Two-dimensional models of disordered systems with a relativistic spectrum: exact results and the 1/*N* expansion

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(Submitted 8 April 1986)
Zh. Eksp. Teor. Fiz. **91**, 1767–1775 (November 1986)

The applicability of the replica method is elucidated using the example of two-dimensional exactly solvable theories; for the case in which the replica procedure has been found to be correct, perturbation theory in $1/N$ (N is the number of colors) is constructed. The connection between the models investigated and the experimental situation is discussed.

INTRODUCTION

Many results obtained in recent years in the theory of disordered systems owe their origin to the replica method, formulated independently by Edwards and de Gennes.[1] The basis of this method lies in the fact that the average over fluctuations is performed in a system that is repeated R times; in the final result it is necessary to let R tend to zero. Of course, all the calculations are performed for integer $R\geqslant 1$; therefore, the correctness of the replica limit (i.e., the limit $R\to 0$) is unclear and evidently depends on the specific problem. Searches for exactly solvable models might clarify the situation. The first such model was solved by Kardar and Nelson.[2]

In the present paper we give examples of two-dimensional theories of disordered systems that can be solved by means of the Bethe method. Some of these have a replica limit and others do not, and this is not apparent, of course, in the framework of perturbation theory.

The plan of the paper is as follows: In Sec. 1 we recall the content of the replica method and formulate the models, Sec. 2 is devoted to an exact solution by means of the Bethe method, and in Secs. 3 and 4 we develop perturbation theory in $1/N$ (N is the number of colors) for the particular model that has a replica limit and is, at the same time, the most physical; here we also calculate the density of states and estimate the conductivity.

1. FORMULATION OF THE MODELS. THE REPLICA METHOD

We shall consider a $1+2$-dimensional theory of noninteracting particles situated in a random field $v(x, y)$ distributed in a Gaussian manner:

$$P\{v\}=A^{-1}\exp\Big(-\int d^2x\, v^2(x,y)/2g\Big). \tag{1}$$

As a rule, in the theory of disordered systems we are interested in the following Green's functions. The first is

$$\hat{G}_1(\omega)=\overline{(i\omega-\hat{H})^{-1}} \tag{2}$$

(the bar denotes averaging over the random field v distributed in accordance with the law (1), and $\hat{H}$ is the system Hamiltonian and depends linearly on v), and this function can be used to describe the thermodynamics of the system. The second Green's function of interest is

$$\hat{G}_2(\varepsilon,\omega)=\overline{(\omega-i0-\hat{H})^{-1}[(\omega+\varepsilon+i0)-\hat{H}]^{-1}}, \tag{3}$$

in terms of which the conductivity at frequency ε can be expressed.

We write the Green's function (2) in the form

$$G(\omega;x,x')=A^{-1}\int Dv\,\frac{\delta^2\ln Z\{v,\varepsilon\}}{\delta\varepsilon(x)\,\delta\bar{\varepsilon}(x')}\exp\Big(-\frac{1}{2g}\int v^2\,d^2x\Big), \tag{4}$$

where

$$Z\{v,\varepsilon\}=\int D\bar{\psi}\,D\psi\exp[-\bar{\psi}(i\omega-\hat{H})\psi+\varepsilon\bar{\psi}+\psi\bar{\varepsilon}]$$

is a generating functional of the fermion fields ψ. (Since the particles do not interact with each other, we can also write the Green's functions in the form of a functional integral over boson fields, but fermions seem to us to be more convenient.)

The replica method consists in using the limit

$$\ln Z=\lim_{R\to 0}\frac{Z^R-1}{R},$$

for the average over $v(x, y)$ in formula (4).

