Kinetic theory of the thermal conductivity of He II in capillaries

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The thermal conductivity of He II in capillaries is found as a function of the temperature and the thickness of the capillary. The results have been obtained for both low pressures (when the phonon spectrum is slightly unstable) and relatively high pressures (when the decay processes are forbidden).

1. INTRODUCTION

The Landau-Khalatnikov theory of kinetic phenomena in He II (Refs. 1–3) is based on the assumption that the phonon spectrum is stable. A refinement of the phonon dispersion law shows that, up to pressures P of roughly 15 or 18 atm

$$\gamma=\frac{1}{6c}\left(\frac{\partial^3\varepsilon}{\partial p^3}\right)_{p=0}>0,$$

(ε and p are the phonon energy and momentum, respectively, and c is the velocity of sound), and, thus, the decay processes (the processes $1\rightarrow2$) and not forbidden.[4–7] But because the parameter γ is so small, the processes $1\rightarrow2$ are small-angle processes (i.e., all the three phonons have almost collinear momenta). Therefore, these processes cause insignificant changes in the previously found[1–3] values and temperature dependences of the kinetic coefficients, the computation of which was based on the assumption that the fastest processes in He are the establishment of equilibrium in the roton gas and the establishment of equilibrium—in energy terms—for the phonons with a given momentum direction. Phonon scattering by the rotons is the process that establishes total equilibrium in the phonon gas. Nevertheless, as will become clear below, the change in sign of the parameter γ (at $P\sim18$ atm), a change that results in the stability of the phonons, has a slight effect on the magnitude of the kinetic characteristics, but does not change their temperature dependence.

In the presence of a temperature gradient in superfluid He, besides the macroscopic liquid motion, which is accompanied by heat transport by the normal component, a nonreversible heat flow occurs:

$$\mathbf{q}=-\varkappa\nabla T.$$

The thermal conductivity coefficient $\varkappa$ is the sum of the thermal conductivity coefficients for the photons and rotons: $\varkappa=\varkappa_p+\varkappa_r$. Above 0.9 °K the thermal conductivity coefficient $\varkappa_r$ for the rotons predominates; it is determined by the roton-roton scattering.[3,8] For $T<0.9$ °K the dominant contribution to the thermal conductivity is made by the phonons. In the region 0.6–1.4 °K the phonon-governed thermal conductivity is determined by the scattering of the phonons by the rotons, which is characterized by a relaxation time $\tau_{\rm pr}$. According to Khalatnikov,[3]

$$\varkappa_{\rm p}={}^1/_3C_{\rm p}cL_{\rm pr}\,(1-ST/\rho_nc^2)^2;$$

$$L_{\rm pr}=c\tau_{\rm pr},\qquad \tau_{\rm pr}=\frac{\hbar^7\rho^2c^5(kT)^{-9/2}e^{\Delta/kT}}{(2\pi)^{3/2}p_0{}^4\mu^{1/2}\Gamma}, \tag{1}$$

$$\Gamma=\frac{2}{9}+\frac{1}{25}\left(\frac{p_0}{\mu c}\right)^2+\frac{2A}{9}\frac{p_0}{\mu c}+A^2,$$

$$A=\frac{\rho^2}{p_0c}\left[\frac{\partial^2\Delta}{\partial\rho^2}+\frac{1}{\mu}\left(\frac{\partial p_0}{\partial\rho}\right)^2\right].$$

Besides $L_{\rm pr}=c\tau_{\rm pr}$, all the designations are conventional designations ($C_{\rm p}$ and S are the phonon specific heat and entropy per unit volume; ρ_n and ρ are the normal-component and total helium densities; Δ, p_0, and μ are the parameters of the roton energy spectrum; and k is the Boltzmann constant).

The formula (1) was derived under the assumption that the phonons moving in one direction are in a state of quasi-equilibrium. For $T<1$ °K the thermal conductivity can indeed be computed under the assumption that the equilibrium—in energy terms—for the phonons moving in the same direction (at $P<18$ atm, the process $1\rightarrow2$) is established instantaneously, but for $T>1$ °K it is necessary to allow for the finiteness of the time $\tau_{1\rightarrow2}$. Adamenko and Slyusarev[9] have obtained a modified expression for the phonon thermal conductivity coefficient, using Maris's[10] results for the time $t_{1\rightarrow2}(\varepsilon)$. For $\tau_{1\rightarrow2}\rightarrow0$, the corresponding Adamenko-Slyusarev formula[9] naturally goes over into (1). Allowance for the finiteness of the time $\tau_{1\rightarrow2}$ leads to an approximately 10% increase in $\varkappa_{\rm p}$ at 1 °K and to as much as 80% increase at 1.4 °K. But since the relative contribution of the phonons to the total thermal conductivity decreases with increasing temperature, the effect of the finiteness of the time $\tau_{1\rightarrow2}$ is not too important.

