Influence of electron-electron interaction on the critical current in a Josephson junction

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The vertex part of the electron-electron (*e-e*) interaction in a "dirty" superconductor is found by considering all types of *e-e* interactions in a Cooper pair condensate. It is shown that the *e-e* interaction causes fluctuations in the phase and modulus of the order parameter, screens the Coulomb interaction dynamically, and gives rise to a correction to the critical current due to interference between the fluctuations in the order parameter and scalar potential. The order parameter (renormalized for the *e-e* interaction) is calculated, and the corrections to the critical current due to the *e-e* interaction are found for a Josephson junction. In addition to their temperature dependence, the corrections also depend nontrivially on the size and geometry of the junction.

The effects of electron-electron (*e-e*) interactions on the thermodynamic and kinetic properties of disordered electron systems have recently been widely discussed in the literature.[1] In addition to corrections to the specific heat, conductivity, magnetic resistance, and other physical properties, *e-e* interactions also significantly influence the *I-V* characteristics of tunnel junctions. For example, it was shown in Ref. 2 that dynamic screening of the electron-electron Coulomb potential (diffusion *e-e* interaction channel) is responsible for the anomalous behavior of the *I-V* characteristics for junctions at zero bias. If one allows for interaction processes involving superconducting fluctuations (the Cooper *e-e* interaction channel) in tunnel junctions at temperatures above the critical point,[3] one finds that the anomaly at $V=0$ is also accompanied by a distinctive "pseudogap" minimum at $\mathrm{eV} \sim k_B\,(T-T_c)$. In Ref. 4 it was shown that for $T>T_c$, the superconducting fluctuations perturb the Josephson component of the current in a tunnel junction by producing a rapidly oscillating additional current which causes the junction to emit electromagnetic waves (fluctuation radiation).

The purpose of this paper is to analyze how *e-e* interactions alter the properties of a Josephson junction below the critical temperature. The situation here is more complicated than for $T>T_c$ because the corrections to the diffusion and Cooper channels cannot be considered independently. The condensed Cooper pairs permit *e-e* interactions that do not conserve the number of uncondensed particles before and after the interaction, and the diffusion and Cooper channels are thus coupled.

In Sec. 1 we examine *e-e* interaction in a "dirty" superconductor by means of the temperature diagram technique. In Sec. 2 we calculate the first-order correction to the one-electron Green's function for a superconductor and find the average value of the order parameter after renormalization for the *e-e* interaction. The *e-e* interactions cause the modulus and phase of the order parameter to fluctuate, screen the Coulomb interaction dynamically (the scalar potential fluctuates) when a condensate is present, and give rise to an interference contribution to the critical current due to interference between the fluctuations in the phase and in the scalar potential. In Sec. 3 we express the total current through the Josephson junction in terms of correlation functions for the temperature Greeen's functions. This expression is used to analyze how the *e-e* interaction alters the critical current for a Josephson film junction. The resulting correction to the critical current alters the temperature dependence near T_c. In addition, the correction depends nontrivially on the dimensions of the junction, because large-scale phase fluctuations decrease the average order parameter for each of the electrodes.

1. *e-e* INTERACTION IN A DISORDERED SUPERCONDUCTOR

To a certain extent, electron-electron interactions in superconductors have been considered previously—for example, in connection with the Carlson-Goldman experiments[5] and in the theory of collective oscillations in superconductors.[6] Specifically, the one-electron state density and results from tunnel experiments were considered in Ref. 7, where a generalization of the standard approach[8] was suggested for describing the fluctuations in a gapless superconductor. However, this generalization did not treat the coupling between the diffusion and Cooper channels. In fact, all the *e-e* interaction processes in the superconductor phase must be considered together, and the method proposed in Ref. 9 makes it possible to do this systematically. The contribution from the *e-e* interaction to the linear response and dielectric permittivity of superconductors was considered somewhat later in Refs. 10–12, where experimental results[5] were also analyzed. However, the choice of the representation formalism employed there was unfortunate because it necessitated extremely elaborate and physically obscure calculations.

We use the temperature diagram technique to analyze the *e-e* interaction in a disordered superconductor (for which the electron mean free path l satisfies the condition $p_F^{-1} \lesssim l \ll v_F/T_c$). The *e-e* interactions mentioned above can all be described in a unified way by introducing a suitable

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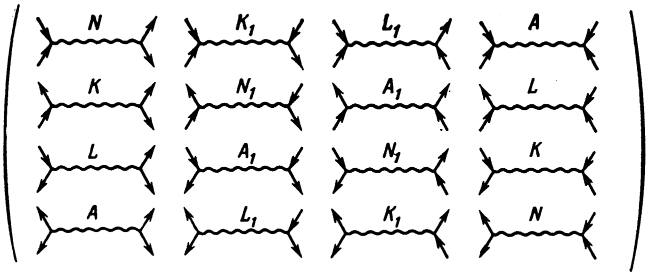

FIG. 1. Diagram representation of the electron-electron interaction matrix for a superconductor.

vertex part $\tilde{L}$ (Refs. 9 and 10), which is expressible as a rank-two tensor on the four-dimensional Euclidean space R_4. Figure 1 shows the corresponding matrix in the convenient "arrow" representation which describes the interaction for $T > T_c$ (it was also used in Ref. 10). The structure of this matrix can be analyzed as follows. Above T_c, the matrix elements $\tilde{L}_{00}$, and $\tilde{L}_{33}$ correspond to the ordinary fluctuation propagator,[8] while $\tilde{L}_{11}$, $\tilde{L}_{12}$, $\tilde{L}_{21}$, and $\tilde{L}_{22}$ correspond to the dynamically screened Coulomb interaction[1] (the elements $\tilde{L}_{12}$, $\tilde{L}_{21}$ describe interactions between a particle and a hole with total spin $S=0$, while $\tilde{L}_{11}$ and $\tilde{L}_{22}$ do the same for $S=1$). The presence of the condensate is responsible for the other off-diagonal elements, which describe processes that do not conserve particle number.