Ascribing to the fields ψ the replica indices $\alpha=1,\dots,R\to 0$, we obtain

$$G(\omega;x,x')=\int Dv\lim_{R\to 0}\frac{1}{R}\Big\{\frac{\delta^2}{\delta\varepsilon(x)\,\delta\bar{\varepsilon}(x')}$$

$$\times\int\exp\Big[-\Big(\sum_{\alpha=1}^{R}\bar{\psi}_\alpha(x)\,[i\omega\delta(x,x')-\hat{H}(x,x')]\,\psi_\alpha(x')$$

$$+\bar{\psi}_\alpha(x)\,\varepsilon(x)$$

$$+\bar{\varepsilon}(x)\,\psi_\alpha(x)\Big)\Big]D\bar{\psi}_\alpha D\psi_\alpha\Big\}\exp\Big(-\frac{1}{2g}\int v^2\,d^2x\Big)$$

$$=\lim_{R\to 0}\frac{1}{R}\frac{\delta^2\overline{Z^R}}{\delta\varepsilon\,\delta\bar{\varepsilon}}=\frac{\delta^2}{\delta\varepsilon(x)\,\delta\bar{\varepsilon}(x')}\ln Z_{eff}\{\varepsilon\}. \tag{5}$$

The effective action already contains a term of fourth order in ψ_α; this term has appeared as a result of averaging over the disorder, and the replica indices in it are interchanged.

We remark once again that the existence of the limit as $R\to 0$ in formula (5) is certainly not obvious. Below, for the example of exact solutions, it will be seen that models that apparently differ only slightly from each other can have different behavior as $R\to 0$.

We shall formulate the models with which we shall be

 0038-5646/86/111046-05$04.00

working. We consider a $1+2$-dimensional massless relativistic color Fermi field $\psi_n(\omega, x, y)$, transforming according to the fundamental representation of a group G in an external static random vector potential[1] $A_\mu^a \tau^a$ (τ^a are the generators of the algebra of G), distributed in accordance with the Gaussian law

$$\overline{A_\mu^a(x, y) A_\nu^b(x', y')} = g\delta^{ab}\delta_{\mu\nu}\delta^2(\mathbf{x}-\mathbf{x}').$$

The two-dimensional action for the Fourier harmonics ψ_ω (henceforth we shall omit the subscript ω) is given by the formula

$$S = \int d^2x \{ i\bar{\psi}_n \gamma_\mu (i\partial_\mu \delta_{nm} + A_\mu^a \tau_{nm}^a)\psi_m + i\omega\bar{\psi}_n\psi_n \} \qquad (6)$$

$$\gamma_0 = \sigma_x, \quad \gamma_1 = \sigma_y, \quad \bar{\psi} = i\psi^{*T}\sigma_x.$$

After averaging over the replicas in the spirit of formula (5) and going over to the Hamiltonian formalism, we obtain a system described by the effective Hamiltonian

$$H_{eff} = \lim_{R\to 0} \frac{1}{R} \int dx \Big\{ i\psi^{\cdot}_{Rn\alpha} \frac{d}{dx} \psi_{Rn\alpha} - i\psi^{\cdot}_{Ln\alpha} \frac{d}{dx} \psi_{Ln\alpha}$$
$$+ \omega (\psi^{\cdot}_{Rn\alpha}\psi_{Ln\alpha} + \psi^{\cdot}_{Ln\alpha}\psi_{Rn\alpha})$$
$$+ g(\psi^{\cdot}_{Rn\alpha}\tau^a_{nn'}\psi_{Rn'\alpha})(\psi^{\cdot}_{Lm\beta}\tau^a_{mm'}\psi_{Lm'\beta}) \Big\} \qquad (7)$$

(here the indices R and L denote the right-handed and left-handed components of the two-dimensional spinors).

For $\omega = 0$ the model (7) is integrable for all simple groups; for a finite $R > 1$ it was considered in Ref. 6 for the group $G = SU(N)$ and in Ref. 7 for all the other simple groups. The S-matrix of the theory (7) has the symmetry $G \times GL(R)$ (R is the number of replicas); a generally accepted hypothesis, of which a rigorous proof exists at present only for $G = SU(N)$,[6] is that one can replace this theory by the theory with the S-matrix of the group G in the representation with the dominant weight $(R, 0,\ldots,0)$. (This hypothesis was used in Ref. 7 to calculate S-matrices of chiral fields; the results obtained coincide with the formulas of phenomenological scattering theory.)