For $P\gtrsim18$ atm, $\tau_{1\rightarrow2}\rightarrow\infty$, and all the remaining phonon times increase approximately by two orders of magnitude (Adamenko and Tsyganok[11]). As shown in Refs. 11 and 12, in the region of high pressures (18–24 atm) an important role is played in the kinetic phenomena by the relaxation time $t(\epsilon)$ due to the emission (absorption) of phonons by the rotons, and we must, in computing the thermal conductivity coefficient allow for four relaxation times: $t(\varepsilon)$, $t_{\rm pr}(\varepsilon)$, $t_{2\rightarrow2}(\varepsilon)$ (four-phonon small-angle scattering), and $t^{\perp}_{2\rightarrow2}(\varepsilon)$ (four-phonon large-angle scattering). In Ref. 11 an expression for $\varkappa_p$ is derived which is valid for this region of pressures.

Above we discussed the thermal conductivity of an infinite volume of He. The influence of the size effects on the kinetic coefficients was first taken into consideration by Atkins,[13] who considered the case when the roton motion is similar to the flow of a viscous liquid and the phonons constitute a Knudsen gas, i.e., when the following inequalities are valid

$$L_r \ll D \ll L_p,$$

where D is the thickness of the plane-parallel capillary and $L_{r(p)}$ is the roton (phonon) mean free path. Using Atkins's hydrodynamic equations[13] for He, we have derived expressions for the phonon- and roton-transported heat fluxes, and used them to derive expressions for the effective thermal conductivity coefficients:

$$\varkappa_p^{\text{eff}} = \frac{2S_pTD}{f\rho_p c p}(\rho_p S+\rho_s S_p), \qquad \varkappa_r^{\text{eff}} = \frac{S_r TD^2}{12\eta_r \rho}(\rho_r S+\rho_s S_r).$$

Here $S_{p(r)}$, $\rho_{p(r)}$, and η_r are the phonon (roton) parts of the entropy, density, and viscosity, respectively, ρ_s is the density of the superfluid component of helium, and f is the fraction of phonons diffusely reflected from the walls ($f \leqslant 1$). When we go over to a circular capillary, we should replace D by R (R is the radius of the capillary) and the factor 1/12 in the second formula by 1/8. We should, however, bear in mind that, strictly speaking, $\varkappa^{\text{eff}}$ describes not the thermal conductivity proper, but the transfer of heat as a result of the flow of the normal component (see above). As far as we know, the influence of the size effects on the true thermal conductivity of He II has not been estimated.

In the present paper we report the results of a theoretical investigation of the influence of the size effect on the thermal conductivity in the temperature region from 0.6 to 1.4 °K. The size effects in this region are important only for the phonon-governed thermal conductivity, since in capillaries of reasonable dimensions (10^{-5}–10^{-3} cm) the manifestation of the size effects in the roton system can be expected at $T < 0.9$ °K and at these temperatures the contribution of the rotons to the thermal conductivity is negligibly small ($\varkappa_r \ll \varkappa_p$).

In this paper we assume that the capillary is sufficiently wide, so that the quantization of the transverse phonon motion can be ignored, i.e., $D/\lambda \gg 1$, where λ is the wavelength of a phonon with energy $\varepsilon \sim 7kT$. Hence

$$kTD/c\hbar \gg 1 \quad \text{or} \quad T(K) \gg 10/D(\text{Å}).$$

It can be seen that, in the case of capillaries with 10^{-5} cm $\lesssim D \lesssim 10^{-3}$ cm, this condition does not impose significant limitations.

Underlying the analysis is the assumption made about the nature of the interaction of the phonons with the surface: the scattering by the walls is assumed to be entirely diffuse scattering. This assumption should hold on extremely rough walls. It enables us to determine the maximum influence of the size effects on the thermal conductivity,[1)] making the collisions with the wall the cause of the required dissipation of the momentum of the directed motion of the phonon gas.

The existence of several types of relaxation processes in a phonon gas can lead to a peculiarity of size effects in He II, since the scattering by the rough wall should "split off" those dissipation mechanisms for which the mean free path L is significantly greater than the capillary thickness D. But as an analysis of the electrical conductivity of thin plates shows,[15,16] because of the quasiparticles flying parallel to the boundaries (of the plate or capillary), the "memory" of the internal dissipation mechanism (with mean free path L) is retained in the expression for the kinetic coefficient: when $L \gg D$, the kinetic coefficients contain the characteristic factor $\ln(L/D)$. This phenomenon, which has been studied on conduction electrons, is investigated in the present paper for phonons—quasiparticles obeying the Bose-Einstein statistics, and therefore requiring that allowance be made for their energy distribution.

