Superconducting superlattices

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Using the BCS model, we have investigated superlattices consisting of superconducting and normal metals in alternating layers whose thicknesses are small compared to the superconducting correlation length. It is assumed that the one-electron properties of these layers are identical, and that the layers differ only in the parameter which describes the effective electron-electron attraction. We have computed the phase transition temperature and single-electron density of states. In a purely coherent system, the superconductivity of a superlattice must be gapless; we have evaluated the density of states in the gap and the low-temperature specific heat.

As a result of progress in the technology of vacuum deposition it is already possible to fabricate nearly perfect superlattices consisting of alternating layers of similar elements.[1–7] Much interest attaches to semiconducting superlattices, which are now being studied intensively both theoretically and experimentally (see Ref. 1), Superconducting superlattices[2,3] were obtained somewhat later; in the preparation of these systems, layer thicknesses are maintained to atomic-layer precision, while the coherence length of the structure across the layers can be tens of periods. At present, the superconducting superlattices which have been studied experimentally in some detail consist of Nb/Cu and Pb/Cu (Nb and Cu are the superconductors). Layer thickness in such systems is varied over a wide range, roughly from 10 Å to several thousand Å.[4–6] Systems have now been created consisting of alternating layers of the superconducting and magnetic metals V and Ni (where Ni is an itinerant ferromagnet),[5] and also of superconducting metals of different kinds, i.e., Nb/Zr.[7]

It is clear that if the layer thickness d in the superlattice exceeds the coherence length ξ of the superconductor, then the superconducting characteristics of the individual layers are little changed. In fact, in superlattices of the type S/N or S/S' we are dealing with two weakly-interacting systems S and N, or S and S', respectively. The influence of a normal metal on superconductivity, i.e., the "proximity effect", has been discussed for these systems by DeGennes and Guyon,[8,9] and by Werthamer[10]; their approach gives a good description of the experimental data for long-period superlattices.[11] A theoretical description of the properties of these systems was given using the Landau-Ginzburg functional in Ref. 12 and in a recent review (Ref. 13), where a thorough analysis of later experimental data can also be found.

Qualitatively new superconducting characteristics appear for $d \ll \xi$, i.e., the case of short-period S/N superlattices, where we can speak only of the superconducting system as a whole rather than of its individual layers. In the simplest model—free motion of electrons across the layers—the critical temperature of the system is determined by an average of the Cooper pairing constant $\langle \lambda \rangle$ over the layers,[14,15] By now a certain amount of experimental information about the properties of such systems has been accumulated, and it can be assumed that the number of novel short-period superconducting superlattices will grow rapidly.

As regards theoretical descriptions of superconducting superlattices, up until now only isolated inhomogeneities consisting of material whose Cooper pairing constant is larger than that of the volume have been studied,[16–18] along with periodic sequences of thin layers of such material in a superconducting host.[19,20]

In this paper we will investigate a superlattice model with S and N layers which are thin compared to the superconducting correlation length but thick relative to the interatomic spacing. We will assume that the single-electron properties of the S and N layers are identical, i.e., that the layers S and N differ only in the Cooper pairing constant λ, which is zero in the N layer. In this situation the Landau-Ginzburg theory is inapplicable; however, we can use the quasiclassical description. Using this simple model we will find how the critical temperature depends on the layer thickness d for the impure- and pure-superconductor cases. We will show that the electronic spectrum for a pure system becomes gapless, and find its density of states and specific heat.

Let us note at once, however, that the model we are investigating is only a first step in the study of short-period superconducting superlattices. In particular, it does not take into account the possibility that new electronic states may appear, i.e., bands with their own densities of states; nor does it deal with reflection of electrons from the boundaries between the S and N layers, which can affect the formation of the superconducting state, or the change of the system's phonon spectrum which may either weaken or enhance its superconducting properties. This latter effect is doubtless important in, e.g., the superlattice Au/Ge, in which superconductivity has recently been observed with $T_c = 1.5$ K for layer thicknesses of Au equal to 10 Å and of Ge equal to 13 Å (Ref. 21), although neither of the original elements Au or Ge is a superconductor.

