Properties of an anisotropic disordered system of superconductors in a normal metal

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(Submitted 10 April 1986)
Zh. Eksp. Teor. Fiz. **91**, 1776–1788 (November 1986)

We investigate the properties of a system of superconducting filaments in a normal metal matrix. For low concentrations of superconducting material, the critical parameters of the system and their dependence on magnetic field are found from percolation theory. We show that when the fraction of superconducting material is appreciable, a high current density is achievable in strong fields due to surface pinning at the interface between normal and superconducting phases. We find conditions under which the system is capable of carrying a supercurrent with density exceeding the "pair-breaking" current density.

1. INTRODUCTION

Recently there has been much interest in the study of disordered systems in which superconducting inclusions are distributed randomly in a matrix made of dielectric or normal metal. When the concentration of the superconducting phase is low, the average properties of the system, e.g., its transition temperature, critical current and critical field, can be found by percolation theory.[1] The percolation threshold for which an infinite superconducting-phase cluster can be formed corresponds to a volume fraction of superconductor $f_{s0} \approx 0.15$ (Ref. 2). Above the percolation threshold, the system becomes a hard superconductor, whose properties are determined by pinning at inhomogeneities.

In the present paper we investigate an anisotropic disordered system in which superconducting filaments, highly elongated in one direction, are located in a normal metal. Such systems are of practical interest since they can carry high current densities in strong fields without appreciable dissipation of energy.[3–6] Their electrodynamic properties are regulated by the volume fraction of superconductor f_s, the concentration N of superconducting inclusions and the degree of anisotropy, which is conveniently characterized by the ratio of the length of a filament to its radius $R = 1/b$. For given values of f_s and N, the quantity R can be varied by metallurgical drawing of the sample.

In the percolation regime, when the spacing d between filaments is large compared to the coherence length ξ_N in the normal metal, the critical current is exponentially small (the exponent is $\sim d/\xi_N$). Therefore, elongation of the superconducting inclusions causes an increase in the surface over which the supercurrent can flow in and along the normal metal as a result of the proximity effect between relatively widely-spaced inclusions. As a result, the degree of coupling between the inclusions increases, which gives rise to the appearance of a large pre-exponential factor in the expression for the critical current density. A magnetic field, whose influence in the case of a "dirty" superconductor can be investigated using the Usadell equations,[7] suppresses the proximity effect. The coupling region is compressed transverse to the field direction, and the critical current density decreases exponentially as the magnitude of the field increases.

Above the percolation threshold the critical current density is determined by the disruption of superconductivity in the inclusions themselves, and depends on their transverse dimensions. If these dimensions are large compared to the penetration depth λ of the magnetic field, the critical current is determined by volume pinning within the inclusions. However, a situation is possible in which the critical current is determined by strong pinning at the boundaries between the superconducting and normal phases, and can be close to the "pair-breaking" current of the superconductor for high values of the magnetic field, i.e., on the order of the upper critical field. For this to occur, the transverse dimensions of the inclusions must be small compared to the magnitude of λ, in contrast to the case of volume pinning, while the inclusions themselves should be isolated from one another (the supercurrent in the normal matrix must be small compared to the pair-breaking current). In addition (naturally) the filament diameters must not be smaller than the coherence length ξ_s in the superconductor; this ensures that the proximity effect will not destroy the superconductivity (usually $\xi_s \ll \xi_N$).

Under certain special conditions, the system investigated here can carry a supercurrent with a density even higher than the superconductor's pair-breaking current. This is because the supercurrent in the "dirty superconductor" limit is proportional to the metallic conductivity in the normal state, and for the normal matrix σ_N is usually much larger than the value σ_s in the superconductor. Therefore, when the spacing between filaments is small compared to ξ_N and there is some superconductivity present in the normal metal due to the proximity effect, the supercurrent flowing in the normal metal can be larger than the current in the superconducting inclusions. However, in this case the suppression of the critical current by a magnetic field is stronger than for surface pinning.

2. CRITICAL CURRENT DENSITY FOR LOW CONCENTRATIONS OF THE SUPERCONDUCTING PHASE

If the volume fraction of superconductor satisfies $f_s \ll 1$, the system consists of a normal matrix with widely-spaced superconducting inclusions, which are assumed to be highly

 0038-5646/86/111051-08$04.00

elongated along one direction (the z-axis), $R \gg 1$. Let the spacing d between filaments be large compared to the length ξ_N, so that the supercurrent between them is exponentially small. Under these circumstances, the critical current of the whole system can be studied by percolation theory.

Each inclusion can be surrounded by a surface consisting of points a distance $d_1/2$ from the inclusion. As d_1 increases, the surfaces begin to overlap; let us refer to these inclusions as "conditionally-coupled." For some value d_c, an infinite cluster of conditionally-coupled inclusions forms. This quantity (in units of ξ_N) also determines the exponent in the expression for the critical current density of the system. It is clear that the quantity d_c is on the order of the average distance d between filaments. Its exact value is found from the condition that the volume external to the surfaces described above is equal to the quantity $B_c/8N$, where the parameter B_c depends weakly on the shape of the coupling region; for spheres, it is close to 2.7 (Ref. 2). The case $d_c \gg 1$ differs little from a system of small superconducting spheres in a normal matrix discussed in Ref. 1. For $d_c \ll 1$ the volume of the coupling region equals $\pi d_c^2 l/4$, and in this case, which corresponds to strong anisotropy ($R \gg f_s^{-1/2}$), it turns out that the critical spacing d_c decreases rapidly as the filament length increases:

$$d_c=(B_c/2\pi N l)^{1/2}. \quad (1)$$

For a cylinder-shaped inclusion, this expression can be written in the form

$$d_c=\pi^{-1/3}(B_c/2)^{1/3}N^{-1/3}R^{-1/3}f_s^{-1/6}, \quad R\gg f_s^{-1/2}. \quad (2)$$

Thus, marked elongation of a sample accompanied by a decrease in the spacing between inclusions and a strengthening of the degree of coupling between them leads to exponential growth of the critical current and efficient stimulation of superconductivity in the system.