In the ladder approximation, the vertex part $\tilde{L}$ is given by the familiar formula

$$\tilde{L}=\tilde{L}^{(0)}+\tilde{L}^{(0)}\tilde{\Pi}\tilde{L}. \tag{1}$$

The matrix of bare vertices $\tilde{L}^{(0)}$ in (1) describes e-e processes that occur to first order in perturbation theory. Thus, the matrix elements $\tilde{L}^{(0)}_{00} = \tilde{L}^{(0)}_{33} = \lambda$, where λ is the effective e-e interaction constant for large momentum transfers, while the elements $\tilde{L}^{(0)}_{12} = \tilde{L}^{(0)}_{21} = 4\pi e^2/q^2$ correspond to the bare Coulomb interaction. In the next approximation, the first-order elements $\tilde{L}_{11}$ and $\tilde{L}_{22}$ describe two distinct interaction processes[13] and correspond to large and small momentum transfers, respectively. We may therefore write

$$\tilde{L}^{(0)}_{11}=\tilde{L}^{(0)}_{22}=\lambda+4\pi e^2/q^2.$$

The bare vertices vanish for the remaining matrix elements $\tilde{L}_{ij}$, because they correspond to processes that do not conserve particle number.

The polarization operator $\tilde{\Pi}_{ik}$ in Eq. (1) is represented by the matrix whose elements are all possible loops consisting of normal and anomalous Green's functions for the superconductor (averaged over the spatial impurity distribution).

It is a very tedious process to calculate the vertex part $\tilde{L}$ in this "arrow" representation.[10] On the other hand, a similar polarization operator in a more convenient block representation was used in Ref. 14 to describe collective oscillations in a superconductor. We can recover this representation for $\tilde{\Pi}$ in (1) by transforming by the matrix

$$S=2^{-1/2}\begin{pmatrix} 0 & 1 & -1 & 0\\ -1 & 0 & 0 & 1\\ -i & 0 & 0 & -i\\ 0 & -1 & -1 & 0\end{pmatrix}, \tag{2}$$

so that $\tilde{\Pi}$ decomposes into block matrices. Writing L and $\tilde{\Pi}$ for the vertex part and the polarization matrix in this representation, we obtain

$$L=S\tilde{L}S^{-1}=\begin{bmatrix} L_A & 0\\ 0 & L_B\end{bmatrix}$$

$$=\begin{pmatrix} N_1-A_1 & 0 & 0 & 0\\ 0 & N-A & 0 & 0\\ 0 & 0 & N+A & -i(K_1+L_1)\\ 0 & 0 & i(K+L) & N_1+A_1\end{pmatrix}, \tag{3}$$

$$\Pi_{ik}(\mathbf{q},\Omega_k)=S_{il}\tilde{\Pi}_{lm}S_{mk}^{-1}=\begin{bmatrix}\Pi_A & 0\\ 0 & \Pi_B\end{bmatrix}$$

$$=-\frac{1}{2}\,\mathrm{Sp}\,T\sum_{\omega_n}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\{\hat{\tau}_i\hat{G}^{(0)}(\mathbf{p}+\mathbf{q},\omega_n+\Omega_k)$$

$$\times\hat{\Gamma}_k(\mathbf{q},\omega_n+\Omega_k,\omega_n)\hat{G}^{(0)}(\mathbf{p},\omega_n)\}. \tag{4}$$

Here

$$\hat{G}^{(0)}(\mathbf{p},\omega_n)=\mu_l(\mathbf{p},\omega_n)\,\hat{\tau}_l=\frac{(-i\omega_n\hat{\tau}_0+\Delta\hat{\tau}_1)\,\eta_{\omega_n}-\xi_{\mathbf{p}}\hat{\tau}_3}{(\omega_n^2+\Delta^2)\,\eta^2_{\omega_n}+\xi_{\mathbf{p}}^2} \tag{5}$$

is the one-electron Green's function for a dirty superconductor in the Nambu formalism (here and throughout, a summation is understood over repeated indices). The factor $\eta_{\omega n} = 1+\nu/2(\omega_n^2+\Delta^2)^{1/2}$ $(\nu=\tau^{-1})$ takes into account the averaging over the impurity distribution,[15] Δ is the order parameter, and $\xi_{\mathbf{p}}$ is the electron energy measured relative to the Fermi level. The Green's function $\hat{G}^{(0)}(\mathbf{p},\omega_n)$ is defined on the space S_2 of two-dimensional matrices and is expressible in terms of the Pauli basis matrices

$$\hat{\tau}_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \hat{\tau}_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

$$\hat{\tau}_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \hat{\tau}_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{6}$$