2. EXACT SOLUTION

We shall consider in detail the solution of the model with symmetry $SU(N) \times SU(R)$. The Bethe equations in the massive sector (we shall discuss the Goldstone mode separately in Sec. 3) have the form[6,7]

$$[e_R(\lambda_\alpha^{(1)} + 1/g)]^{N_0} [e_R(\lambda_\alpha^{(1)} - 1/g)]^{N_0} \prod_{\beta=1}^{M^{(2)}} e_1(\lambda_\alpha^{(1)} - \lambda_\beta^{(2)})$$
$$= \prod_{\beta=1}^{M^{(1)}} e_2(\lambda_\alpha^{(1)} - \lambda_\beta^{(1)}),$$
$$\prod_{\beta=1}^{M^{(j-1)}} e_1(\lambda_\alpha^{(j)} - \lambda_\beta^{(j-1)}) \prod_{\beta=1}^{M^{(j+1)}} e_1(\lambda_\alpha^{(j)} - \lambda_\beta^{(j+1)}) = \prod_{\beta=1}^{M^{(j)}} e_2(\lambda_\alpha^{(j)} - \lambda_\beta^{(j)});$$
$$j = 1, \ldots, N-1, \qquad (8)$$
$$E = \frac{N_0}{2iR} \sum_{\alpha=1}^{M^{(1)}} \ln [e_R(\lambda_\alpha^{(1)} + 1/g)/e_R(\lambda_\alpha^{(1)} - 1/g)],$$

where N_0 is the number of right-handed particles, equal to the number of left-handed particles, $2N_0 > M^{(1)} > \ldots > M^{(N-1)}$, and

$$e_n(x) = (x - in/2)/(x + in/2).$$

The equations (8) are written for the bare-particle vacuum. In order to take the limit $R \to 0$, it is necessary to rewrite them in terms of the rapidities of the excitations and to let $N_0 \to \infty$. The limit $R \to 0$ exists only for excitations above a filled vacuum. The vacuum in the model (8) consists of strings of all colors ($j = 1,\ldots,N-1$) with label R. Following the procedure described in Refs. 6–8, we shall write out the Bethe equations over the physical vacuum (a detailed derivation for precisely this case was first given in Ref. 6):

$$\exp[iML\,\mathrm{sh}(2\pi\Lambda_\alpha/N)] = \prod_{\beta=1}^{n} S(\Lambda_\alpha - \Lambda_\beta)$$
$$\times \prod_{\beta=1}^{m(1)} e_1(\Lambda_\alpha - \lambda_\beta^{(1)}) \prod_{\beta=1}^{l(1)} \frac{\mathrm{sh}[\pi R^{-1}(\Lambda_\alpha - u_\beta^{(1)} - i/2)]}{\mathrm{sh}[\pi R^{-1}(\Lambda_\alpha - u_\beta^{(1)} + i/2)]}, \qquad (9a)$$
$$\prod_{\beta=1}^{m(j-1)} e_1(\lambda_\alpha^{(j)} - \lambda_\beta^{(j-1)}) \prod_{\beta=1}^{m(j+1)} e_1(\lambda_\alpha^{(j)} - \lambda_\beta^{(j+1)}) = \prod_{\beta=1}^{m(j)} e_2(\lambda_\alpha^{(j)} - \lambda_\beta^{(j)}),$$
$$(9b)$$

where $\{\lambda^{(0)}\} \equiv \{\Lambda\}$, $j = 1,\ldots,N-1$, and

$$\prod_{\beta=1}^{l(j-1)} \frac{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j-1)} - i/2)]}{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j-1)} + i/2)]}$$
$$\times \prod_{\beta=1}^{l(j+1)} \frac{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j+1)} - i/2)]}{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j+1)} + i/2)]}$$
$$= \prod_{\beta=1}^{l(j)} \frac{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j)} - i)]}{\mathrm{sh}[\pi R^{-1}(u_\alpha^{(j)} - u_\beta^{(j)} + i)]}, \quad \{u^{(0)}\} \equiv \{\lambda\}, \qquad (9c)$$

where

$$\frac{E}{L} = \frac{M}{R} \sum_{\alpha=1}^{n} \mathrm{ch}\left(\frac{2\pi\Lambda_\alpha}{N}\right), \quad M = \frac{N_0}{L} \exp\left(-\frac{2\pi}{Ng}\right), \qquad (9d)$$

$$S(\Lambda) = \exp\Big\{ 2i \int_0^\infty d\omega \frac{\sin\omega\Lambda}{\omega}$$
$$\times \Big[\frac{2\,\mathrm{sh}[(N-1)\omega/2]\,\mathrm{sh}(\omega/2)}{\mathrm{sh}(N\omega/2)(1-e^{-R\omega})} - 1 \Big] \Big\}. \qquad (10)$$