2. THE τ APPROXIMATION

In order to get qualitative ideas about the role of the size effects, we shall use the τ approximation, i.e., we shall replace the true collision integral $I\{n\}$ by the expression $-n_1/\tau(\varepsilon)$, where $n_1 = n - n_0$ is the correction to the equilibrium phonon distribution function

$$n_0=\{\exp\,[(cp-\mathbf{p}(\mathbf{v}_n-\mathbf{v}_s))/kT]-1\}^{-1},$$

produced by the temperature gradient $\partial T/\partial x$; $\tau(\varepsilon)$ is a phenomenological parameter—the energy-dependent phonon-relaxation time; and $\mathbf{v}_n$ and $\mathbf{v}_s$ are the velocities of the normal and superfluid components. Let us write down the linearized kinetic equation for the function n_1, assuming that the temperature gradient is directed along the x axis, while the normal to the surface of the plane-parallel capillary of thickness $D = 2d$ is oriented along the z axis:

$$v_z\frac{\partial n_1}{\partial z} - I\{n_1\} = \frac{c^2 n_0' p_x}{kT^2}\left(1-\frac{ST}{\rho_n c^2}\right)\frac{\partial T}{\partial x}. \tag{2}$$

Here $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$, $n_0' = -n_0(n_0+1)$, and $n_0 = (e^{cp/kT}-1)^{-1}$. We have discarded in n_0 the convective term (i.e., the term $-\mathbf{p}(\mathbf{v}_n - \mathbf{v}_s)$ in the expression for the phonon energy). It is convenient to make allowance for it in the final expression for the heat flux, as shown in Ref. 3: to do this we must multiply the expression obtained with the aid of the fluctuation n_1 by $(1 - ST/\rho_n c^2)$. This factor, which occurs also in the right member of the kinetic equation (2), is equal to zero for a phonon gas with a strictly linear dispersion law (see Ref. 3), and we must take this into account when substituting the values of ρ_n and S into all the formulas (in particular, into the formula (1) and the Adamenko-Slyusarev formula,[9] which, naturally, take account of the convective term).

The total diffuseness of the phonon reflection from the capillary walls dictates the boundary conditions for the function n_1 at the walls:

$$n_1(z=d;\ v_z<0) = n_1(z=-d;\ v_z>0) = 0. \tag{3}$$

Replacing the collision integral in the kinetic equation (2) by $-n_1/\tau(\varepsilon)$, and solving the resulting differential equation with allowance for the boundary conditions (3), we obtain

$$n_1=\frac{c^2n_0'}{kT^2}\left(1-\frac{ST}{\rho_nc^2}\right)p_x\tau(\varepsilon)\left[1-\exp\left(-\frac{z\pm d}{v_z\tau(\varepsilon)}\right)\right]\frac{\partial T}{\partial x}.$$

The upper sign corresponds to $v_z>0$; the lower sign, to $v_z<0$. The thermal conductivity coefficient $\langle\varkappa_{\rm p}\rangle$ averaged over the width of the capillary is defined as follows:

$$\langle\varkappa_{\rm p}\rangle=-\frac{1}{2d}\int_{-d}^{d}dz\int\varepsilon v_xn_1\,d^3p\Bigg/\frac{\partial T}{\partial x};\qquad d^3p=\frac{dp_x\,dp_y\,dp_z}{(2\pi\hbar)^3}. \tag{4}$$

Hence, changing the order of integration, and going over to spherical coordinates in **p** space (the angle θ is measured from the z axis), we obtain

$$\langle\varkappa_{\rm p}\rangle=\frac{4\pi(1-ST/\rho_nc^2)^2}{3(2\pi\hbar)^3kT^2c}\int_0^\infty\frac{\varepsilon^4e^{\varepsilon/kT}\tau(\varepsilon)}{(e^{\varepsilon/kT}-1)^2}\Phi(\varepsilon)\,d\varepsilon, \tag{5}$$

where

$$\Phi(\varepsilon)=\frac{3}{2}\int_0^{\pi/2}\left[1-\frac{\tau(\varepsilon)c}{D}\times\left(1-\exp\left\{-\frac{D}{c\tau(\varepsilon)\cos\theta}\right\}\right)\cos\theta\right]\sin^3\theta\,d\theta. \tag{6}$$

It is natural to consider the two limiting cases of a wide ($D\gg c\,\overline{\tau(\varepsilon)}$) and a narrow ($D\ll c\,\overline{\tau(\varepsilon)}$) capillary. Strictly speaking, the mean relaxation time $\overline{\tau(\varepsilon)}$ should be defined more accurately in the computation, but if the integrals obtained as a result of the expansions do not diverge (for this to be the case it is, in particular, necessary that $\tau(\varepsilon)$ not go to infinity as $\varepsilon\to0$ faster than ε^{-1}), then we can use the following procedure. Let $\tau(\varepsilon)=\tau\chi(\varepsilon/kT)$, where τ characterizes the order of magnitude of $\tau(\varepsilon)$ for $\varepsilon\sim kT$. Then we can use the ratio $c\tau/D$, instead of $c\,\overline{\tau(\varepsilon)}/D$, when making estimates.