The averaging of the polarization operator over the impurity distribution is carried out by means of the three-tail vertex $\hat{\Gamma}_k$; the latter is defined by the usual ladder equation[15] which in the present case, however, involves the matrix Green's functions (5) for the superconductor.[14] Since one of its ends corresponds to the vertex part L_{ik} of the e-e interaction, which belongs to the space R_4, while the other two ends correspond to matrix Green's functions in the space S_2, the vertex $\hat{\Gamma}_k$ itself is defined on the direct sum $R_4\oplus S_2$. We point out that $\hat{\Gamma}_k$ contains elements that correspond to "cooperons" and "diffusions" in the superconductor.[1] The vertex $\hat{\Gamma}_k$ can in turn be decomposed in terms of the basis matrices (6),

$$\hat{\Gamma}_k=T_{kl}\hat{\tau}_l, \tag{7}$$

where the matrix (T_{kl}) is again of block form:

$$T=\begin{bmatrix} T_A & 0 \\ 0 & T_B \end{bmatrix}.$$

Using the results in Ref. 14, we find the following expressions for the matrices $T_{A;B}$ and $\Pi_{A;B}$:

$$T_{A;B}=\frac{1}{1-2\gamma}\left[(1-\gamma)\hat{\tau}_0-\frac{\Delta(\omega\pm\omega')}{RR'}\gamma\hat{\tau}_{2;1}+\frac{\Delta^2\mp\omega\omega'}{RR'}\gamma\hat{\tau}_3\right],\tag{8}$$

$$\Pi_{A;B}=\frac{2\pi\rho}{v}T\sum_{\omega_n}\frac{\gamma}{1-2\gamma}\left[\frac{\omega\omega'\mp\Delta^2}{RR'}\hat{\tau}_0+\frac{i\Delta(\omega\pm\omega')}{RR'}\hat{\tau}_2\mp\hat{\tau}_3\right],\tag{9}$$

where $\rho=mp_F/2\pi^2$ is the denisty of states on the Fermi surface,

$$\gamma=\frac{v}{2}\left\langle\frac{R+R'+v}{(R+R'+v)^2+(v\mathbf{q})^2}\right\rangle,\tag{10}$$

$$R=(\omega_n^2+\Delta^2)^{1/2},\quad R'=(\omega'^2+\Delta^2)^{1/2},\quad \omega'=\omega_n-\Omega_k,$$

and the brackets in (10) denote an angular average along the Fermi surface: $\langle\ldots\rangle=\int(d\Omega_p/4\pi)(\ldots)$.

In general, one should sum expression (4) over the frequencies ω_n first before doing the momentum integration.[15] Expression (9), in which the integration is done prior to summing over the frequencies, is therefore not completely correct. However, this error will disappear in what follows, because when evaluating the frequency sums we will express $1/RR'$ and $1/RR'(R+R')$ as integrals[14]

$$\frac{1}{RR'}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\xi^2\,d\xi}{(\xi^2+R^2)(\xi^2+R'^2)}$$

$$\frac{1}{RR'(R+R')}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\xi}{(\xi^2+R^2)(\xi^2+R'^2)}\tag{11}$$

and add over the frequencies before performing the integration over ξ. This leads to an explicit expression for the polarization operator $\Pi(\mathbf{q},\Omega_k)$ which is valid near T_c $(T_c-T\ll T_c)$:

$$\Pi_A(\mathbf{q},\Omega_k)=-\rho\begin{pmatrix}-\dfrac{Dq^2}{|\Omega_k|+Dq^2} & 0\\ 0 & \left[-\ln\dfrac{\omega_D}{2\pi T}+\dfrac{7\zeta(3)\Delta^2}{8\pi^2T^2}\right]+\dfrac{7\zeta(3)\Delta^2}{4\pi^2T^2}\delta_{k,0}+\chi_0\end{pmatrix},\tag{12}$$

$$\Pi_B(\mathbf{q},\Omega_k)=-\rho\begin{pmatrix}\left[-\ln\dfrac{\omega_D}{2\pi T}+\dfrac{7\zeta(3)\Delta^2}{8\pi^2T^2}\right]+\chi_0 & \dfrac{2\Delta(Dq^2\chi_1-|\Omega_k|\chi_0)}{\Omega_k^2-(Dq^2)^2}\operatorname{sgn}\Omega_k\\ -\dfrac{2\Delta(Dq^2\chi_1-|\Omega_k|\chi_0)}{\Omega_k^2-(Dq^2)^2}\operatorname{sgn}\Omega_k & -\dfrac{Dq^2}{|\Omega_k|+Dq^2}\end{pmatrix},\tag{13}$$

where

$$\chi_0=\psi(1/2+(|\Omega_k|+Dq^2)/4\pi T)-\psi(1/2),\tag{14}$$

$$\chi_1=\psi(1/2+|\Omega_k|/2\pi T)-\psi(1/2),\tag{15}$$

$\psi(x)$ is the logarithm of the derivative of the gamma function, and $\zeta(x)$ is the Riemann zeta-function.