The number of transverse cross-sections per unit area through which there flows a current of the same order of magnitude as that which flows through a conditionally-coupled cluster is of order $d_c^{-2}(\xi_N/d_c)^{2\nu}$, where the critical exponent for the correlation radius of an infinite cluster is $\nu = 0.9$ (Ref. 2). Therefore, we obtain the following expressions for the critical current densities

$$j_{cz}\approx j_c(d_c)\frac{bl}{d_c^2}\left(\frac{\xi_N}{d_c}\right)^{2\nu}, \quad j_{cx}=j_{cy}=j_{cz}\frac{d_c}{l}, \quad (3)$$

where $j_c(d)$ is the critical current density between two superconducting cylindrical electrodes located in a normal metal at a distance d from one another.

The critical current of an *SNS* contact in the dirty-superconductor limit, when the electron mean free path is long compared to the characteristic parameters of the problem, can be found with the help of the Usadell equations.[7] To calculate this current we use the expression for the supercurrent density.

$$\mathbf{j}_s=\sigma(2\pi T/e)\sum_{n\geq 0}\mathrm{Im}(F_n^*\partial F_n), \quad \partial=\nabla-2ie\mathbf{A}, \quad (4)$$

where σ is the conductivity of the metal in its normal state, and $\mathbf{A}$ is the vector potential of the magnetic field. The Gor'kov functions integrated over the energy variable are found from the equations

$$D\partial[G_n\partial F_n-F_n\nabla G_n]=2\omega_nF_n-2\Delta G_n,$$
$$G_n=(1-|F_n|^2)^{1/2}, \quad \omega_n=\pi T(2n+1), \quad (5)$$

where D is the diffusion coefficient for electrons and Δ is the superconducting order parameter, which is nonzero only in the superconductor (we treat the matrix as normal for all temperatures). Boundary conditions for Eq. (5) consist of continuity of the functions F_n together with their normal derivatives multiplied by the corresponding conductivities (σ_s is the superconductor, σ_N in the normal metal).[8-10]

In the absence of an external field, the Gor'kov functions, as follows from equations (5), decrease with distance away from the superconducting filaments according to

$$F_n=C_nK_0[(2n+1)^{1/2}\rho/\xi_N], \quad \xi_N=(D_N/2\pi T)^{1/2}, \quad (6)$$

where ρ is the distance from the center of the superconducting cylinder, and $K_0(x)$ is the modified Bessel function of the second kind. The coefficient C_n is found from the boundary conditions by matching with the Gor'kov functions in the superconductor, which vary over a distance $\sim\xi_s = (D_s/2\pi T_c)^{1/2}$ and have a modulus of order unity in the interior of the electrodes (their transverse dimensions b are assumed to be much larger than ξ_s). The critical current density of the contact is estimated from the squared modulus of the Gor'kov function F_0 (which decreases most slowly with increasing ρ), evaluated midway between the electrodes and divided by the characteristic length over which it varies:

$$j_c(d)\approx j_p\gamma\frac{\xi_N}{d}\exp\left(-\frac{d}{\xi_N}\right), \quad \gamma=\frac{\sigma_N\xi_S}{\sigma_S\xi_N}, \quad (7)$$

where

$$j_p\approx T_c\sigma_S/e\xi_S \quad (8)$$

is the pair-breaking current density of the superconductor. Note that the pre-exponential factor in the expression for the critical current of the contact depends on the relationships between the parameters γ, b/ξ_N and 1; for simplicity Eq. (7) is written for the case $\gamma \max\{\xi_N/b, 1\} \ll 1$.

Thus, the critical current density of the system is proportional to $\exp(-d_c/\xi_N)$, and is determined by Eqs. (2), (3) and 7). The expression for the longitudinal critical current density j_{cz} clearly contains the large pre-exponential factor $bl/d_c^2 \sim R f_s$. Its behavior is explained by the fact that as the sample is elongated (R increases for a given f_s and N), not only does the spacing d between filaments decrease, but also the surface of the filaments, through which the current between inclusions flows, increases $\sim bl$. Therefore, for sufficiently long filaments the critical current density in each filament must become of order j_p (and thus the longitudinal current density averaged over a cross-section $\sim j_p f_s$) even for spacings between filaments which are somewhat larger than ξ_N (by virtue of the large logarithm). The corresponding expression for the degree of anisotropy takes the form

$$R_0=\pi^{-1}(B_c/2)^{3/2}f_s^{-1/2}N^{-1}\xi_N^{-3}\ln^{-3}(f_s^{-1/2}N^{-1}\xi_N^{-3}), \quad (9)$$

where a factor has been omitted from inside the logarithm which contains γ and depends on the boundary conditions; we assume that this factor is not too different from unity. As R increases further, for $d < \xi_N$ the critical current of the system can become larger than the pair-breaking current density. At this point the mechanism of current transport changes; the current along the superconducting filaments is shunted by the longitudinal supercurrent induced in the normal matrix as a consequence of the proximity effect. This case will be investigated in detail in the following sections of this paper.