To describe the electronic motion, we will use the model of isotropic diffusive motion and introduce a phenomenological parameter, the mean free path of an electron. Actually, the motion of an electron is clearly anisotropic, and to lowest order we can introduce the longitudinal (along the layer)

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and transverse mean free paths $l_{\|}$ and $l_{\perp}$, respectively. Information about these parameters can be obtained from data on the system resistance along and through the layers. For example, it is well known that in the superlattice Nb/Cu the product ρd (ρ is the resistivity) increases at a roughly linear rate as d is increased for $d > 1000$ Å and $d < 10$ Å, remaining constant on the interval from 10 Å to 100 Å (Ref. 4). This data indicates that the effective mean free path of an electron is on the order of d for 10 Å $< d <$ 100 Å and is independent of d for $d > 1000$ Å, where it is on the order of 100 Å, and for $d < 10$ Å, where it is on the order of 10 Å. Unfortunately, information about the anisotropy in ρ is lacking in Ref. 4.

2. SUPERLATTICE CRITICAL TEMPERATURE

As already noted, we will be discussing a model of superconducting superlattices in which the normal (N) and superconducting (S) layers differ only in the value of the dimensionless Cooper pairing constant: $\lambda(x) = 0$ in the N layer and $\lambda(x) = \lambda$ in the S layer, where the x-axis is directed perpendicular to the layers. The period $d = d_n + d_s$, where d_s is the superconducting layer thickness and d_n is the normal layer thickness, is smaller than the superconducting correlation length ξ, but much larger than the interatomic spacing. This latter circumstance allows us to use the quasiclassical approximation to describe the system, i.e., the Eilenberger equations[22] for the Green's functions $g(\mathbf{r},\mathbf{v})$ and $f(\mathbf{r},\mathbf{v})$. These functions depend on the coordinate $\mathbf{r}$, the frequency $\omega = (2n+1)\pi/T$, and also on the direction of the vector $\mathbf{v}$, where $|\mathbf{v}| = v_F$. In our case, the function g and f depend on the single coordinate x, and the Eilenberger equations take the form

$$\left[\omega + \frac{1}{2\tau}\bar{g}(x) + \frac{1}{2}v_x\frac{\partial}{\partial x}\right]f(x,\mathbf{v}) = \left[\Delta(x) + \frac{1}{2\tau}\bar{f}(x)\right]g(x,\mathbf{v}),$$

$$\left[\omega + \frac{1}{2\tau}\bar{g}(x) - \frac{1}{2}v_x\frac{\partial}{\partial x}\right]f^+(x,\mathbf{v}) = \left[\Delta^*(x) + \frac{1}{2\tau}\bar{f}^+(x)\right]g(x,\mathbf{v}),$$

$$g^2(x,\mathbf{v}) + f(x,\mathbf{v})f^+(x,\mathbf{v}) = 1, \qquad (1)$$

$$\bar{f}(x) = \int\frac{d\Omega}{4\pi}f(x,\mathbf{v}), \quad \bar{g}(x) = \int\frac{d\Omega}{4\pi}g(x,\mathbf{v}),$$

where $\tau = 1/v_F$ is the mean free path of the electrons, $v_x = v_F\cos\theta$ and θ is the angle between the x-axis and the direction $\mathbf{v}_F$, i.e., $f(x,\mathbf{v}) = f(x,\theta)$; the integral over $d\Omega$ is an integration over the direction of the velocity at the Fermi surface. In the absence of a magnetic field, the quantity $\Delta(x)$ can be chosen to be real without loss of generality; then the self-consistency equation for $\Delta(x)$ completes the system (1):

$$\Delta(x) = \lambda(x)\pi T\sum_{\omega}\bar{f}(\omega,x), \qquad (2)$$

where the summation over ω, as usual, is cut off at $\omega = \omega_D$. We note that the function $\Delta(x)$ for the superlattice, which is proportional to $\lambda(x)$, is a rapidly-varying discontinuous function (it varies from zero in the N-region to $\Delta(x) \approx \Delta =$ constant in the S-region), while the functions f and g are continuous.

In deriving the superconducting transition temperature T_c, if we assume that $\Delta \to 0$ we can set $g = \operatorname{sign}\omega$ in (1), and the equation for f then takes the form

$$\left(\omega + \frac{1}{2\tau}\operatorname{sign}\omega + \frac{1}{2}v_x\frac{\partial}{\partial x}\right)f(x,\theta) = \left[\Delta(x) + \frac{1}{2\tau}\bar{f}(x)\right]\operatorname{sign}\omega. \qquad (3)$$

Expanding the periodic functions $\Delta(x)$ and $f(x)$ (of period d) in a Fourier series, we find from (3) that the mth harmonic f_m is

$$f_m = \frac{\left(\Delta_m + \frac{1}{2\tau}\bar{f}_m\right)\operatorname{sign}\omega}{\omega + (1/2\tau)\operatorname{sign}\omega + iv_xk_m/2}, \quad k_m = \frac{2\pi m}{d}, \quad m = 0, \pm 1, \ldots . \qquad (4)$$