Using expressions (12), (13), and Eq. (1), we obtain

$$L_A^{-1}(\mathbf{q},\Omega_k)=-\rho\begin{pmatrix}-\dfrac{1}{|\lambda|\rho}-\dfrac{Dq^2}{|\Omega_k|+Dq^2} & 0\\ 0 & \left[\ln\dfrac{T}{T_c}+\dfrac{7\zeta(3)\Delta^2}{8\pi^2T^2}\right]+\dfrac{7\zeta(3)\Delta^2}{4\pi^2T^2}\delta_{k,0}+\chi_0\end{pmatrix},\tag{16}$$

$$L_B^{-1}(\mathbf{q},\Omega_k)=-\rho\begin{pmatrix}\left[\ln\dfrac{T}{T_c}+\dfrac{7\zeta(3)\Delta^2}{8\pi^2T^2}\right]+\chi_0 & \dfrac{2\Delta(Dq^2\chi_1-|\Omega_k|\chi_0)}{\Omega_k^2-(Dq^2)^2}\operatorname{sgn}\Omega_k\\ -\dfrac{2\Delta(Dq^2\chi_1-|\Omega_k|\chi_0)}{\Omega_k^2-(Dq^2)^2}\operatorname{sgn}\Omega_k & -\dfrac{q^2}{8\pi e^2\rho}-\dfrac{Dq^2}{|\Omega_k|+Dq^2}\end{pmatrix}.\tag{17}$$

Formula (16) shows that the elements $L_{00}=N_1-A_1$ is nonsingular, and for $|\lambda|\rho\ll 1$ it gives a negligible correction to the Green's function. For $T>T_c$, the off-diagonal matrix elements in L vanish, L_{11} becomes equal to L_{22} and corresponds to an ordinary fluctuation propagator,[8] and L_{33} reduces to the expression in Ref. 1 for dynamic Coulomb screening in a "dirty" metal.

We are interested in the case $T<T_c$, for which the expressions in the square brackets in (16), (17) vanish. When we invert the matrix $L^{-1}(\mathbf{q},\Omega_k)$ in the block L_A remains diagonal; the elements of greatest interest to us is

$$L_{11}(\mathbf{q},\Omega_k)=-\frac{1}{\rho}\,\frac{1}{2\ln(T_c/T)\delta_{k,0}+\chi_0}.\tag{18}$$

When $k=0$, the block L_B is also diagonal:

$$L_B(\mathbf{q},0)=\begin{pmatrix} -\frac{1}{\rho}\frac{8T}{\pi Dq^2} & 0 \\ 0 & \frac{8\pi e^2}{q^2+\varkappa^2} \end{pmatrix}. \qquad (19)$$

The singularity in (19) at low momenta corresponds to the generation of a Goldstone boson (L_{22}) in the superconductor for $T<T_c$. The element L_{33} determines the statically screened Coulomb interaction, which for $T\approx T_c$ is the same as for a normal metal ($\varkappa=(8\pi e^2\rho)^{1/2}$ is the reciprocal of the Debye screening radius).

If $\Omega_k\neq 0$ then

$$L_B(\mathbf{q},\Omega_k\neq 0)=\begin{pmatrix} -\frac{1}{\rho\chi_0} & \frac{2\Delta\,\mathrm{sign}\,\Omega_k(Dq^2\chi_1-|\Omega_k|\chi_0)}{\chi_0(\Omega_k^2-(Dq^2)^2)}L_{33} \\ -\frac{2\Delta\,\mathrm{sign}\,\Omega_k(Dq^2\chi_1-|\Omega_k|\chi_0)}{\chi_0[\Omega_k^2-(Dq^2)^2]}L_{33} & L_{33} \end{pmatrix}. \qquad (20)$$

$$L_{33}=\left[\frac{q^2}{8\pi e^2}+\frac{\rho Dq^2}{|\Omega_k|+Dq^2}\right]^{-1}\approx\frac{|\Omega_k|+Dq^2}{\rho Dq^2}.$$

This shows that the coupling between the Cooper and diffusion channels shows up as soon as $\Omega_k\neq 0$ (i.e., the off-diagonal matrix elements L_{23} and L_{32} are turned on), while L_{33} describes a dynamically screened Coulomb interaction.[1]

We note that Eqs. (12)–(20) above were derived only for the Matsubara frequencies $\Omega_k=2\pi Tk$ ($k=0,\pm 1,\pm 2,...$) and differ markedly for zero and nonzero frequencies, so that (12) cannot be analytically continued in an unambiguous way. However, Eqs. (18)–(20) at the Matsubara frequencies suffice to calculate the corrections to the thermodynamic quantities and, in particular, the critical current in the Josephson junction.

2. RENORMALIZATION OF THE ORDER PARAMETER

We now discuss how the e-e interaction alters the temperature dependence of the order parameter. In the case $H=0$ which we consider, the self-consistency equation can be written in the form

$$\Delta=\frac{1}{2}|\lambda|\rho T\sum_{\varepsilon_n}\mathrm{Sp}\{\hat g(\varepsilon_n)\hat\tau_1\}, \qquad (21)$$

where

$$\hat g(\varepsilon)=\int d\xi_{\mathbf{p}}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\hat G(\mathbf{p},\varepsilon_n)$$

is the energy-integrated one-electron Green's function for the superconductor. Figure 2a,b shows the correction $\delta\hat g(\varepsilon_n)$ to

$$\hat g^{(0)}(\varepsilon_n)=\pi(-i\varepsilon_n\hat\tau_0+\Delta\hat\tau_1)(\varepsilon_n+\Delta^2)^{-1/2} \qquad (22)$$

to first order in the e-e interaction. The analytic expression corresponding to the diagram in Fig. 2a is