The equations (9) and (10) have been derived for $R > 1$. Analytical continuation into the region $R < 1$ requires care. The fundamental particles for $R > 1$ have mass M/R (as $R \to 0$ the energy per replica should be finite, and that is why R is present in formula (9d), i.e., at first sight the mass of the particles tends to infinity as $R \to 0$. However, as R passes through unity the spectrum acquires a new branch of excitations: bound states with rapidities Λ and $\lambda^{(1)},\ldots,\lambda^{(n-1)}$:

$$\Lambda = \theta + i[(N+1)/2 - k](1-R), \quad k = 1,\ldots,N,$$
$$\lambda^{(j)} = \theta + i[(N-j+1)/2 - k](1-R), \quad k = 1,\ldots,N-j. \qquad (11)$$

The mass of this bound state is equal to

$$m=\frac{M}{R}\sum_{j=0}^{N-1}\cos\left[\frac{2\pi}{N}(1-R)\left(\frac{N-1}{2}-j\right)\right]$$

$$=\frac{M}{R}\frac{\sin\pi R}{\sin[\pi(1-R)/N]}\to NM$$

$$\text{as } R\to 0$$

and remains finite in the replica limit.

Thus, as $R\to 0$ the former fundamental particles cease to be excited, since their mass tends to infinity like $1/R$, and it is their bound states, whose mass is finite as $R\to 0$, that are excited. Formally, what is operating here is the same confinement mechanism that was discovered by Wiegmann in his solution of the problem of the $O(3)$-symmetric σ-model.[8] It was Wiegmann who noticed that the solution of this model can be obtained by taking the limit $R\to 0$ in Eqs. (9) for $N=2$.

Substituting (11) into Eqs. (9) and (10) and letting $R\to 0$, we obtain equations for the rapidities of the bound states:

$$\exp[imL\,\mathrm{sh}(2\pi\theta_\alpha/N)]$$
$$=\prod_{\beta=1}^{n}\frac{\theta_\alpha-\theta_\beta+i}{\theta_\alpha-\theta_\beta-i}\prod_{\beta=1}^{m^{(1)}}\frac{\theta_\alpha-\lambda_\beta^{(1)}-iN/2}{\theta_\alpha-\lambda_\beta^{(1)}+iN/2},$$
$$\prod_{\beta=1}^{n}e_N(\lambda_\alpha^{(1)}-\theta_\beta)\prod_{\beta=1}^{m^{(2)}}e_1(\lambda_\alpha^{(1)}-\lambda_\beta^{(2)})=\prod_{\beta=1}^{m^{(1)}}e_2(\lambda_\alpha^{(1)}-\lambda_\beta^{(1)}),$$
$$\vdots\qquad\vdots\qquad\vdots \tag{12}$$
$$\prod_{\beta=1}^{m^{(N-2)}}e_1(\lambda_\alpha^{(N-1)}-\lambda_\beta^{(N-2)})=\prod_{\beta=1}^{m^{(N-1)}}e_2(\lambda_\alpha^{(N-1)}-\lambda_\beta^{(N-1)}),$$
$$n>m^{(1)}>\ldots>m^{(N-1)},\quad E/L=m\sum_{\alpha=1}^{n}\mathrm{ch}(2\pi\theta_\alpha/N).$$

These equations, which determine the spectrum of the effective Hamiltonian, are a proof of the mathematical correctness of the replica procedure for the model (6) with the group $SU(N)$.