We also present expressions for the critical temperature of the system. It must be on the order of the coupling energy between superconducting regions[1]:

$$T_{cn} \approx \Phi_0 b l j_c(d_c) \qquad (10)$$

and is usually much lower than the critical temperature T_c of the superconducting inclusions themselves ($\Phi_0 = \pi/e$ is a flux quantum).

The last thing we will investigate in this section is the influence of an external magnetic field on the critical current density in the percolation region (the field H will be assumed to be directed along the y-axis). The field begins to suppress the superconductivity of the system when it reaches values on the order of Φ_0/dl, at which point a single flux quantum Φ_0 can flow in the normal interlayers between inclusions. This leads to oscillations in the current between filaments and to a power-law decrease in the critical current density:

$$j_{cz}(H) \approx j_{cz}(0)\, \Phi_0/Hdl, \quad H \gg \Phi_0/dl. \qquad (11)$$

Thus, in weak fields, until suppression of the proximity effect itself occurs, only the pre-exponential factor changes in the expression for the critical current. However, the exponent also begins to change even at fields $\sim \Phi_0/\xi_N^{1/2} d^{3/2}$, so that the Gor'kov functions rapidly decrease as we move away from the inclusions.

This effect can be analyzed to exponential accuracy with the help of Eqs. (5) for the functions F_0, setting $A = A_z = Hx$ and locating the superconducting filament at the coordinate origin along the z-axis:

$$\frac{\partial^2 F_0}{\partial x^2} + \frac{\partial^2 F_0}{\partial y^2} - (4e^2H^2x^2 + \xi_N^{-2})F_0 = 0. \qquad (12)$$

Let us look for a solution to Eq. (12) in the form $F_0 = e^{-S}$, where $S \gg 1$; we then obtain the following equation for the function $S(x,y)$:

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = 4e^2H^2x^2 + \xi_N^{-2}. \qquad (13)$$

The critical current density, just as in the case of no magnetic field, is determined by the infinite cluster near the percolation threshold which is made up of surfaces of constant values of the function S surrounding each inclusion. Twice the corresponding value of S_0 yields the exponent in the expression for j_{cz}, and is found from the condition

$$NV_{S_0} = B_c/8, \qquad (14)$$

where V_{s0} is the volume excluding the interior of the constant—S_0 surfaces.

For $H \ll \Phi_0/d\xi_N$, the surface of constant S is close to a cylinder. The correction to the value of S connected with the magnetic field is relatively small and can be found by perturbation theory. Going to polar coordinates in Eq. (13) and making the substitution $S = (\rho/\xi_N) + \delta S$, we obtain the following expression for the correction δS:

$$\delta S(\rho, \varphi) = {}^2/_3 e^2 H^2 \rho^3 \xi_N \cos^2\varphi. \qquad (15)$$

After calculating the volume

$$V_{S_0} = \pi l \xi_N^2 S_0^2 \left[1 - (\pi S_0)^{-1} \int \delta S(d_c/2, \varphi)\, d\varphi \right], \qquad (16)$$
$$S_0 = d_c/2\xi_N$$

and making use of Eq. (14), we find the following expression for the critical current density in a magnetic field:

$$j_{cz}(H) \approx j_{cz}(0) \exp\left[-(\pi^2/12) H^2 d_c^3 \xi_N/\Phi_0^2\right], \qquad (17)$$
$$\Phi_0/d^{3/2}\xi_N^{1/2} \ll H \ll \Phi_0/d\xi_N,$$

where the left-hand inequality corresponds to the condition $\delta S \gg 1$. We see that in the field interval under discussion the exponent in the expression for the critical field j_{cz} acquires a correction which is relatively small but is large in absolute value.

As the magnitude of the field H increases further, the surfaces $S = S_0$ are strongly elongated in the y direction (the strongest suppression of the proximity effect occurs in a plane perpendicular to the field direction). For $H \gg \Phi_0/d\xi_N$, this surface is given by the expression

$$S = eHx^2 + |y|/\xi_N, \qquad (18)$$

and we obtain the functional dependence for the decrease of the critical current from Eq. (14) in the form

$$j_{cz}(H) \approx j_{cz}(0) \exp\left[-\pi (d_c/\xi_N)(9Hd_c\xi_N/128\Phi_0)^{1/3}\right], \qquad (19)$$
$$H \gg \Phi_0/d\xi_N.$$

Thus, in this region the exponent is determined by the magnetic field, which defines the effective magnitude of the coherence length in the normal metal.

3. PROPERTIES OF THE SYSTEM FOR SIGNIFICANT VOLUME FRACTIONS OF SUPERCONDUCTOR

An increase in the volume fraction of superconductor f_s leads to a change in the numerical coefficients in expressions (1) and (2) for the critical spacing between inclusions d_c. This is related to the fact that the finiteness of the volume of superconductor compared to the normal matrix begins to make itself felt, and so the percolation problem is no longer purely a problem of random point elements. However, the critical current is determined as previously by the proximity effect, and for $d_c \gg \xi_N$ is given by Eqs. (3) and (7).