Averaging (4) over the angle θ, we obtain an expression for f_m; substituting it into the self-consistency equation, we rewrite (2) in the form

$$\Delta_m = \pi T\lambda_m\sum_{\omega}\frac{\Delta_0}{|\omega|} + \frac{Td}{v_F}\sum_{n\neq 0}\lambda_{m-n}\frac{\Delta_n}{|n|} \times \sum_{\omega}\left[\operatorname{arctg}^{-1}\frac{v_F|k_n|}{2|\omega| + 1/\tau} - \frac{1}{\tau v_F|k_n|}\right]^{-1}, \qquad (5)$$

where λ_m is a harmonic of the function $\lambda(x)$: $\lambda_0 = \lambda d_s/d$, while for $m \neq 0$ we have $\lambda_m = i\lambda[\exp(-ik_m d_s) - 1]/2\pi m$.

To lowest order in the layer thickness, as is clear from (5), the critical temperature is determined by the average Cooper pairing constant $\lambda_0 = \lambda d_s/d$ (Refs. 14, 15):

$$\bar{T}_c = 1.14\omega_D e^{-1/\lambda_0}. \qquad (6)$$

To first order in d, corrections to this result are determined by the parameter $d\omega_D/v_F$ in a pure system and $d^2\omega_D v_F l$ in an impure system.

Solving (5) in the limit $d\omega_D/v_F \ll 1$, we arrive at the following dependence of the critical temperature on layer thickness (for simplicity we will assume from here on that $d_s = d_n = d/2$):

$$T_c = \bar{T}_c\left[1 + \frac{d\omega_D}{v_F}f\left(\frac{d}{l}\right)\right], \quad \bar{T}_c = \frac{T_{c0}^2}{1{,}14\omega_D}, \quad T_{c0} = 1{,}14\omega_D e^{-1/\lambda}, \qquad (7)$$

where the function f is determined by the expression

$$f(z)=\frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\left[\operatorname{arctg}^{-1}\frac{2\pi(2n+1)}{z}-\frac{z}{2\pi(2n+1)}\right]^{-1}. \quad (8)$$

Here, $f(z)\to 0.42$ for $z\to 0$ and $f(z)\to 0.12z$ for $z\to\infty$. We note that the question of whether or not the BCS model can be used when the Cooper pairing constant $\lambda(x)$ varies with x over distances smaller than v_F/ω_D is not fully resolved and requires additional analysis.

In the case $d\omega_D/v_F\gg 1$ (but $dT_{c0}/v_F\ll 1$), the function $f(x)$ also varies weakly over the superlattice, and Equation (5) can then be solved to logarithmic accuracy. To calculate T_c, we can replace the second sum over ω in (5) by an integration (this is correct when the parameter $dT_c/v_F=d/\xi\ll 1$). In a pure superconductor Equation (5) can be cast in the form:

$$\Delta_m=\pi T\lambda_m\sum_{\omega}\frac{\Delta_0}{|\omega|}+\sum_{n\neq 0}\lambda_{m-n}\Delta_n\ln\left(\frac{d\omega_D e}{\pi v_F|n|}\right). \quad (9)$$

To first order, Δm equals the first term on the right side of (9). Substituting it into the equation for Δ_0, we find that the critical temperature of the superlattice is determined by an effective Cooper pairing constant

$$\lambda_{\text{eff}}=\lambda_0[1+\lambda_0\ln(\omega_D d/v_F)]. \quad (10)$$

Unfortunately, this result is correct only to logarithmic accuracy, and to this accuracy $T_c\sim\bar{T}_c\omega_D d/v_F\sim T_{c0}^2 d/v_F$. An analogous investigation shows that in the impure-superconductor limit $l\ll d$

$$T_c\sim\bar{T}_c\frac{\omega_D d^2}{v_F l}\sim T_{c0}^2\frac{d^2}{v_F l}.$$

At this time, experimental data on the dependence of T_c for superlattices on the period of the latter $d=2d_n=2d_s$ is available only for the structure Nb/Cu.[4] As d decreases from 5000 Å to 5 Å, the magnitude of T_c decreases from 9 K to about 2.5 K. The initial portion of the falloff of T_c (for $d>300$ Å) is well described by the theory in Ref. 10, taking into account the proximity effect. The further decrease of T_c for $d<300$ Å is more rapid due to averaging of λ. However, the decrease in T_c for $d<300$ Å happens more slowly than is predicted by a model of freely-moving electrons in the superlattice. In this model, λ_{eff} decreases by a factor of 2 as we pass from $d\gg\xi_0$ to $d\ll\xi_0$, and $T_c(d\ll\xi)\approx T_c^2(d=\infty)/\omega_D$. In the Nb/Cu system, this relation is not obeyed, which can be related to reflection of electrons off the boundaries between N and S layers, which reduces the effective averaging of λ. In addition, one cannot exclude an enhancement mechanism for the superconductive pairing due to modification of the phonon spectrum in the short-period superlattices.