We now consider the same problem but with the group $O(2N)$. The Bethe equations have almost the same form as Eqs. (9), but in place of the last two equations [the $(N-2)$th and $(N-1)$th equations], we have

$$\prod_{\beta=1}^{m^{(N-2)}}e_2(\lambda_\alpha^{(N-2)}-\lambda_\beta^{(N-2)})=\prod_{\beta=1}^{m^{(N-3)}}e_1(\lambda_\alpha^{(N-2)}-\lambda_\beta^{(N-3)})$$
$$\times\prod_{\beta=1}^{m^{(+)}}e_1(\lambda_\alpha^{(N-2)}-\lambda_\beta^{(+)})\prod_{\beta=1}^{m^{(-)}}e_1(\lambda_\alpha^{(N-2)}-\lambda_\beta^{(-)}), \tag{13a}$$
$$\prod_{\beta=1}^{m^{(N-2)}}e_1(\lambda_\alpha^{(\pm)}-\lambda_\beta^{(N-2)})=\prod_{\beta=1}^{m^{(\pm)}}e_2(\lambda_\alpha^{(\pm)}-\lambda_\beta^{(\pm)}). \tag{13b}$$

The equations over the physical vacuum are

$$\exp\{iLM\,\mathrm{sh}[\pi\Lambda_\alpha/(N-1)]\}=\prod_{\beta=1}^{n}S(\Lambda_\alpha-\Lambda_\beta)\prod_{\beta=1}^{m^{(1)}}e_1(\Lambda_\alpha-\lambda_\beta^{(1)})\prod_{\beta=1}^{l^{(1)}}\frac{\mathrm{sh}[\pi R^{-1}(\Lambda_\alpha-u_\beta^{(1)}-i/2)]}{\mathrm{sh}[\pi R^{-1}(\Lambda_\alpha-u_\beta^{(1)}+i/2)]}. \tag{14}$$

The rapidities $\{\lambda^{(1)},\ldots,\lambda^{(\pm)}\}$ are determined by Eqs. (9b) and (13), while the rapidities $\{u^{(1)},\ldots,u^{(\pm)}\}$ are determined by the same equations but with the trigonometric kernels

$$S(\Lambda)=\exp\left\{2\int_0^\infty d\omega\frac{\sin\omega\Lambda}{\omega}\times\left[\frac{2\,\mathrm{sh}(\omega/2)}{1-e^{-R\omega}}\frac{\mathrm{ch}[(N-2)\omega/2]}{\mathrm{ch}[(N-1)\omega/2]}-1\right]\right\}.$$

The energy of the system is

$$\frac{E}{L}=\frac{M}{R}\sum_{\alpha=1}^{n}\mathrm{ch}\frac{\pi\Lambda_\alpha}{N-1},\quad M=\frac{N_0}{L}\exp\left(-\frac{\pi}{(N-1)g}\right). \tag{15}$$

The most important difference from the preceding case is the absence of the factor of two in the argument of the hyperbolic cosine in formula (15). Because of this, the mass of the bound state

$$\Lambda=\theta^{(\pm)}+i\left(\frac{N+1}{2}-k\right),\quad k=1,\ldots,N,$$
$$\lambda^{(j)}=\theta^{(\pm)}+i\left(\frac{N-j}{2}-k\right),\quad k=N-j,\ldots,1;\quad j=1,\ldots,N-2, \tag{16}$$
$$\lambda^{(\pm)}=\theta^{(\pm)}$$

is equal to

$$m=\frac{M}{R}\sum_{j=0}^{N-1}\cos\left[\frac{\pi}{N-1}\left(\frac{N-1}{2}-j\right)(1-R)\right]$$
$$\to\left\{R\sin\frac{\pi}{2(N-1)}\right\}^{-1}$$

and does not have finite limit as $R\to 0$. Confinement does not occur, and the replica procedure is not correct.

An analogous situation obtains for the group $O(2N+1)$. As regards the last classical group $\mathrm{Sp}(2N)$, here there is also no replica limit, but for a different reason. From the equations obtained in Ref. 4 it can be seen that in this case the strings with labels $R-1$ and $R+1$ drop out into the condensate. Correspondingly, the trigonometric functions in the equations over the physical vacuum have the form

$$\mathrm{sh}\left[\frac{\pi}{R\pm 1}(\lambda-i/2)\right]\Big/\,\mathrm{sh}\left[\frac{\pi}{R\pm 1}(\lambda+i/2)\right].$$

Here, bound states appear, although not as $R\to 0$ but as $R\to 1$.