When the concentration of superconductor exceeds a critical value $f_{s0} \approx 0.15$, the superconducting phase forms an infinite cluster. The current density rises rapidly, and even for a very small increase in f_s over the value f_{s0}, it begins to shunt the *SNS* contact; the critical current density increases according to[3]

$$j_c \approx j_{c0}(f_S - f_{S0})^{2\nu}, \quad f_S - f_{S0} \ll f_{S0}, \qquad (20)$$

where j_{c0} is the critical current density of the superconducting inclusion. As f_s increases further, the results of numerical calculations show[2] that even for $f_s \gtrsim 2 f_{s0}$ almost all the inclusions enter into the infinite cluster and the critical current density

$$j_c \approx f_s j_{c0}. \tag{21}$$

If the inclusions are large compared to the penetration depth λ of the magnetic field, the critical current density of the inclusions j_{c0} is determined by volume pinning within the inclusions.[11] Here, we are investigating the opposite limiting case, which is specific to the system at hand. Let us first consider the situation where the supercurrent through the normal matrix can be neglected, and where each inclusion can be regarded as coupled only to the other superconductors (in the interstice between coupled point elements, the electrodynamics is the same as for superconducting cylinders in vacuum). Then a strong surface pinning appears, and the critical current density turns out to be close to the pair-breaking current density j_p for very large values of the magnetic field perpendicular to the axis of the cylinder.

If the transverse size of the inclusion b (the radius of the cylinder in the simplest case) is small compared to λ, then the supercurrent in the absence of an external field is distributed uniformly over the cross-section. This Meissner state remains stable up to the current density j_p, since in this case, just as in films, there is a barrier to the entry of vortices.[12] In an external field directed perpendicular to the filament, the current density becomes a linear function of the transverse coordinate. Therefore, when the current density at the boundaries of the cylinder reaches the value of j_p and the Meissner state begins to be disrupted,[13–15] the average density is smaller than j_p (the magnetic field suppresses the superconductivity). In magnetic fields above the value $H_0 \sim \Phi_0 / b\xi_s$, Abrikosov vortices are found even in the filament located along the field. The critical value of the current corresponds to a steady-state scenario, in which a certain portion of the filament volume is filled with an immobile lattice of Abrikosov vortices (in this case the density of transported current equals zero), while the remaining portion is in the Meissner state and carries a longitudinal transport current. At the point where the density of this current reaches a value of j_p at the edge of the filament, it already corresponds to a much smaller average value of the current density.

In order to calculate the function $j_{c0}(H)$, let us consider the generalized London equation[16] for the supercurrent j, which for a radius $b \ll \lambda$ of the cylinder (when we can neglect the self-field of the current) has the form

$$4\pi\lambda^2 dj/dx = H - n\Phi_0, \tag{22}$$

where n is the density of vortices and the axis of the cylinder, as before, corresponds to the z-axis; the magnetic field H is directed along the y-axis. The critical current of the cylinder corresponds to the current density j reaching a value j_p at some point in the cross-section of the cylinder.

As long as the field H is not too large, there are no vortices in the cylinder. The expression for the critical current density obtained from Equation (22) (the local current density peaks at $x = b$) takes the form

$$j(x) = j_p + H(x-b)/4\pi\lambda^2, \quad H < H_0 = 2\pi\lambda^2 j_p / b, \qquad j_{c0} = j_p - Hb/4\pi\lambda^2. \tag{23}$$

For $H > H_0$, the region $-b < x < x_0$ is filled with vortices; in it we have $j = 0$ for the transport current while for $x > x_0$ we have

$$j(x) = (H/4\pi\lambda^2)(x - x_0) = j_p + (H/4\pi\lambda^2)(x-b), \qquad x_0 = b(1 - 2H_0/H), \quad H > H_0. \tag{24}$$

The current density equals the critical value j_p once more at the edge of the cylinder for $x = b$, and for the critical current density averaged over the cylinder cross-section we obtain

$$j_{c0}(H) = \frac{2}{\pi b^2}\int_{x_0}^{b} j(x)\,(b^2 - x^2)^{1/2}\,dx \approx \left(\frac{32}{15\pi}\right) j_p \left(\frac{H_0}{H}\right)^{1/2}, \qquad H \gg H_0, \tag{25}$$

i.e., the critical current density decreases as a power of the field.

Thus, the critical current density j_{c0} for transverse dimensions of the inclusions $b \ll \lambda$ is of the same order of magnitude as the pair-breaking current density up to a field of order $H_0 \sim \Phi_0 / b\xi_s$. This field can attain values on the order of the vortex critical current H_{c2} for inclusion dimensions $\sim \xi_s$, corresponding to the beginning of suppression of superconductivity in the inclusion itself due to the proximity effect.

In this investigation, the critical current is determined by the properties of an individual superconducting filament. Its isolation from other filaments ensures that the supercurrent in the normal matrix will be small compared to the current in the filaments. However, other cases are possible, which we will investigate below. If the conductivity σ_N of the normal matrix is large compared to the conductivity of the superconductor σ_s in its normal state, then we see from Eq. (4) that for large absolute values of the Gor'kov functions F_n in the normal metal, the supercurrent density j_s in the normal metal can significantly exceed the current density in the parallel-connected superconductor. When the spacing between superconducting inclusions is small compared to ξ_N and the Gor'kov functions are not exponentially small in the greater part of the normal matrix, the longitudinal critical current density j_c of the system, thanks to the superconductivity induced in the normal metal, can be far larger than the pair-breaking current density j_p of the superconductor.