3. QUASIPARTICLE DENSITY OF STATES IN A SUPERLATTICE: ANALYSIS WITHIN THE FRAMEWORK OF THE EILENBERGER EQUATIONS

In a superlattice consisting of normal and superconducting layers, in the limit $d\ll\xi$ the superconductivity has a homogeneous character, since the functions f and g to a first approximation do not depend on the coordinate x. This implies that, e.g., the density of superconducting electrons is practically uniform, and the superlattice exhibits exactly the same Meissner effect as the usual bulk superconductor. However, in contrast to a normal superconductor, the quasiparticle spectrum in a pure-superconductor superlattice can be gapless. This is related to the fact that Cooper pairs moving along the layers in the N regions in fact do not participate in superconductive pairing (this pairing is mediated only by those pairs which cross the S layers in their motion). We will calculate this effect first in the framework of the Eilenberger equations. Let us assume that the system is pure ($l\to\infty$), since it is clear that gapless superconductivity can occur only in this case.

For $d\ll v_F/T_c$, the gap parameter $\Delta(x)$ varies only slightly within an S layer, and we can use for $\Delta(x)$ the piecewise-constant function $\Delta(x)=\Delta=\text{const}$. in the S layer, and $\Delta(x)=0$ in the N layer.

The Eilenberger equations for a pure superconductor can be solved exactly in this case. Let us solve the system (1) for $\tau=\infty$ separately for the S and N layers, and impose on the solutions the requirement of periodicity. For the S layer, system (1) takes the form

$$\omega f+{}^1/_2 v_x f'=\Delta g,\quad \omega f^+-{}^1/_2 v_x f^{+\prime}=\Delta g,\quad g^2+ff^+=1, \quad (11)$$

where f and g are functions of the arguments x, v and ω, while f' denotes differentiation with respect to x. From (11) we obtain a third-order equation for the function g:

$$g'''-\frac{4(\omega^2+\Delta^2)}{v_x^2}g'=0. \quad (12)$$

Solutions to Equation (12) have the form:

$$g(\theta,x)=C_1e^{kx}+C_2e^{-kx}+C_3, \quad (13)$$

where $k=2(\omega^2+\Delta^2)^{1/2}/v_x$, while the constants C_1, C_2 and C_3 are determined from the conditions of continuity and periodicity of the functions $g(x)$ and $f(x)$. Using the matching conditions, we finally obtain for the S region ($0<x<d_s$)

$$g(\theta,x)=C(e^{-kx}+e^{k(x-d_s)}+S),$$
$$C=\left[\frac{(\omega^2+\Delta^2)}{\omega^2}S^2+4e^{-kd_s}\frac{(\omega^2+\Delta^2)}{\Delta^2}\right]^{-1/2}, \quad (14)$$
$$S=\frac{\omega^2}{\Delta^2}(1+e^{-kd_s})[1+(1+\Delta^2/\omega^2)^{1/2}\operatorname{th}(kd_s/2)\operatorname{cth}(\omega d_n/2)].$$

The rather complicated form of the Green's function is related to the special role of trajectories with angles $\theta\approx\pi/2$: for these trajectories, electrons undergo long-period oscillations in the pairing potential, and effective averaging of $\Delta(x)$ does not occur. As for the Green's functions outside this narrow band, they coincide to lowest order with the Green's functions for a normal superconductor; however, the role of the superconducting parameter is played not by Δ, but by its average value $\Delta=\Delta d_s/d$. The equation for determining Δ coincides to first approximation in d/ξ with the usual self-consistency equation for a homogeneous super-

conductor, with an average pairing constant $\lambda = \lambda d_s/d$. Therefore, to first order in d/ξ the relation between the gap E_g and T_c will be the same as that for homogeneous samples predicted by the BCS model.