$$\delta\hat g^{(1)}(\varepsilon_n)$$
$$=-\frac{1}{2}\int d\xi_{\mathbf{p}}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}T\sum_{\Omega_k}\int(d\mathbf{q})\hat G^{(0)}(\mathbf{p},\varepsilon_n)\hat\Gamma_i(\mathbf{q},\varepsilon_n,\varepsilon_n-\Omega_k)$$
$$\times\hat G^{(0)}(\mathbf{p}-\mathbf{q},\varepsilon_n-\Omega_k)\hat\Gamma_j(\mathbf{q},\varepsilon_n-\Omega_k,\varepsilon_n)\hat G^{(0)}(\mathbf{p},\varepsilon_n)L_{ij}(\mathbf{q},\Omega_k). \qquad (23)$$

If we use the expansions (5) and (7) for $\hat G^{(0)}$ and $\hat\Gamma_i$ in terms of the Pauli matrices, we obtain

$$\delta\hat g^{(1)}(\varepsilon_n)=-\frac{1}{2}T\sum_{\Omega_k}\int(d\mathbf{q})\Lambda^{st}(q,\varepsilon_n,\varepsilon_n-\Omega_k)\times M^{lmn}(\mathbf{q},\varepsilon_n,\varepsilon_n-\Omega_k)\hat E^{st}_{lmn},$$

where

$$\Lambda^{st}(\mathbf{q},\varepsilon_n,\varepsilon_n-\Omega_k)=T_{is}(\mathbf{q},\varepsilon_n,\varepsilon_n-\Omega_k)L_{ij}(\mathbf{q},\Omega_k)T_{jt}(\mathbf{q},\varepsilon_n-\Omega_k,\varepsilon_n), \qquad (25)$$

$$M^{lmn}(\mathbf{q},\varepsilon_n,\varepsilon_n-\Omega_k)=\int d\xi_{\mathbf{p}}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\mu^l(\mathbf{p},\varepsilon_n)\mu^m(\mathbf{p}-\mathbf{q},\varepsilon_n-\Omega_k)\mu^n(\mathbf{p},\varepsilon_n), \qquad (26)$$

$$\hat E^{st}_{lmn}=\hat\tau_l\hat\tau_s\hat\tau_m\hat\tau_t\hat\tau_n. \qquad (27)$$

The correction $d\hat g^{(2)}$ shown in Fig. 2b contains a four-tail impurity vertex D_{ij}, which like $\hat\Gamma_i$ is expressible in terms of the matrix T_{ij}:

$$D_{ij}(\mathbf{q},\omega_1,\omega_2)=\frac{\nu}{2\pi\rho}H^{ik}T_{kj}(\mathbf{q},\omega_1,\omega_2), \qquad (28)$$

where

$$H^{ij}=\begin{bmatrix}\tau_3 & 0\\ 0 & \hat\tau_3\end{bmatrix}.$$

We can now express $\delta\hat g^{(2)}$ in terms of the correction $\delta\hat g^{(1)}$ found above:

$$\delta\hat g^{(2)}(\varepsilon_n)=K^{(i)}(\varepsilon_n)D_{ij}(0,\varepsilon_n,\varepsilon_n)\times\int d\xi_{\mathbf{p}}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\hat G^{(0)}(\mathbf{p},\varepsilon_n)\hat\tau^j\hat G^{(0)}(\mathbf{p},\varepsilon_n), \qquad (29)$$

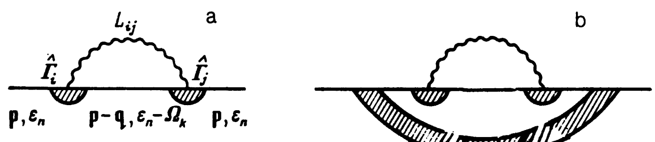

FIG. 2. Corrections to the one-electron Green's function for a dirty superconductor to first order in the e-e interaction. The heavy line shows the Green's function $\hat G$ for the superconductor; the wavy line corresponds to the vertex part L_{ij} of the e-e interaction; the vertices with three and four tails (hatched regions) allow for averaging over the position of the impurities.

where we write

$$K^{(i)}(\varepsilon_n)=\frac{1}{2}\operatorname{Sp}\{\delta\hat{g}^{(1)}(\varepsilon_n)\hat{\tau}^i\}. \tag{30}$$

Inserting (23) and (30), (31) into Eq. (21) and calculating the trace, we find

$$\Delta=|\lambda|\rho T\sum_{\varepsilon_n}\left\{\frac{\pi\Delta}{(\varepsilon_n^2+\Delta^2)^{1/2}}+K^{(1)}(\varepsilon_n)+\frac{\nu}{2|\varepsilon_n|}\left[K^{(1)}(\varepsilon_n)-\frac{i\Delta}{\varepsilon_n}K^{(0)}(\varepsilon_n)\right]\right\}. \tag{31}$$