We shall not investigate exceptional groups.

3. THE 1/N EXPANSION FOR THE MODEL WITH THE GROUP *SU(N)*. CALCULATION OF THE DENSITY OF STATES

We shall reformulate the theory (7) with the group $SU(N)$ in a form convenient for the $1/N$ expansion. Introducing the tensor field $Q_{\alpha\beta}$, we rewrite the effective action of the theory

$$S_{eff}=\frac{1}{R}\int d^2x\{i\bar\psi_{n,\alpha}\gamma_\mu\partial_\mu\psi_{n,\alpha}+i\omega\bar\psi_{n,\alpha}\psi_{n,\alpha}+2gJ_{nm}{}^{\mu}J_{mn}{}^{\mu}\}$$
$$\times(J_{nm}{}^{\mu}=\bar\psi_{n,\alpha}\gamma^\mu\psi_{m,\alpha}), \tag{17}$$

in the form

$$S_{eff}=\frac{1}{R}\int d^2x\left\{\frac{1}{2g}\operatorname{Tr}(\hat{Q}-\omega\hat{I})(\hat{Q}^+-\omega I)\right.$$

$$+\left[i\bar{\psi}_{n,\alpha}\gamma_\mu\partial_\mu\psi_{n,\alpha}+\bar{\psi}_{n,\alpha}\left(\frac{1}{2}(1-\gamma_5)Q_{\alpha\beta}\right.\right.$$

$$\left.\left.\left.+\frac{1}{2}(1+\gamma_5)Q_{\beta\alpha}^*\right)\psi_{n,\beta}\right]\right\}. \quad (18)$$

Integrating over the fermions, we obtain the effective action for the tensor $Q_{\alpha\beta}$:

$$S_{eff}=\frac{1}{R}\int d^2x\left\{\frac{1}{2g}\operatorname{Tr}QQ^+-\frac{\omega}{2g}\operatorname{Tr}(Q+Q^+)\right.$$

$$\left.-N\operatorname{Tr}\ln[i\gamma_\mu\partial_\mu\hat{I}+i\hat{Q}(1-\gamma_5)/2+i\hat{Q}^+(1+\gamma_5)/2]\right\}; \quad (19)$$

the $1/N$ expansion presupposes that the functional $S_{\text{eff}}\{Q\}$ has a minimum. This minimum is realized at $Q_{\alpha\beta}=Q\delta_{\alpha\beta}$, where Q is determined from the equation

$$Q-\frac{Ng}{\pi}Q\ln\frac{\Lambda}{|Q|}=\omega. \quad (20)$$

It can be seen from Eq. (20) that the natural scale for measurement of the frequency is the quantity NgQ_0/π, where $Q_0=\Lambda\exp(-\pi/Ng)$. It is convenient to rewrite Eq. (20) in the form

$$\frac{Q}{Q_0}\ln\frac{|Q|}{Q_0}=\frac{\omega\pi}{NgQ_0}. \quad (21)$$

For $|\omega|\ll NgQ_0$,

$$\frac{Q}{Q_0}=1+\frac{\pi\omega}{NgQ_0}+O\left(\left(\frac{\omega}{NgQ_0}\right)^3\right). \quad (22)$$

For $|\omega|\gg NgQ_0$,

$$Q=\omega\{1-(Ng/\pi)\ln[\Lambda Ng/|\omega|]\}^{-1} \quad (23)$$

—this is the result of summing the parquet diagrams.

The expansion about the saddle point is formulated conveniently by writing the tensor $Q_{\alpha\beta}$ in the form

$$\hat{Q}=U^+\Lambda e^{i\varphi}U, \quad (24)$$

where $U^+U=\hat{I}$, U is the matrix of the group $SU(R)$, Λ is a diagonal matrix, normalized by the condition

$$\operatorname{Im}\prod_{\alpha=1}^{R}\Lambda_\alpha=0,\quad \operatorname{Re}\prod_{\alpha=1}^{R}\Lambda_\alpha>0, \quad (25)$$

and φ is a general phase;