Knowledge of the Green's functions allows us to determine the electronic density of states of the system by analytic continuation of these functions ($\omega \to -iE + \delta$). In this section we will investigate the density of states in the immediate vicinity of the Fermi energy E_F for $E \ll \Delta$, where E is measured from E_F. For $\omega \ll \Delta$, expression (14) for $g(\omega,\theta,x)$ is notably simplified

$$g(\omega,\theta,x) \cong \frac{\omega}{\Delta} + \operatorname{th}\left(\frac{\omega d}{2v_x}\right)\frac{\operatorname{ch}(k(d/4-x))}{\operatorname{sh}(kd/4)}. \qquad (15)$$

After averaging the Green's function g over a period, we obtain for the density of states $N(E)$

$$N(E) = -\frac{1}{2}N(0)\operatorname{Re}\int_0^{\pi} d\theta \sin\theta \langle g(\omega \to -iE+\delta,\theta)\rangle, \qquad (16)$$

where $N(0)$ is the density of states at the Fermi level in the normal state. As follows from the expression of the averaged function $g(\omega,\theta)$

$$\langle g(\omega,\theta)\rangle = \operatorname{th}\left(\frac{\omega d}{2v_x}\right)\left[\frac{v_x}{\Delta d} + \frac{1}{2}\operatorname{cth}\frac{\Delta d}{2v_x}\right] + \frac{\omega}{\Delta}, \qquad (17)$$

after analytic continuation, the only contribution to the density of states will be given by the poles of the function th $\omega d/2v_x$. As a result, we are led to the following representation for the density of states

$$\frac{N(E)}{N(0)} = \frac{d}{4\pi v_F}\sum_{k=0}^{\infty}\frac{E\operatorname{cth}[\pi(k+1/2)\Delta/E]}{(k+1/2)^2} \cong \frac{\pi}{8}\frac{d|E|}{v_F}, \qquad (18)$$

where the summation is taken over all poles of the function (Eq. F).

An important peculiarity of superconducting superlattices, as is clear from Equation (18), is the absence of a gap in the spectrum of electronic excitations. This gapless character of the superconductivity is related to the presence of electron trajectories with $\theta \approx \pi/2$, for which Cooper pairing is absent.

The summation over the poles in (18) in essence implies the presence of discrete electronic levels. However, the periodic character of the pairing potential $\Delta(x)$ allows one to suppose that bands, not levels, must appear in the spectrum. This is in fact the case (see Section 4), and here we are up against the special properties of the quasiclassical approximation based on the Eilenberger approximation. The Green's function $g(\mathbf{v},\mathbf{r})$ used in the Eilenberger equations is related to the exact Green's function $G(r,r')$, which depends on two coordinates for an inhomogeneous system, through the transformation

$$g(\mathbf{v},\mathbf{r}) = i\int\frac{d\xi}{2\pi}\int d\rho G\left(\mathbf{r}+\frac{\rho}{2},\ \mathbf{r}-\frac{\rho}{2}\right)e^{ip\rho}, \qquad (19)$$

where $\xi = v_F(p - p_F)$. This transformation is not invertible (i.e., we cannot reconstruct the exact Green's function from the function g). The integration in (19) over the relative coordinate ρ leads to a loss of information in the function g concerning the band motion of electrons. This latter circumstance also explains why the Eilenberger equations do not give rise to a band-like character in the quasiparticle spectrum. A more direct procedure, which allows one to obtain a band picture of the quasiparticle spectrum, can be based on the quasiclassical Bogolyubov equations (see the next section). However, the bands which arise turn out to be exponentially narrow, and so the expression obtained in the present section for the density of states (18) is still correct.

4. THE BOGOLYUBOV EQUATIONS IN THE QUASICLASSICAL APPROXIMATION

The exact Bogolyubov equations which describe a superconductor with a stepwise parameter $\Delta(x)$ (i.e., in essence a superlattice) were investigated in Ref. 23, where the conclusion was reached that there are states within the energy gap. However, it was not possible to obtain an expression for the quasiparticle density of states or to delineate the nature of the energy spectrum in Ref. 23, because of the extreme complexity and awkwardness of an investigation based on the full Bogolyubov equations.

In this section we will consider a quasiclassical variant of the Bogolyubov equations, which essentially simplifies the investigation, gives full information on the energy spectrum and allows us to find the superlattice density of states. Use of this quasiclassical approximation is correct under the condition that the superlattice period is large compared to the interatomic spacing

The Bogolyubov equations for the functions $u(\mathbf{r})$ and $v(\mathbf{r})$ have the form (see, e.g., Ref. 24):

$$\begin{aligned}\hat{\xi}u_\nu(\mathbf{r}) - \Delta(\mathbf{r})v_\nu(\mathbf{r}) &= E_\nu u_\nu(\mathbf{r}),\\ \hat{\xi}v_\nu(\mathbf{r}) + \Delta(\mathbf{r})u_\nu(\mathbf{r}) &= -E_\nu v_\nu(\mathbf{r}),\end{aligned} \qquad (20)$$

where $\hat{\xi} = p^2/2m - \mu$ and E_ν is the quasiparticle energy.