Let us find expressions for the critical current of the normal matrix that are accurate up to a numerical coefficient for temperatures not too close to the critical temperature T_c of the superconductor (in the Appendix we present an exact calculation of the longitudinal critical current density of a planar *SNS* "sandwich," which illustrates this effect). In the absence of an external magnetic field we can set $A = 0$ in Eqs. (4) and (5), and assume that the phase χ of the order parameter and of the functions F_n vary linearly along the direction of the transport current (the z-axis), which is parallel to the inclusion. If the spacing between superconducting filaments satisfies $d \ll \xi_N$, then it follows from Eq. (5) that the absolute value $\mathcal{F}_n$ of the Gor'kov functions F_n in the normal region vary weakly, and are close to the value $\mathcal{F}_{nN}$, while their transverse gradients are

$$\partial \mathcal{F}_n/\partial \rho \sim \mathcal{F}_{nN} d[(2\omega_n/D_N)+(\nabla\chi)^2], \quad (26)$$

where $\nabla\chi$ is the gradient of the phase χ, which determines the supercurrent through Eq. (4).

In the superconductor, the Gor'kov function differs in absolute value from $\mathcal{F}_{nN}$ over distances $\sim\xi_s$ at the boundary with the normal metal up to its equilibrium value

$$\mathcal{F}_{ns}=\Delta_\infty/(\omega_n^2+\Delta_\infty^2)^{1/2}$$

deep in the filament, where $\Delta_\infty \sim T_c$ is the equilibrium value of the order parameter (we recall that the transverse dimensions of the filaments is $b \gg \xi_s$). The magnitudes of the Gor'kov functions in the normal interlayer $\mathcal{F}_{nN}$, which determine the supercurrent in the matrix according to expression (4), are found by matching these functions with the functions $\mathcal{F}_n$ in the superconductor, making use of the boundary conditions in Eqs. (5) and (26); they depend significantly on the magnitude of the parameter $\gamma d/\xi_N$ ($\gamma = \sigma_N \xi_s/\sigma_s \xi_N$).

In the limit $\gamma d/\xi_N \gg 1$, the Gor'kov functions in the normal metal are small compared to their equilibrium values in the superconductor:

$$\mathcal{F}_{nN} \sim \frac{\xi_N}{\gamma d}\frac{\mathcal{F}_{ns}}{(2n+1)+(\nabla\chi\xi_N)^2}. \quad (27)$$

Substituting these expressions into Eq. (4) and estimating their maximum along $\nabla\chi$ (this maximum is attained for $\nabla\chi_m \sim (\xi_N)^{-1}$), we obtain for the critical current of the normal matrix

$$j_{cN} \approx \frac{\sigma_N T}{e\xi_N}\left(\frac{\xi_N}{\gamma d}\right)^2, \quad \frac{\gamma d}{\xi_N} \gg 1. \quad (28)$$

If $\gamma d/\xi_N \ll 1$, then for moderate currents ($\nabla\chi \ll \nabla\chi_m$) the functions $\mathcal{F}_n$ are everywhere close to the equilibrium value $\mathcal{F}_{ns}$, and their deviations $\delta\mathcal{F}_n$ from these values are given by the relations

$$\delta\mathcal{F}_n \sim \mathcal{F}_{ns}(\gamma d/\xi_N)[(2n+1)+(\nabla\chi\xi_N)^2]. \quad (29)$$

The critical current corresponds to that value of the phase gradient $\Delta\chi$ for which the ratio $\delta\mathcal{F}_n/\mathcal{F}_{ns}$ ceases to be small, i.e., $\nabla\chi_m \sim (\gamma d\xi_n)^{-1/2}$. For the critical current density j_{cN} we obtain the expression

$$j_{cN} \approx \frac{\sigma_N T}{e\xi_N}\left(\frac{\xi_N}{d\gamma}\right)^{3/2}, \quad \frac{T}{T_c} \ll \frac{\gamma d}{\xi_N} \ll 1, \quad (30)$$

$$j_{cN} \approx \frac{\sigma_N T}{e\xi_N}\left(\frac{\xi_N}{d\gamma}\right)^{1/2}, \quad \frac{\gamma d}{\xi_N} \ll \frac{T}{T_c}. \quad (31)$$

Here we have taken into account the fact that in the first case [Eq. (30)] the sum in expression (4) for the current decreases for $n \sim (\gamma d/\xi_N)^{-1} \ll T_c/T$, while in the second case it decreases for $n \sim T_c/T$; in the intermediate range of temperatures $T \sim T_c$, Eq. (30) has no region of applicability, and we must use Eq. (31). A comparison of the expressions obtained for j_{cN} with the pair-breaking current density j_p of the superconductor determined by Eq. (8) shows that when the spacings d between superconducting inclusions is small there is a very broad range of values of the parameter γ for which $j_{cN} \gg j_p$, and practically all the supercurrent is carried in the normal matrix via the induced proximity-effect superconductivity. In the intermediate-temperature region, using Eqs. (28) and (31) we obtain

$$j_{cN}/j_p \sim \xi_N^2/\gamma d^2, \quad \xi_N/d \ll \gamma \ll (\xi_N/d)^2,$$
$$j_{cN}/j_p \sim (\gamma\xi_N/d)^{1/2}, \quad d/\xi_N \ll \gamma \ll \xi_N/d. \quad (32)$$

In a magnetic field H, together with the term $\nabla\chi$ in Eqs. (26), (27) and (29) there appears a field term $\sim eHd$ determined by the vector potential. In fields

$$H_N \sim \Phi_0/d\xi_N \text{ for } \gamma d/\xi_N \gg 1$$

and

$$H_N \sim \Phi_0/(\gamma\xi_N)^{1/2} d^{3/2} \text{ for } \gamma d/\xi_N \ll 1,$$

corresponding to approximate equality between the variation in the vector potential over a distance $\sim d$ and the critical value of the phase gradient $\nabla\chi_m e$, suppression of the proximity effect begins (i.e., decrease of the quantities $\mathcal{F}_{nN}$ as the field increases). A similar picture of the suppression of superconductivity by a field obtains in thin films of type-I superconductor.[7] As is clear from the results obtained in the Appendix, the critical current density decreases as a power law:

$$j_{cN}(H) \approx j_{cN}(0)(H_N/H)^3, \quad H \gg H_N, \quad (33)$$

which is correct in the region of moderate temperatures.