$$S_{eff}=\frac{1}{R}\int d^2x\left\{\frac{1}{2g}\sum_{\alpha=1}^{R}\Lambda_\alpha^2-\frac{\omega}{g}\cos\varphi\sum_{\alpha=1}^{R}\Lambda_\alpha\right.$$

$$\left.+N\mathscr{L}\{U,\Lambda,\varphi\}\right\}. \quad (26)$$

The transformation $\psi\to\exp(i\gamma_5\varphi/2)\psi$ takes φ out of the mass term in the expression (18) but adds to the action the term $i\partial_\mu\varphi J_\mu^5/2$. Thus, in lowest order in $\partial_\mu\varphi$ we find that the contribution of φ to the Lagrangian $\mathscr{L}\{U,\Lambda,\varphi\}$ is

$$\frac{1}{2}\int\int d^2x\,d^2x'\,\partial_\mu\varphi(x)\langle\langle J_\mu^5(x)J_\nu^5(x')\rangle\rangle\partial_\nu\varphi(x')$$

$$\approx\frac{N}{8\pi}\int d^2x(\partial_\mu\varphi)^2,$$

since a calculation with the saddle-point Green's functions (a calculation valid for $N\to\infty$) gives

$$\langle\langle J_+(x)J_-(x')\rangle\rangle=\frac{N}{4\pi}\delta(x-x')+\frac{1}{Q^2}O(\delta''(x-x')).$$

Thus, at small momenta the field φ is decoupled from the other fields. This decoupling also occurs in the term $\omega g^{-1}\cos\varphi\operatorname{Tr}\Lambda$. As we have seen, the exact solution for $\omega=0$ gives masses $\sim Q$ for all excitations except the Goldstone field φ. Therefore, for $|\omega|/gN\ll Q$, when the mass of the field φ is much smaller than the masses of all the other fields, we can replace $R^{-1}\operatorname{Tr}\Lambda$ by Q. In this case the effective action for the field Q is

$$S\{\varphi\}=\int d^2x\left[\frac{N}{8\pi}(\partial_\mu\varphi)^2-\frac{\omega}{g}Q\cos\varphi\right]. \quad (27)$$

This result can also be generalized to $N\sim 1$; in this case, however, we can no longer be certain about the value of the numerical factor multiplying $(\partial_\mu\varphi)^2$.

For $N\Rightarrow\infty$ the correlator

$$\int\langle\langle\varphi(x)\varphi(0)\rangle\rangle e^{ipx}d^2x=(Np^2/8\pi+\omega Q/2g)^{-1} \quad (28)$$

has the form of the propagator of the diffusion mode familiar in the theory of localization. The diffusion coefficient in our case is equal to

$$D=gNQ^{-1}/4\pi.$$

The asymptotic form of the correlator $\langle\langle Q_{\alpha\beta}(x)Q_{\gamma\delta}(0)\rangle\rangle$, in terms of which, in particular, the density of states can be expressed (see below), can be calculated for $\omega=0$ for all N and for $N\to\infty$ and $\omega\neq 0$:

$$\langle\langle Q_{\alpha\beta}(x)Q_{\gamma\delta}(0)\rangle\rangle$$

$$=\delta_{\alpha\beta}\delta_{\gamma\delta}Q_0^2(Q_0|x|^{-2/N}\quad(Q_0^{-1}\ll|x|\ll Q_0^{-1}[Q_0Ng/|\omega|]^{1/2});$$

$$\int e^{ipx}\langle\langle Q_{\alpha\beta}(x)Q_{\gamma\delta}(0)\rangle\rangle d^2x=\delta_{\alpha\beta}\delta_{\gamma\delta}\frac{g}{N\omega Q}\left(1-\frac{Ng}{4\pi}\frac{p^2}{\omega Q}+\ldots\right)$$

for $N\gg 1$ and small momenta. The density of states is expressed in terms of the correlator $\operatorname{Tr}\hat{Q}$:

$$\rho(\varepsilon)=\frac{1}{\pi}\operatorname{Im}\sum_n\langle\bar{\psi}_n(x)\psi_n(x)\rangle_{\omega=-i\varepsilon+0}=\frac{1}{\pi g}\langle\operatorname{Tr}Q(x)\rangle$$

$$\approx\frac{\varepsilon}{\pi g^2}\int\frac{d^2y}{S}\langle\langle\operatorname{Tr}Q(x)\operatorname{Tr}Q(y)\rangle\rangle=\frac{2Q_0}{\pi g}+O(\varepsilon^2). \quad (29)$$

We note that the bare density of states (in the absence of disorder) vanishes as $\varepsilon\to 0$:

$$\rho(\varepsilon)=N|\varepsilon|. \quad (30)$$

A comparison of formulas (29) and (30) shows that the change of regime occurs, as we should expect, at $\varepsilon\sim Q_0/Ng$.