In the quasiclassical approximation we can assume that the functions $u(\mathbf{r})$ and $v(\mathbf{r})$ are modulated free-electron functions at the Fermi level, i.e., $u(r) \sim \exp(i\mathbf{p}_F\mathbf{r})\tilde{u}(\mathbf{r})$ and $v(\mathbf{r}) \sim \exp(i\mathbf{p}_F\mathbf{r})\tilde{v}(\mathbf{r})$, where the functions $\tilde{u}$ and $\tilde{v}$ vary slowly on an interatomic scale. This separation of rapidly- and slowly-varying parts, and the neglect of second derivatives of the functions $\tilde{u}$ and $\tilde{v}$, allows us to write (20) as

$$\begin{aligned}iv_x\tilde{u}' + \Delta(x)\tilde{v}(x) &= -E\tilde{u}(x),\\ iv_x\tilde{v}' - \Delta(x)\tilde{u}(x) &= E\tilde{v}(x).\end{aligned} \qquad (21)$$

Here, as before, $v_x = v_F\cos\theta$; it must be noted that the division into "rapidly-varying" and "slowly-varying" functions is correct only for $mv_x \gg d^{-1}$, i.e., the present quasiclassical approximation is correct only for angles $\cos\theta \gg a/d$.

Using the stepwise approximation for $\Delta(x)$ (which is correct for $d \ll v_F/\omega_D$) and solving (21) separately for the S and N regions, we obtain from the requirement of continuity of $\tilde{u}$ and $\tilde{v}$, and the Bloch condition of periodicity, the following dispersion relation for the spectrum:

$$\cos qd = \cos\left(\frac{Ed_n}{v_x}\right)\operatorname{ch}\frac{(\Delta^2-E^2)^{1/2}}{v_x}d_s - \frac{E}{(\Delta^2-E^2)^{1/2}} \times \sin\frac{Ed_n}{v_x}\operatorname{sh}\frac{(\Delta^2-E^2)^{1/2}}{v_x}d_s, \quad (22)$$

where q is the wave vector along the x-axis ($0<q<2\pi/d$). For $v_x/d \gg \Delta \sim E$, i.e., for $\cos\theta \gg d/\xi$ where $\xi = v_F/\Delta$, we obtain from (22):

$$\cos qd = 1 + \frac{\Delta^2 d_s^2}{2v_x^2} - E^2\frac{d^2}{2v_x^2}. \quad (23)$$

From (23) it follows that for angles not close to $\pi/2$, there is a gap in the excitation spectrum $E_g = \Delta d_s/d$, which coincides with the average value of $\Delta(x)$. However, a special role is played by the directions $\cos\theta < d/\xi$. In this case,

$$\cos qd = {}^1/_2 \exp(\Delta d_s/v_x)\cos(Ed_n/v_x) \quad (24)$$

and the spectrum consists of a set of mini-bands with exponentially narrow widths:

$$E_n(q) = \pi(n+{}^1/_2)\frac{v_x}{d_n} - \frac{2v_x}{d_n}\exp(-\Delta d_s/v_x)\cos qd \quad (25)$$

($n = 0, \pm 1, \ldots$); these mini-bands lie within the forbidden gap E_g. We recall that an approach based on the Eilenberger equations leads for $\cos\theta \ll d/\xi$ to a discrete picture of the spectrum, and does not lead to a band picture. However, because of the exponentially small width of the bands, in order to calculate the energy of states we can assume the levels are discrete, and the density of states for $E \ll E_g$ is

$$N(E) = \sum_{n,q,\theta}\delta[E - E_n(\theta, q)] = \sum_{n=-\infty}^{\infty}\frac{4\pi}{(2\pi)^3} \times \int\delta\left[E - \pi(n+{}^1/_2)\frac{v\cos\theta}{d_n}\right](mv\cos\theta)\,d(mv\cos\theta)\,dq = N(0)\frac{\pi}{2}\frac{d_n^2}{(d_n+d_s)}\frac{E}{v_F}, \quad (26)$$

which coincides (for $d_n = d_s = d/2$) with the result (18).

In a normal superconductor the density of states equals zero within the gap $|E| < E_g = \Delta$, and diverges as an inverse (square root as $|E| \to \Delta + 0$, i.e., $N(E) = N(0)E_g(E^2 - E_g^2)^{-1/2}$.

