t}\right)=auE_1\sin(\delta_0-\delta_1-\delta_u), \tag{20}$$

$$E_1\left(\frac{\partial\delta_1}{\partial x}+\frac{1}{c}\frac{\partial\delta_1}{\partial t}\right)=auE_0\sin(\delta_0-\delta_1-\delta_u), \tag{21}$$

$$u\left(\frac{\partial\delta_u}{\partial x}+\frac{1}{c}\frac{\partial\delta_u}{\partial t}\right)=bE_0E_1\sin(\delta_0-\delta_1-\delta_u). \tag{22}$$

At the start of the parameteric process, a solution arises with a zero phase difference, and is preserved in time and in space by virtue of (20)–(22). At the point where the wave amplitude vanishes, however, the phase is not defined, a phase slip can occur, and the solution of the complete system must be determined anew after passage through such a point.

For stationary SBS, such a point is $x_0=(2\alpha/ab\mathscr{E}^2)\ln(b\mathscr{E}^2/\alpha u_0)$. At this point, the total energy has already been transferred to the scattered waves. For coordinates larger than x_0 we choose a solution with a conserved phase difference equal to π. The solution of the equations for the amplitudes is given by Eqs. (6) and (7), and by (8) taken with a minus sign. It must be emphasized that no exponential parametric growth of the wave amplitudes occurs in this coordinate region. In the absence of such a growth, separation of the interaction of only three waves has no physical meaning. The parametric connection is therefore "drowned in the noise" for stationary SBS at $x > x_0$.

In nonstationary SBS, the evolution of the process is somewhat different. The amplitude E_0 vanishes for each coordinate at some definite instant of time, and a phase shift by π is again possible. The solution for the amplitudes is then given by (17), (18), and by (19) with a minus sign. This solution corresponds to reverse intensity transfer, at infinity, from the scattered into the incident wave. At long times, however, the sound damping should lower the effectiveness of such a reverse transfer.

APPENDIX

Since the inverse scattering transform method is applicable to functions that do not increase at infinity, we present a modification of Eq. (12), cutting off the initial value of the funciton $\varphi(x,t=0)$ at $x\to\infty$:

$$\varphi(x,t=0)=\begin{cases} 0, & x<0 \\ (2au_0/\gamma)[1-\exp(-\gamma x)], & x\geqslant 0 \end{cases}. \quad (A1)$$

We show now a procedure for finding a desired solution that does not depend on the cutoff parameter.

The inverse scattering transform method reduces the solution of the Cauchy problem for a nonlinear equation to a solution of a set of linear problems. The first problem involves consideration of the system $(x>0)$

$$d\psi^{(1)}(x,\lambda)/dx=i\lambda\psi^{(1)}(x,\lambda)+iau_0e^{-\gamma x}\psi^{(2)}(x,\lambda), \quad (A2)$$

$$d\psi^{(2)}(x,\lambda)/dx=-i\lambda\psi^{(2)}(x,\lambda)+iau_0e^{-\gamma x}\psi^{(1)}(x,\lambda). \quad (A3)$$

The role of the potential is assumed in this system by $0.5\partial\varphi(x,t=0)/\partial x$. We must find a unimodular transition matrix that relates the exponentially observed linearly independent solutions at $x\to\pm\infty$. At $x<0$, since there is no potential, the solutions of the system are free. The system (A2), (A3) can be easily reduced to one second-order equation for each of the functions $\psi^{(n)}(x,\lambda)$. Its solution is simply expressed in terms of the variable $\zeta=(au_0/\gamma)e^{-\gamma x}$:

$$\psi^{(1)}(x,\lambda)=C_1\zeta^{1/2}J_{1/2+i\lambda/\gamma}(\zeta)+C_2\zeta^{1/2}J_{-1/2-i\lambda/\gamma}(\zeta), \quad (A4)$$

$$\psi^{(2)}(x,\lambda)=C_1i\zeta^{1/2}J_{-1/2+i\lambda/\gamma}(\zeta)-C_2i\zeta^{1/2}J_{1/2-i\lambda/\gamma}(\zeta). \quad (A5)$$

The constants are determined from the continuity of Eqs. (A4) and (A5) at $x=0$.

The asymptotic solution as $x\to+\infty$ is

$$\psi^{(1)}(x,\lambda)\to\frac{C_2 2^{1/2}\exp[-i(\lambda/\gamma)\ln(au_0/2\gamma)]}{\Gamma(1/2-i\lambda/\gamma)}e^{i\lambda x},$$

$$\psi^{(2)}(x,\lambda)\to\frac{C_1 2^{1/2}(-i)\exp[i(\lambda/\gamma)\ln(au_0/2\gamma)]}{\Gamma(1/2+i\lambda/\gamma)}e^{-i\lambda x}. \quad (A6)$$

By this token, the transition matrix is completely determined. The final expression for the reflection coefficient $r(\lambda)$ is

$$r(\lambda)=i\frac{\Gamma(1/2+i\lambda/\gamma)J_{1/2+i\lambda/\gamma}(au_0/\gamma)}{\Gamma(1/2-i\lambda/\gamma)J_{-1/2-i\lambda/\gamma}(au_0/\gamma)}\times\exp\left[-2i\frac{\lambda}{\gamma}\ln\frac{au_0}{2\gamma}\right]. \quad (A7)$$

This equation shows that $r(\lambda)$ has no poles, i.e., there is no discrete spectrum in the problem. This means also that the solution of the Cauchy problem is not connected with the soliton solutions of the (sine-Gordon) equation (14).