Formulas (24)–(27) yield the following expressions for $K^{(0)}$ and $K^{(1)}$:

$$K^{(0)}(\varepsilon_n)=-\pi i\nu^{-2}T\sum_{\Omega_k}\int(d\mathbf{q})\times\{T_{11}{}^2L_{11}+T_{22}{}^2L_{22}-T_{33}{}^2L_{33}\}\operatorname{sgn}\varepsilon_n, \tag{32}$$

$$\begin{aligned}K^{(1)}(\varepsilon_n)=\pi\nu^{-2}T\sum_{\Omega_k}\int(d\mathbf{q})\Big\{&L_{11}T_{11}\Big[2iT_{01}\operatorname{sgn}\varepsilon_{n-k}+\Delta T_{11}\Big(\frac{2}{|\varepsilon_n|}+\frac{1}{|\varepsilon_{n-k}|}\Big)\Big]+L_{22}T_{22}\Big[-2T_{23}\operatorname{sgn}\varepsilon_{n-k}\\&+\Delta T_{22}\Big(\frac{2}{|\varepsilon_n|}-\frac{1}{|\varepsilon_{n-k}|}\Big)\Big]+L_{33}T_{33}\Big[2T_{23}\operatorname{sgn}\varepsilon_{n-k}\\&-\Delta T_{33}\Big(\frac{2}{|\varepsilon_n|}+\frac{1}{|\varepsilon_{n-k}|}\Big)\Big]+2T_{22}T_{33}L_{23}\operatorname{sgn}\varepsilon_{n-k}\Big\}.\end{aligned} \tag{33}$$

If we substitute the explicit expressions (8) for the matrix elements T_{ij} into (32) and (33) and sum over the fermion frequency ε_n in (31), we get

$$\begin{aligned}1=|\lambda|\Big\{&\frac{7\zeta(3)}{16\pi^2T}\int(d\mathbf{q})\,[3L_{11}(\mathbf{q},0)+L_{22}(\mathbf{q},0)]\\&-T\sum_{\Omega_k}\int(d\mathbf{q})\Big[L_{33}\Big(\frac{Dq^2}{|\Omega_k|}\frac{(Dq^2)^2-3\Omega_k^2}{[\Omega_k^2-(Dq^2)^2]^2}\chi_1\\&+\frac{2\Omega_k^2}{[\Omega_k^2-(Dq^2)^2]^2}\chi_0+\frac{Dq^2}{4\pi T}\frac{\psi'\Big(\frac{1}{2}+\frac{|\Omega_k|}{2\pi T}\Big)}{[\Omega_k^2-(Dq^2)^2]^2}\Big)\\&+\frac{|\Omega_k|\chi_0-Dq^2\chi_1}{\Omega_k^2-(Dq^2)^2}\operatorname{sgn}\Omega_k\frac{L_{23}}{\Delta}\Big]\Big\},\end{aligned} \tag{34}$$

where the frequency $\Omega_k=0$ gives the dominant contribution to the terms containing L_{22} and L_{11}.

We point out that the propagator matrix element L_{22} has a singularity $\sim q^{-2}$ at small momenta. For an infinite homogeneous film or wire specimen, this term does not produce any divergence because the system possesses a long-range order. However, for a bounded specimen the integration in (34) must be replaced by a summation over the eigenvalues of the momentum:

$$\int(d\mathbf{q})(\ldots)\to\frac{1}{L_xL_yd}\sum_{q_xq_yq_z}(\ldots), \tag{35}$$

where $L_x\geqslant L_y\geqslant d$ are the linear dimensions of the specimen.

The momentum quantization conditions depend strongly on the boundary conditions. If the system is not subject to any external effects, the only condition is that the flux across the superconductor/vacuum interface must vanish; this leads to the quantization condition $q_n=\pi n/L$ ($n=0,\pm1,\pm2,\ldots$). The zeroth harmonic of the phase fluctuations in the order parameter is then found to give a divergent contribution—by Hohenberg's theorem,[16] the average value of the order parameter vanishes.

However, the boundary conditions are different if we want to perform a measurement on the specimen (e.g., connect it in an electric circuit). The zeroth harmonic now disappears and the divergence is replaced by a cutoff at $q_{\min}\sim1/L$.

Expression (34) can thus be transformed in various ways, depending on the relative linear dimensions. The effects of the e-e interaction are most interesting for one- or two-dimensional specimens such as thin films and wires.

We consider a "dirty" superconducting film of thickness $d\ll L_y,L_x$ whose edges (perpendicular to the x axis) are connected to two massive electrodes in which fluctuations can be neglected. The quantization condition then read[1]

$$q_x=\frac{\pi n}{L_x},\quad q_y=\frac{\pi k}{L_y},\quad q_z=\frac{\pi m}{d} \tag{36}$$

(where n, k, and m are integers and $n\neq0$), and the divergence in (34) at long wavelengths is removed:

$$\int(d\mathbf{q})\frac{1}{q^2}\to\frac{1}{8L_xL_yd}\sum_{q_xq_yq_z}\frac{1}{q^2}=\frac{1}{8L_yd}\sum_{q_yq_z}\left\{\frac{\operatorname{cth}[L_x(q_y^2+q_z^2)^{1/2}]}{(q_y^2+q_z^2)^{1/2}}-\frac{1}{L_x(q_y^2+q_z^2)}\right\}. \tag{37}$$

Since $d\ll L_y,L_x$, the dominant contribution is from the zeroth harmonic $q_z=0$. Summing over q_y in (37), we obtain

$$\frac{1}{L_xL_yd}\sum_{q_xq_yq_z}\frac{1}{q^2}=\frac{2}{\pi d}\left\{\ln\frac{L_y}{\max\{d,L_T\}}+\frac{\pi L_x}{6L_y}\right\}, \tag{38}$$

where $L_T=(D/T)^{1/2}\sim\xi_{\rm bd}$ is the diffusion length.