As follows from (23), the density of states of a superconducting superlattice also behaves in an analogous fashion as $E \to \Delta d_s/d + 0$; now, however, the quantity $\Delta d_s/d = E_g$ plays the role of a gap in the superlattice. When the energy lies within the gap E_g, a contribution to the density of states appears only for angles $\cos\theta \lesssim d/\xi$ and the density of states for the superlattice will be nonzero even for $|E| < E_g$. So, for $|E| \to 0$ it is determined by the formula (18).

For $|E| \to E_g - 0$, the relation (22) for angles $\cos\theta \ll d/\xi$ is transformed to the form ($d_n = d_s = d/2$)

$$\exp\left(-\frac{3^{1/2}}{2}\frac{d\Delta}{v_x}\right)\cos qd = \cos\frac{Ed}{v_x} - 3^{-1/2}\sin\frac{Ed}{v_x}. \quad (27)$$

Neglecting the width of the mini-bands, we obtain from (27) $E_n = \pi v_x(n + 1/2)/d$ for $n \gg 1$. Substituting this expression for E_n into (26), we obtain

$$N(E \to E_g - 0) \sim N(0)(d\Delta/v_F)\sum_n n^{-2}.$$

Our approximation is correct for $n \gg 1$, but qualitatively it is true also for $n \sim 1$. Thus, to order of magnitude the density of states for $E \to E_g - 0$ comes to $N(0)d\Delta/v_F \sim N(0)d/\xi$.

The schematic form of the density of states in a superlattice is shown in the figure.

The presence of gapless superconductivity in a superlattice (in the pure-superconductor limit) must make itself known in tunneling measurements, as well as in the nature of the temperature dependence of the specific heat. Whereas in an ordinary superconductor the presence of a forbidden gap leads to an exponentially small electronic heat for low temperatures $T \ll \Delta$, in a superlattice the presence of levels within the gap leads to a square-law dependence of the specific heat on temperature:

$$C = 4\frac{\partial}{\partial T}\int_0^{\infty} Ee^{-E/T}N(E)\,dE = 12\pi\frac{T^2}{v_F}\frac{d_n^2}{d}N(0). \quad (28)$$

At this time, no evidence of the presence of states in the gap in the superlattice Nb/Cu has been detected in the data on tunneling measurements.[4,5] This could be related to the fact that the mean free path in this superlattice is apparently of the same order of magnitude as the layer thickness (see the estimates at the end of the Introduction). In such a case, levels within the gap must be absent—indeed, the trajectories required for their presence are absent, i.e., those with $\cos\theta < d/\xi$, for which an electron traverses a path of order ξ through the N layers; impurity scattering for $l \ll \xi$ must also disrupt this regime. We note also the extremely small density of states within the gap for superlattices, which makes the experimental observation of this effect difficult.

5. A DESCRIPTION OF IMPURE SUPERCONDUCTING SUPERLATTICES BASED ON THE USADELL EQUATIONS

In this section we will investigate the properties of superconducting superlattices at low temperatures $T \ll T_c$ in the impure-superconductor limit $l \ll \xi$. As was shown by Usadell,[25] the Eilenberger equations become considerably simpler in this case. It is necessary to note that in our situation, i.e., in the presence of a system with a periodic potential $\Delta(x)$ with period $d \ll \xi$, application of the Usadell equations requires in addition the fulfillment of the more stringent condition $l \ll d$. This condition for the correctness of the Usadell equations is very general; the physical content of this requirement consists of an averaging of the system properties over the impurities within a period of the external field (in our case, its role is played by the potential $\Delta(x)$).

The Usadell equations are written in terms of $F(\mathbf{r})$ and $G(\mathbf{r})$, i.e., angular averages of the functions $f(\mathbf{v},\mathbf{r})$ and $g(\mathbf{v},\mathbf{r})$ respectively. In the superlattice these functions de-

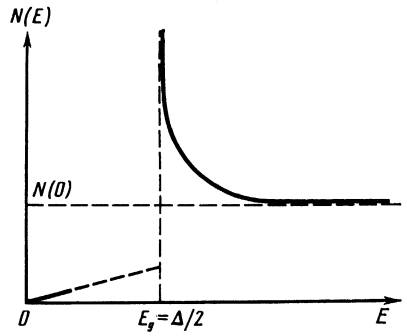

FIG. 1. Schematic form of the density of states in a superconducting superlattice.