According to the general theory, the dependence of the sine-Gordon equation on the time $\tau=abw\mathscr{E}^2$ is universal for the reflection coefficient

$$r(\lambda,\tau)=r(\lambda)\exp(-i\tau/2\lambda). \quad (A8)$$

Strictly speaking, the inverse scattering transform method pertains to the following linear problem—solving a system of integral equations with a specified kernel $r(\lambda,\tau)$:

$$y^{(1)}(\lambda)=1-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar r(\lambda',\tau)\exp(-2ix\lambda')}{\lambda'-\lambda-i\delta}y^{(2)}(\lambda')\,d\lambda',$$

$$\delta\to 0, \quad (A9)$$

$$y^{(2)}(\lambda)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{r(\lambda',\tau)\exp(2ix\lambda')}{\lambda'-\lambda+i\delta}y^{(1)}(\lambda')\,d\lambda', \quad \delta\to 0. \quad (A10)$$

The apostrophe denotes a complex conjugate. The dependence on space and time enters in the function $y^{(n)}(\lambda)$ as a dependence on the parameters. From the solution of this system of equations we can determine the spatial derivative of the function $\varphi(x,\tau)$ at $\tau>0$:

$$\frac{1}{2}\frac{\partial\varphi(x,\tau)}{\partial x}=\frac{1}{\pi i}\int_{-\infty}^{+\infty}r(\lambda,\tau)\exp(2ix\lambda)y^{(1)}(\lambda)\,d\lambda. \quad (A11)$$

Let us determine the form of the reflection coefficient, letting $\gamma\to 0$. At $\lambda\gg\gamma$ and $au_0\gg\gamma$ we have

$$r(\lambda)=\left[\frac{au_0}{\lambda+(\lambda^2+a^2u_0^2)^{1/2}}\right]\exp\left\{\frac{2i}{\gamma}\left[(\lambda^2+a^2u_0^2)^{1/2}-\lambda-\lambda\ln\left(\frac{\lambda+(\lambda^2+a^2u_0^2)^{1/2}}{2\lambda}\right)\right]\right\}. \quad (A12)$$

The square root must be determined as an analytic function of the complex variable λ, a function positive when $\lambda>0$. Only the phase of $r(\lambda)$ depends on the γ cutoff. It should be noted that expression (A12) for the reflection coefficient can be obtained from the solution of the system (A2) and (A3) by using a classical method, known from quantum mechanics.

At $\lambda\gg au_0$ the phase of $r(\lambda)$ takes the simpler form $a^2u_0^2/2\lambda\gamma$, so that by choosing γ small, satisfying the inequalities

$$\lambda\gg au_0\gg\gamma\gg a^2u_0^2/2\lambda, \quad (A13)$$

we make the phase small and can neglect it completely. In this limiting case the expression for $r(\lambda)$ becomes greatly simplified:

$$r(\lambda)=au_0/2\lambda. \quad (A14)$$

There is no cutoff parameter in (A14), and the integral equations have a solution determined by such values of the kernel and independent of the cutoff parameter artificially introduced into the problem.

We seek the solutions of the system (A9), (A10) at large $(x\tau)^{1/2}$, where the nonlinearity of the SBS problem manifests itself. The corresponding method consists of determining the integrals in (A9) and (A10) by the saddle-point method. To this end it is necessary to draw the integration contour in the corresponding planes through the points $\lambda_0 = \pm i(\tau/4x)^{1/2}$. For these values of λ we can use the simple expression (A14), since $au_0/2|\lambda_0| = au_0 x/(x\tau)^{1/2} \ll 1$ and in this fraction the numerator is small and the denominator large.