Applying the above procedure to expression (34) for a film ($d\ll L_T;L_T\ll L_y\sim L_x\sim L$) and performing the remaining summation over Ω_k, we get the final result

$$\Delta^2=\frac{8\pi^2T^2}{7\zeta(3)}\left\{\frac{T_{c0}-T}{T_{c0}}-\frac{63\zeta(3)}{4\pi^2p_F{}^2ld}\ln\frac{T_{c0}}{2(T_{c0}-T)}-\frac{21\zeta(3)}{2\pi^2p_F{}^2ld}\ln\frac{L}{L_T}-\frac{2}{p_F{}^2ld}\ln^3\frac{L_T}{d}\right\}. \tag{39}$$

The first correction to Δ^2 is due to fluctuations in the modulus of the order parameter and is primarily responsible for the temperature dependence; the second correction comes from fluctuations in the phase of the order of parameter and depends on the longitudinal dimensions of the specimen. The last term receives contributions both from fluctuations in the scalar potential and from the interference term between the phase and the scalar potential. We observe that the last term may be neglected except for thin films with $d\ll L_T$. Indeed, for thicker films ($d\gtrsim L_T$) the last term is small and lies within the critical region.

3. INFLUENCE OF *e-e* INTERACTION ON THE CRITICAL CURRENT IN A JOSEPHSON JUNCTION

We now calculate the corrections to the critical current due to the *e-e* interaction. In the diagram formalism the total current through the junction is given by

$$I(V)=-2e\ \mathrm{Im}\ \{(K_0^R+K_3^R)+\exp\{i(\varphi+2eVt)\}(K_3^R-K_0^R)\}. \tag{40}$$

Here K_i $(i=0,3)$ is the correlation function for the Green's functions of the electrodes forming the junction (cf. Fig. 3) and is continued analytically into the upper halfplane of the complex frequency $(i\omega_\nu\to\omega=eV$, where V is the junction voltage):

$$K_i(\omega_\nu)=\frac{1}{2}\ \mathrm{Sp}\ T\sum_{\varepsilon_n}\sum_{\mathbf{pk}}|T_{\mathbf{pk}}|^2 \times\{\hat{\tau}_i\hat{G}_{\mathrm{I}}(\mathbf{p},\varepsilon_n+\omega_\nu)\hat{\tau}_i\hat{G}_{\mathrm{II}}(\mathbf{k},\varepsilon_n)\}$$
$$=\frac{1}{8\pi e^2R_n}\ \mathrm{Sp}\ T\sum_{\varepsilon_n}\hat{\tau}_i\hat{g}_{\mathrm{I}}(\varepsilon_n+\omega_\nu)\hat{\tau}_i\hat{g}(\varepsilon_n) \tag{41}$$

(there is no summation over the index i in this formula). $T_{\mathbf{pk}}$ is the matrix element of the tunnel Hamiltonian which takes an electron from momentum state $\mathbf{p}$ in the first electrode to momentum state $\mathbf{k}$ in the second electrode and gives it the boson frequency ω_ν, with respect to which the analytic continuation is performed.[3] R_N is the normal resistance of the junction.

This formula can be readily verified by using the Lehmann representation for the Green's functions as in Refs. 3 and 17; expression (40) then reduces to the Ambegaokar-Baratoff formula.[18] We emphasize that (40) contains all the currents passing through the Josephson junction—the tunnel and the Josephson currents, as well as the interference current for the quasi-particles and Cooper pairs (the $\cos\varphi$ contribution).

For $T>T_c$, the Josephson component of the current vanishes identically, and the first term in (40) describing the tunnel current agrees with the result found previously in Ref. 3. For $T<T_c$ and $V=0$, expression (40) reduces to the expression

$$I_c=4eT\sum_{\varepsilon_n}\sum_{\mathbf{p},\mathbf{k}}|T_{\mathbf{pk}}|^2F_{\mathrm{I}}(\mathbf{p},\varepsilon_n)F_{\mathrm{II}}^+(\mathbf{k},\varepsilon_n) \tag{42}$$

for the critical current derived in Ref. 19.

We next consider a Josephson junction with "dirty" superconducting film electrodes at subcritical temperatures $T<T_c$ (Fig. 4). In this case, in addition to contributing a correction to the Green's functions in (41), the *e-e* interaction also leads to a renormalization of the order parameter

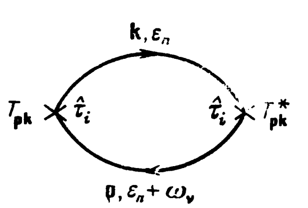

FIG. 3. Diagram representation for the correlation function $K_i(\omega_\nu)$.