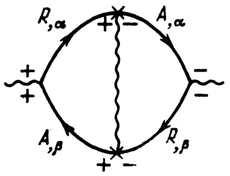

FIG. 1.

4. ESTIMATE OF THE CONDUCTIVITY

Since the conductivity is expressed in terms of retarded and advanced Green's functions, it is necessary to introduce into the effective action (6) a further fermion-field index; the fermion now carries three indices:

$$\psi = \psi^{(i)}_{\alpha,n}, \qquad i=R,A.$$

Generally speaking, the R and A fermions have different frequencies, but we shall limit ourselves immediately to calculating the conductivity at zero frequency:

$$\sigma(0)=\frac{e^2}{S}\lim_{R\to 0}\frac{1}{R^2}\int d^2x \langle\langle \psi^{*R}_{+\alpha n}(x)\,\psi^{A}_{+\beta n}(x)\,\psi^{*A}_{-\alpha m}(0)\,\psi^{R}_{-\beta m}(0)\rangle\rangle.$$

The saddle-point condition for the tensor $\hat{Q}$ has the form

$$Q_{\alpha\beta}{}^{ij}=Q_0\delta_{\alpha\beta}\sigma_{ij}{}^3$$

(σ^3 is a Pauli matrix).

For the field φ introduced by formula (24), the effective action coincides with (27). The first nonvanishing diagram in $1/N$ for the conductivity is depicted in the figure. A wavy line corresponds to the correlator $\langle\langle e^{i\varphi(x)}e^{-i\varphi(0)}\rangle\rangle$.

If instead of this correlator we had the correlator $\langle\langle\varphi(x)\varphi(0)\rangle\rangle$, this diagram would contain a logarithmic divergence $\sim N^{-1}\ln(Q_0/\omega)$. But since, for $\omega \ll q \ll Q_0$, the behavior of the Green's function is different:

$$\langle\langle e^{i\varphi}e^{-i\varphi}\rangle\rangle_q=\frac{4\pi}{Nq^2}\left(\frac{q}{Q_0}\right)^{2/N},$$

the logarithm is cut off and the whole diagram remains of order unity. Thus, the small factor $1/N$ disappears and the conductivity cannot be calculated by means of the $1/N$ expansion. Nevertheless, it seems plausible to us that, because of the effective cutoff of the logarithm, the conductivity remains finite: $\sigma(0)\sim e^2N$.

CONCLUSION

Thus, we have considered several models in which the particles carry two indices—a color index and a replica index. These models differ only in their symmetry group. We have calculated the spectra of the Hamiltonians for an integer number of replicas $R\geqslant 1$, and have then made a formal analytical continuation into the region $R\to 0$, i.e., we have followed the standard replica procedure. In the framework of this approach it turns out that the limit $R\to 0$ exists only for the group $SU(N)$. In this case we have constructed the $1/N$ expansion for the calculation of the density of states.

Unfortunately, we do not know of a physical realization of the theories that we have considered, although two-dimensional fermions with a relativistic spectrum are encountered in the description of many solid-state systems (see Refs. 3–5). It is possible that the models that we have considered have some relation to systems of charge-density and spin-density waves.

The author is grateful to A. I. Larkin, L. D. Faddeev, D. E. Khmel'nitskiĭ, P. B. Wiegmann, N. Yu. Reshetikhin, M. V. Feigel'man, L. B. Ioffe, and F. A. Smirnov for valuable discussions and debates.

[1)]Models using a relativistic description for different solid-state systems have been proposed recently by Semenoff,[3] Volkov and Pankratov,[4] and Fisher and Fradkin.[5]

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Translated by P. J. Shepherd