The saddle-point integration method can, however, be used only if the oscillations of the function $y^{(n)}(\lambda)$ do not suppress the oscillations of the kernels of the integral equations. Unfortunately, the functions $y^{(n)}(\lambda)$ must contain such rapidly oscillating parts. We separate in the functions $y^{(n)}(\lambda)$ the slowly varying functions of the variable λ:

$$y^{(1)}(\lambda)=z^{(1)}(\lambda)+\bar{r}(\lambda)\exp(-2ix\lambda+i\tau/2\lambda)\,z^{(3)}(\lambda), \quad \text{(A15)}$$

$$y^{(2)}(\lambda)=z^{(2)}(\lambda)+r(\lambda)\exp(2ix\lambda-i\tau/2\lambda)\,z^{(4)}(\lambda). \quad \text{(A16)}$$

That the functions $z^{(n)}(\lambda)$ vary slowly can be proved by considering the system of integral equations which they satisfy:

$$z^{(1)}(\lambda)=1-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{r}(\lambda')\exp(-2ix\lambda'+i\tau/2\lambda')}{\lambda'-\lambda+i\delta}z^{(2)}(\lambda')\,d\lambda'$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{|r(\lambda')|^2}{\lambda'-\lambda-i\delta}z^{(4)}(\lambda')\,d\lambda' \quad (\delta\to 0), \quad \text{(A17)}$$

$$z^{(3)}(\lambda)=-z^{(2)}(\lambda), \quad \text{(A18)}$$

$$z^{(2)}(\lambda)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{r(\lambda')\exp(2ix\lambda'-i\tau/2\lambda')}{\lambda'-\lambda-i\delta}z^{(1)}(\lambda')\,d\lambda'$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{|r(\lambda')|^2}{\lambda'-\lambda+i\delta}z^{(3)}(\lambda')\,d\lambda' \quad (\delta\to 0), \quad \text{(A19)}$$

$$z^{(4)}(\lambda)=z^{(1)}(\lambda). \quad \text{(A20)}$$

In this system we can already obtain the integrals of the rapidly oscillating functions by the saddle-point method:

$$z^{(1)}(\lambda)+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{|r(\lambda')|^2}{\lambda'-\lambda-i\delta}z^{(1)}(\lambda')\,d\lambda'$$

$$=1-\frac{au_0\exp(4x\tau)^{1/2}}{4\pi^{1/2}(x\tau)^{1/4}[\lambda-i(\tau/4x)^{1/2}]}z^{(2)}\left[\lambda=i\left(\frac{\tau}{4x}\right)^{1/2}\right] \quad (\delta\to 0), \quad \text{(A21)}$$

$$z^{(2)}(\lambda)-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{|r(\lambda')|^2}{\lambda'-\lambda+i\delta}z^{(2)}(\lambda')\,d\lambda'$$

$$=\frac{au_0\exp(4x\tau)^{1/2}}{4\pi^{1/2}(x\tau)^{1/4}[\lambda+i(\tau/4x)^{1/2}]}z^{(1)}\left[\lambda=-i\left(\frac{\tau}{4x}\right)^{1/2}\right] \quad (\delta\to 0). \quad \text{(A22)}$$

The solution of this system does not depend on the cutoff: the integral term contain the absolute value of the reflection coefficient, and expression (A14) was used for the saddle-point values.

We can now verify that the integral terms in (A21) and (A22) always make a contribution that is small as $au_0/|\lambda|$, so that these terms can be neglected. We have justified by the same token the possibility of neglecting the difference between the functions $y^{(n)}(\lambda)$ and $z^{(n)}(\lambda)$.

For the saddle-point values $z^{(n)}(\lambda_0)$ we obtain the simple system of algebraic equations:

$$z^{(1)}[\lambda=-i(\tau/4x)^{1/2}]=[1+(a^2u_0^2/16\pi)x^{1/2}\tau^{-3/2}\exp(16x\tau)^{1/2}]^{-1}, \quad \text{(A23)}$$

$$z^{(2)}\left[\lambda=i\left(\frac{\tau}{4x}\right)^{1/2}\right]=\left(\frac{iau_0}{4\pi^{1/2}}\right)x^{1/4}\tau^{-3/4}$$

$$\times\frac{\exp(4x\tau)^{1/2}}{[1+(a^2u_0^2/16\pi)x^{1/2}\tau^{-3/2}\exp(16x\tau)^{1/2}]}. \quad \text{(A24)}$$

The integral (A11) is also calculated by the saddle-point method, demonstrating the need for only the value of the function (A23):

$$\frac{\partial\varphi(x,\tau)}{\partial x}=\frac{\pi^{-1/2}au_0(x\tau)^{-1/4}\exp(4x\tau)^{1/2}}{1+(a^2u_0^2/16\pi)x^{1/2}\tau^{-3/2}\exp(16x\tau)^{1/2}}. \quad \text{(A25)}$$

This equation can be integrated with respect to coordinate approximately, accurate to terms small in $(x\tau)^{-1/2}$, but this is also the accuracy of the saddle-point method used to obtain Eq. (A25):

$$\varphi(x,\tau)=4\,\mathrm{arctg}\,[(16\pi)^{-1/2}(au_0x^{1/4}\tau^{-3/4})\exp(4x\tau)^{1/2}]. \quad \text{(A26)}$$

To satisfy the initial and boundary condition we match the solutions (A25) and (A26) to the linear solutions (9) and (10). Such a matching to the asymptotics of (9) and (10) is obvious.

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