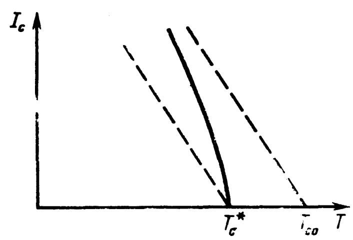

FIG. 4. Temperature dependence of the critical current in a symmetric Josephson junction corrected for the *e-e* interaction. The dashed lines show the Ambegaokar-Baratoff dependences calculated for the critical temperatures T_{c0} and T_c^*, respectively.

appearing in the lowest-order expression (22) for the Green's functions. The former correction alters the critical current by an amount which can be calculated using (40), (24), and (29):

$$\delta I_c^{(1)}=\frac{1}{4\pi eR_N}\ \mathrm{Re}\ T\sum_{\varepsilon_n}\mathrm{Sp}\{\delta\hat{g}(\varepsilon_n)\hat{g}^{(0)}(\varepsilon_n) -\hat{\tau}_3\delta\hat{g}(\varepsilon_n)\hat{\tau}_3\hat{g}^{(0)}(\varepsilon_n)\}. \tag{43}$$

We note that $\mathrm{Im}(K_0-K_3)$ vanishes to all orders in the *e-e* interaction; this corresponds to the fact that when $V=0$, the *e-e* interaction alone is incapable of producing current corrections proportional to $\cos\varphi$.

Expression (43) can be evaluated in the same way as (31); we find that the correction has a logarithmic singularity for T close to the transition temperature. The second correction to the critical current (associated with the renormalization of the order parameter) is readily found directly by substituting Δ from (39) into the zeroth-order F-functions in (42); the result is

$$\frac{\delta I_c^{(2)}}{I_c}=\frac{\delta\Delta^2}{\Delta_0^2}=-\frac{21\zeta(3)}{4\pi^2p_F^2ld}\left(\frac{T_{c0}}{T_{c0}-T}\right) \times\left[3\ln\frac{T_{c0}}{2(T_{c0}-T)}+2\ln\frac{L}{d}\right]. \tag{44}$$

This correction is the dominant one near T_c, because it contains the additional factor $T_{c0}/(T_{c0}-T)$ which is not present in $\delta I_c^{(1)}$.

Figure 4 shows how the critical current depends on temperature after correction for the *e-e* interaction. According to Ref. 18, the critical current I_c^0 for a symmetric Josephson junction must vanish at T_{c0}, and the dependence $I_c^0(T)$ is linear for $T\approx T_{c0}$. However, the temperature T_{c0} here is purely formal, since only T_c^* is experimentally accessible. We can find T_c^* by setting $\Delta(T_c^*)=0$ in (39). Because of the *e-e* interaction, the curve $I_c(T)$ lies somewhat higher than the line $I_c^0(T)$ drawn through the point T_c^*.

CONCLUSIONS

The principal results of this paper may be summarized as follows.

We constructed the vertex part of the electron-electron interaction in a "dirty" superconductor for $T<T_c$ which treats all interaction processes, including ones that change

the number of particles not in the condensate. It was shown that the e-e interaction in a superconductor causes the modulus and phase of the order parameter to fluctuate, screens the Coulomb interaction dynamically when condensed particles are present, and gives rise to an interference contribution to the critical current due to interference between the fluctuations in the order parameter and scalar potential (the interference couples the Cooper and the diffusion channels in the superconductor).

We used the vertex part to calculate the one-electron Green's function for a superconductor to first order in the e-e interaction. When this Green's function is inserted into the self-consistency equation, one obtains a renormalization of the average order parameter for the "dirty" superconductor. If the superconductor is bounded in at least one dimension, large-scale fluctuations in the phase of the order parameter cause the resulting expressions to diverge formally at low momenta. According to Hohenberg's well-known result,[16] this divergence is due to a breakdown of the long-range order in unbounded one- and two-dimensional systems. For systems of finite size, this divergence is eliminated through the imposition of boundary conditions. The average value of the order parameter in this case depends on the longitudinal dimensions of the system. However, according to Ref. 20 a supercurrent can flow in the system even when the average value of the order parameter is now well-defined—it suffices merely for the correlation function of the order parameter to behave as a power of r in the long-range limit $r\to\infty$.

On the other hand, in order for a Josephson current to flow through the junction a single phase must be present along the entire barrier. It is therefore clear that phase fluctuations will decrease the critical current density, so that the corrections due to the e-e interaction become dependent on the size and shape of the junction.

If the dimensions satisfy $\ln(L_y/d)\gg L_x/L_y$ then the phase fluctuations are two-dimensional: $\delta I_c \propto -(1/p_F^2 ld)\cdot\ln(L_y/d)$, and the critical current drops due to phase fluctuations along the barrier itself. Of course, δI_c depends on L_y only when $L_y \lesssim \lambda_J$ (where λ_J is the Josephson penetration depth), because for large (wide) junctions the current distribution becomes nonuniform. In addition, correlations between the phase fluctuations in the two electrodes across the barrier may be significant for large junctions, and this is neglected in the above theory. However, for very long thin junctions, $L_x \gtrsim L_y \ln(L_y/d)$, the asymptotic behavior of (39) changes and the fluctuations in the phase of the order parameter are one-dimensional. The phase fluctuations along the electrodes themselves decrease the critical current in this case:

$$\delta I_c \propto -\frac{1}{p_F^2 ld}\left(\frac{L_x}{d}\right) \sim -\frac{L_x}{p_F^2 Sl}.$$

One must of course remember that all of these results were derived to first order in the e-e interaction, and for the one-dimensional case they are valid only when $L_x \ll p_F^2 Sl$. However, even for slender wires with diameter $d\sim 1\,\mu$m and mean free path $l\sim 10^{-6}$ cm the corresponding lengths are of the order of 1 m, while for two-dimensional thin films they are exponentially large.

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