4. CONCLUSIONS

An anisotropic system of superconductors in normal metal exhibits a series of interesting properties.

When the concentration f_s of the superconducting phase, the coupling between individual superconducting filaments comes about via the proximity effect: a supercurrent flows in the normal matrix between filaments. If the spacing between filaments satisfies $d \gg \xi_N$, the critical current of the system is exponentially small and its density is given by Eqs. (2), (3) and (7). However, a current with density $\sim j_p$ can flow in the filaments even when f_s is small if the spacing between inclusions satisfies $d \gtrsim \xi_N$; in this case the system carries a current with density $\sim j_p f_s$. In this case the volume fraction of superconductor must not be too small: $f_s \gg (\xi_s/\xi_N)^2$, so that the transverse dimension of the filaments b exceeds the coherence length ξ_s and superconductivity is not suppressed in the inclusions themselves.

The critical current density for small values of f_s is very sensitive to an external magnetic field. Suppression of superconductivity begins in comparatively weak fields $\sim\Phi_0/dl$, where l is the length of a filament, because magnetic flux quanta which penetrate into the SNS contact give rise to oscillations in the current density and decrease its average value. The power-law dependence of the critical current density given by Eq. (11) is a result of this effect. For large fields the dependence of the critical current is no longer exponential because the magnetic field begins to suppress the proximity effect (this suppression is anisotropic). The critical current density in fields larger than $\Phi_0/d^{3/2}\xi_N^{1/2}$ is determined by Eqs. (17), (19).

When the concentration of superconductor satisfies $f_s > 0.15$, the current is carried in a superconducting cluster formed by direct contact between filaments. For these concentrations the transport current density also can reach values $\sim j_p$; when the transverse dimension b of an inclusion is small compared to the penetration depth λ, this current is weakly suppressed by an external field. In this case, the strong surface pinning hinders the penetration of Abrikosov vortices into the filament and ensures stability of the Meissner state up to fields $\sim \Phi_0/b\xi_s$. If the size of an inclusion is a few times ξ_s (which is necessary for superconductivity to exist in the inclusions themselves), the field can attain values on the order of H_{c2}. Suppression of the critical current is described by Eqs. (23) and (25).

It is well-known that surface pinning is weakened in the presence of sharp corners, which give rise to appreciable local increases in the current density. For example, if the sides of a corner form an obtuse angle α, with $\pi < \alpha < 2\pi$, and the round-off radius of the corner and its depth $\sim b \ll \lambda$, then the critical current density will be decreased roughly by a factor $(b/r)^{1-\pi/\alpha}$ (Ref. 14). However, a system of superconducting filaments in a normal matrix is usually created by rolling, for which the probability of producing a sharp "dent," i.e., a region of current "concentrators," is small. For inclusions which occur in real systems, the complex shape of a typical transverse cross-section does not introduce strong gradients in the current distribution when the transverse dimensions of the inclusions are small compared to the penetration depth, and consequently do not weaken the surface pinning efficiency.

An interesting situation can arise in the practically important case where the conductivity σ_N of the normal matrix is much larger than σ_s of the superconductor. For spacings between filaments smaller than ξ_N, the longitudinal supercurrent induced in the normal matrix can shunt the current along the superconducting inclusions because of the proximity effect. If the characteristic parameter $\gamma = \sigma_N\xi_s/\sigma_s\xi_N$ satisfies the condition $\xi_s/\xi_N \ll \gamma \ll (\xi_N/\xi_s)^2$, which is usually fulfilled in real systems, then there is a wide interval of spacings d in which $j_c \gg j_p$ [Eqs. (28) and (30)–(32)]. This effect can be detected through the characteristic temperature dependence of the critical j_{cN} which obtains for not too small γ. In contrast to the pair-breaking current in the superconductor, it is clear from Eq. (28) that j_{cN} grows as $T^{-1/2}$ as the temperature falls. This dependence is preserved down to a temperature $\sim T_c[\xi_N(T_c)/\gamma(T_c)d]$, which is small compared to T_c; after this the current approaches a constant [Eq. (30)]. The current j_{cN}, however, is suppressed in a magnetic field

$$\sim(\Phi_0/d\xi_N)\max\{1, (\xi_N/d\gamma)^{1/2}\}.$$

In stronger fields the behavior of the system is determined either by the disruption of the coupling between isolated inclusions (for small f_s), after which the critical current decreases exponentially with growth of the field, or by surface pinning ($f_s > 0.15$); in this latter case the critical current density j_p is preserved up to high fields, and then decreases according to a power law.

Let us note that the degree of anisotropy in practical cases usually is characterized by the parameter $R^* = (l/l_0) \approx R^{2/3}$, where l_0 is the characteristic size of the inclusions before the sample is drawn. By rolling we can elongate the inclusions greatly (R^* can reach tens of thousands) while leaving unchanged the volume fraction f_s of superconductor and concentration N of inclusions, which are prescribed initial parameters. The above investigation shows that in such composites it is possible to attain high "in situ" values of the critical current density $\sim j_p$ in high fields $\sim H_{c2}$. With the help of the equations for the local critical current density obtained in this paper, we can use the Maxwell equations to investigate the electrodynamics of real cables for specific fields.