pend only on the single coordinate x, and the equations take the form

$$\omega F+D(FG''-GF'')=\Delta(x)G, \tag{29a}$$

$$F^2+G^2=1, \tag{29b}$$

$$\Delta(x)=\pi\lambda(x)T\sum_{\omega}F_{\omega}, \tag{29c}$$

where $D=v_F l/3$ is the diffusion coefficient; we can take the function in (29) to be real. Thanks to the condition (29b), we can substitute $F=\sin\varphi(x)$, $G=\cos\varphi(x)$, where the function $\varphi(x)$ satisfies the equation

$$\omega\sin\varphi-D\varphi''=\Delta(x)\cos\varphi. \tag{30}$$

In the case of a small-period superlattice, the function $\varphi(x)$ is slowly varying, and we can express it in the form $\varphi=\varphi_0+\varphi_1(x)$ where $\varphi_1\ll\varphi_0$. Substituting this expression for φ into (30) and expanding in φ_1, we obtain

$$\varphi_1''-\varphi_1\left[\frac{\omega}{D}\cos\varphi_0+\frac{\Delta(x)}{D}\sin\varphi_0\right]+\frac{\Delta(x)\cos\varphi_0-\omega\sin\varphi_0}{D}=0. \tag{31}$$

The functions $\varphi(x)$ and $\Delta(x)$ are periodic with period d. Representing them in the form of Fourier series (for which φ_0 corresponds to the fundamental), we find

$$\Delta_0\cos\varphi_0-\omega\sin\varphi_0-\sin\varphi_0\sum_k\varphi_k\Delta_{-k}=0, \tag{32}$$

$$\varphi_k=\frac{\Delta_k\cos\varphi_0}{Dk^2+\omega\cos\varphi_0},\qquad k=\frac{2\pi n}{d},\qquad n=0,\pm1,\ldots .$$

Using also an expansion of the self-consistency equation (29c) in a Fourier series, we are led to the following self-consistency equation for $\varepsilon=d^2\omega_D/D\ll1$:

$$\Delta_0=\pi T\left(\lambda_0+\sum_k\frac{\lambda_{-k}\lambda_k}{k^2}\frac{\omega_D}{D}\right)\sum_{\omega}\frac{\Delta_0}{(\omega^2+\Delta_0^2)^{1/2}}. \tag{33}$$

Thus, the zero-order equation in ε coincides with the usual self-consistency equation where Δ_0 and λ_0 play the roles of Δ and λ—i.e., the average values of the corresponding quantities. To first order in ε, according to (33), the effective interaction parameter is renormalized; the same renormalization also figures into expression (7) for the superlattice T_c. In the case $d^2\omega_D/D\gg1$ the gap Δ_0, along with T_c, is determined by an effective constant (see Section 2).

The Green's function for the superlattice $G_0=\cos\varphi_0\approx\omega(\omega^2+\Delta_0^2)^{-1/2}$ to first order in ε coincides with the Green's function for a homogeneous superconductor. As a result, the density of states is the same as for a normal superconductor with a gap $E_g=\Delta_0$. Thus, the BCS relation between E_g and T_c is preserved even in an impure system.

6. CONCLUSION

Let us now summarize the basic results of our investigation for the model of free electron motion through the superlattice S/N (where the layers S and N have identical properties except for the constant λ).

1. The superconductivity has a homogeneous character, i.e., the Green's functions G and F are almost constant for $d\ll\xi$. As a result, the magnetic properties of superconducting superlattices possess no distinctive features of their own. In this connection, let us emphasize the difference between the situation we have investigated and the case of an inhomogeneous superconductor, which was investigated theoretically in detail in Ref. 26. In the latter case, the superconductivity is localized in regions which are randomly distributed throughout the sample, and its characteristics can be described by means or percolation theory.[26]

2. The critical temperature T_c is determined by an effective constant λ_{eff}. The relation between the gap E_g in the tunneling density of states (E_g corresponds to the maximum value of this density of states) and T_c is the same as in homogeneous superconductors within the BCS model.

3. In a pure system there exist states within the gap; their density is determined by Eq. (18) and the corresponding contribution to the specific heat (quadratic in the temperature) is given by Eq. (28). The gapless character of superconductivity in pure superlattices distinguishes them from ordinary homogeneous superconductors.

Experimental data for E_g and T_c for the superlattices Nb/Cu are found to agree with the conclusions of paragraph 2; however, the value of T_c for small layer thicknesses d is higher than that predicted by the model we have investigated. This disagreement can be related to the crudeness of our model of free electron motion, in which reflection of electrons from the boundaries of the S and N layers is neglected. A more realistic model of electronic motion in the superlattice must take into account the reflection of an electron from these boundaries together with scattering of electrons by impurities. The gapless character of the spectrum in a pure superlattice requires a thorough investigation, since limitations on the purity of the sample are rather stringent; over a length ξ there must be neither impurities nor variations in the layer thickness.

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