The authors express their thanks to A. A. Abrikosov, A. I. Larkin, and A. S. Nigmatulin for valuable discussions of the results obtained.

APPENDIX

Let us consider a planar SNS contact parallel to the yz plane with SN boundaries at $x = \pm d/2$. We will assume that the dimensions of the "sandwich" are small compared with the penetration depth of the magnetic field, so that an external field H directed along the y-axis is uniform and can be described by the vector potential $A = A_z = Hx$. Let us calculate the longitudinal (along the z-axis) supercurrent density, assuming the temperature T is close to the critical temperature T_c of the (superconducting) edges. In this case the Gor'kov functions F_n are small in absolute magnitude (so that $G_n \approx 1$), and the Usadell Equation (5), when we substitute $F_n(x) = \mathcal{F}_n(x)\exp(i\nabla\chi z)$ gives rise to the form

$$D_N\mathcal{F}_n'' - (\nabla\chi - 2eHx)^2\mathcal{F}_n = 2\omega_n\mathcal{F}_n, \quad |x| < d/2, \tag{A.1}$$

$$\omega_n = \pi T_c(2n+1),$$

$$D_S\mathcal{F}_n'' = 2\omega_n\mathcal{F}_n - 2\Delta(x), \quad |x| > d/2. \tag{A.2}$$

In Eq. (A.2) terms are omitted which contain the phase gradient and magnetic field, and which are relatively small due to the relation between coherence lengths $\xi_s \ll \xi_N$.

If the thickness d of the normal interlayer is small compared to the coherence length ξ_N, while the magnetic field satisfies $H \ll \Phi_0/d^2$, then, as is clear from Eq. (A.1), the function $\mathcal{F}_n$ in the N region is slowly varying. Its value is found from the matching conditions on the quantity itself and its derivative, multiplied by the ohmic conductivity, with the corresponding values in the S-regions. Since the edges are in equilibrium, the derivatives of the Gor'kov functions have the same value at both boundaries. Solving Eq. (A.1) with these boundary conditions, we can obtain closed boundary conditions for the functions $\mathcal{F}_n$ in the S-region (henceforth it is sufficient to consider only the right-hand electrode $x \geqslant d/2$). They have the form

$$\xi_S\mathcal{F}_n'(d/2) = \gamma\beta_n^2(d/2\xi_N)\mathcal{F}_n(d/2), \quad \xi_{S,N} = (D_{S,N}/2\pi T_c)^{1/2},$$

$$\beta_n^2 = (2n+1) + \beta^2, \quad \beta^2 = (\nabla\chi\xi_N)^2 + (eHd\xi_N)^2/3. \tag{A.3}$$

In this case, deep in the superconductor

$$\mathscr{F}_n(\infty)=\mathscr{F}_{ns}=\Delta_\infty/\omega_n,\quad \Delta_\infty=[8\pi^2/7\zeta(3)]^{1/2}\tau^{1/2}T_c,$$
$$\tau=1-T/T_c\ll 1. \qquad (A.4)$$

The coordinate dependence of the modulus of the order parameter $\Delta(x)$ which enters into Eq. (A.2) is found from the Landau-Ginzburg equations

$$\xi^2(T)\Delta''+\Delta(1-\Delta^2/\Delta_\infty^2)=0,\quad \xi(T)=(\pi/2)\xi_S\tau^{-1/2} \qquad (A.5)$$

and is given by the Eq.

$$\Delta(x)=\Delta_\infty \operatorname{th}\left[\frac{x-d/2}{2^{1/2}\xi(T)}+t\right]. \qquad (A.6)$$

The constant t is found from the boundary conditions for Eq. (A.5)[10]:

$$\Delta'(d/2)=\frac{\gamma p}{\xi_S}\Delta\left(\frac{d}{2}\right),$$
$$\frac{\gamma p}{\xi_S}=\left[\sum_{n=0}^{\infty}\omega_n^{-1}\mathscr{F}_n'\left(\frac{d}{2}\right)\right]\left[\sum_{n=0}^{\infty}\omega_n^{-1}\mathscr{F}_n\left(\frac{d}{2}\right)\right]^{-1}, \qquad (A.7)$$

which, by taking expressions (A.3) and (A.6) into account, can be rewritten in the form

$$p=\frac{d}{2\xi_N}\left(\sum\beta_n^2\omega_n^{-1}\mathscr{F}_{nN}\right)\left(\sum\omega_n^{-1}\mathscr{F}_{nN}\right)^{-1}=\frac{2^{1/2}}{\Gamma \operatorname{sh} 2t},$$
$$\mathscr{F}_{nN}=\mathscr{F}_n\left(\frac{d}{2}\right),\quad \Gamma=\gamma\xi(T)\xi_S^{-1}. \qquad (A.8)$$

The solutions to the Usadell equations for the conditions that $\xi(T)\gg\xi_s$ have the form

$$\mathscr{F}_n(x)=\Delta(x)/\omega_n+B_n\exp[-(2\omega_n/D_S)^{1/2}(x-d/2)]. \qquad (A.9)$$

Once we have determined the constant B_n from the boundary conditions (A.3), we obtain the following expressions for the magnitudes of the Gor'kov functions $\mathscr{F}_{nN}$ in the normal layer, taking into account (A.8):

$$\mathscr{F}_{nN}=\frac{\Delta_\infty}{\omega_n}\frac{2^{1/2}\tau^{1/2}\operatorname{ch}^{-2}t+\pi(2n+1)^{1/2}\operatorname{th}t}{\tau^{1/2}\beta_n^2(\Gamma d/\xi_N)+\pi(2n+1)^{1/2}}$$
$$=\frac{\Delta_\infty}{\omega_n}\frac{\gamma p+(2n+1)^{1/2}}{\gamma\beta_n^2(d/2\xi_N)+(2n+1)^{1/2}}\operatorname{th}t. \qquad (A.10)$$

After finding the functions $\mathscr{F}_{nN}$ by a self-consistent solution to Eqs. (A.8) and (A.10), we determine the longitudinal transport current density by using the following formula derived from expression (4):

$$j_s=\frac{2\pi\sigma_N T}{e}\sum\mathscr{F}_{nN}^2\nabla\chi. \qquad (A.11)$$

The maximum value of this expression for a given phase gradient gives the critical current density j_{cN}. It can be calculated analytically in several limiting cases.

In the absence of a magnetic field, as we showed above, the expressions for the critical current depend on $\gamma d/\xi_N$. If $\gamma d/\xi_N\gg 1$, then the superconductivity is strongly suppressed at the SN boundary (we have the parameter $t\ll 1$). Taking into account that B_n is always $\gtrsim 1$, and [as is clear from (A.8)] that the product $pt=(2^{1/2}\Gamma)^{-1}$, we obtain from formula (A.10) the following expression for the Gor'kov functions $\mathscr{F}_{nN}$

$$\mathscr{F}_{nN}=\frac{\Delta_\infty}{\omega_n}\frac{2^{1/2}\xi_N}{\Gamma d\beta_n^2}. \qquad (A.12)$$

The maximum of expression (A.11) equals

$$j_{cN}=0{,}4j^*\left(\frac{\xi_N}{\Gamma d}\right)^2,\quad j^*=\frac{\sigma_N\Delta_\infty^2}{eT\xi_N}. \qquad (A.13)$$

For $(T_c-T)\sim T_c$, this formula matches with (28), which is accurate to within a numerical coefficient.

In the case $\gamma d/\xi_N\ll 1$, when only small values of n are important in the sum (A.11), we have

$$\mathscr{F}_{nN}=(\Delta_\infty/\omega_n)\operatorname{th}t, \qquad (A.14)$$

and the Eq. for the current takes the form

$$j_s=2\pi\frac{\sigma_N\Delta_\infty^2}{eT}s\nabla\chi\operatorname{th}^2t,\quad s=\sum_{n=0}^{\infty}\frac{1}{\pi^2(2n+1)^2}=\frac{1}{8}. \qquad (A.15)$$

The sum in expression (A.8) for the parameter p contains a logarithmic divergence when substituted into the Gor'kov functions (A.14). As is clear from Eqs. (A.10), for such choices of n the equality (A.14) is violated and the quantities $\mathscr{F}_{nN}$ begin to decrease rapidly with increasing n:

$$p=\frac{d}{2\xi_N}(\beta^2+\varepsilon),\quad \varepsilon=s^{-1}\sum_{n=0}^{n_0}[\pi^2(2n+1)]^{-1}=\frac{8}{\pi^2}\ln\frac{\xi_N}{d\gamma}. \qquad (A.16)$$

Expressing the parameter t in terms of $\nabla\chi$ by using Eqs. (A.8) and taking (A.16) into account, and substituting it into (A.15), we obtain for the critical current density

$$j_{cN}=c_1j^*(\Gamma d/\xi_N)^{-2}\ln^{-3/2}(\xi_N/d\gamma),\quad c_1=3^{3/2}\pi^4/2^{19/2}\approx 0.70,$$
$$\tau^{1/2}\ln^{-1}\tau^{-1}\ll\gamma d/\xi_N\ll 1, \qquad (A.17)$$
$$j_{cN}=c_2j^*(\Gamma d/\xi_N)^{-1/2},\quad c_2=3^{3/4}\pi/10^{5/4}\approx 0.40,$$
$$\gamma d/\xi_N\ll\tau^{1/2}\ln^{-1}\tau^{-1}. \qquad (A.18)$$

For $t\sim 1$, Eq. (A.18) matches with Eq. (31), while the region of applicability of expression (A.17) reduces to zero.

Finally, let us investigate the behavior of the critical current density in a magnetic field, which we will assume is strong: $H\gg\Phi_0\nabla\chi_m/d$, where $\nabla\chi_{m0}$ is the critical value of the phase gradient corresponding to the maximum in Eq. (A.11) in the absence of a field. In this case, as is clear from the definition of β_n in (A.3), the maximum of expression (A.11) along $\nabla\chi$ is attained for $\nabla\chi_m\sim eHd\ll\nabla\chi_{m0}$. Therefore, for $n\sim 1$ the coefficients $\beta_n\approx\beta$ do not depend on n while the parameter $p\approx\beta^2d/2\xi_N$. For all values of the Gor'kov functions and transport current density, Eqs. (A.14) and (A.15) are valid when the parameter $t\ll 1$ and [(as follows from (A.8)] equals

$$t=2^{1/2}/\Gamma d\xi_N[(\nabla\chi)^2+(eHd)^2/3]. \qquad (A.19)$$

As a result, we obtain the functional form of the decrease in critical transport current density in the form

$$j_{cN}(H)=c_3j^*\left(\Gamma\frac{d}{\xi_N}\right)^{-2}\left(\frac{\Phi_0}{d\xi_N H}\right)^3,\quad c_3=\frac{27}{32\pi^2}\approx 0.086, \qquad (A.20)$$

which to an accuracy of a numerical factor coincides with expression (33). It is valid for any value of the parameter $\gamma d/\xi_N$.

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Translated by F.J. Crowne