In the case of a wide capillary ($D\gg c\tau$) we should neglect the exponential function in the expression for $\Phi(\varepsilon)$. Then

$$\langle\varkappa_{\rm p}\rangle\approx\varkappa_{\rm p}\left(1-\frac{3}{8}\frac{c\overline{\tau(\varepsilon)^D}}{D}\right),\qquad c\overline{\tau(\varepsilon)^D}\ll D. \tag{7}$$

Here

$$\overline{\tau(\varepsilon)^D}=\frac{\int_0^\infty\tau^2(\varepsilon)\varepsilon^4n_0'\,d\varepsilon}{\int_0^\infty\tau(\varepsilon)\varepsilon^4n_0'\,d\varepsilon}=\tau\int_0^\infty\frac{\chi^2(x)x^4e^x}{(e^x-1)^2}dx\left[\int_0^\infty\frac{\chi(x)x^4e^x}{(e^x-1)^2}dx\right]^{-1}. \tag{8}$$

The value of the thermal conductivity coefficient for bulk He II ($D\to\infty$) is given by the usual formula [see (1)], but we should substitute in place of $L_{\rm pr}$ the expression

$$L=\frac{15}{4\pi^4}c(kT)^{-5}\int_0^\infty\frac{\varepsilon^4e^{\varepsilon/kT}\tau(\varepsilon)\,d\varepsilon}{(e^{\varepsilon/kT}-1)^2}=\frac{15c\tau}{4\pi^4}\int_0^\infty\frac{x^4e^x\chi(x)\,dx}{(e^x-1)^2}. \tag{9}$$

Notice (and this is one of the principal results of the present section) that the quantity $c\,\overline{\tau(\varepsilon)^D}\neq L$.

In the case of a narrow capillary ($D\ll c\tau$) we can use the asymptotic expression for $\Phi(\varepsilon)$ for ($D\to0$):

$$\Phi(\varepsilon)\approx\frac{3D}{4\tau(\varepsilon)c}\left(\ln\frac{c\tau(\varepsilon)}{D}+0.42\right), \tag{10}$$

whose substitution into (5) yields

$$\langle\varkappa_{\rm p}\rangle=\frac{1}{3}C_{\rm p}c\left(1-\frac{ST}{\rho_nc^2}\right)^2\frac{3}{4}D\left(\ln\frac{c\tau}{D}+\gamma\right), \tag{11}$$

where

$$\gamma=0.42+\frac{15}{4\pi^4}\int_0^\infty\frac{x^4e^x}{(e^x-1)^2}\ln[\chi(x)]dx.$$

Comparing the limiting value of the thermal conductivity coefficient (11) with (1) [with the mean free path L given by the formula (9)], we see that the expression (11) can be represented as

$$\langle\varkappa_{\rm p}\rangle=\varkappa_{\rm p}\frac{3}{4}\frac{D}{L}\left(\ln\frac{c\tau}{D}+\gamma\right),\qquad c\tau\gg D. \tag{12}$$

Let us draw attention to the characteristic logarithmic dependence on the ratio of the mean free path to the capillary thickness, a dependence which is due to the phonons that scarcely collide with the surface (as we have indicated, a similar dependence is observed in the electrical conductivity of thin metallic plates). Interesting information about the phonon energy dispersion is contained in all the expressions in which the integrand contains the function $\chi(x)$. It should, however, be remembered that the formulas derived in this section are of a simplified nature, since they are based on the τ approximation.

Further, even though the relation between D and $c\tau$ will not allow the logarithmic factor to manifest itself fully, the resulting expressions indicate that a natural transition from L to D should occur in the expression for $\langle\varkappa_{\rm p}\rangle$ as the capillary thins down (or as the mean free path increases with decreasing temperature). The dominant temperature dependence of the thermal conductivity in the case of a narrow capillary is determined by the coefficient in front of the logarithm: it does not depend on the dissipation mechanisms; the role of the mean free path is played by the capillary thickness in the case when $c\tau\gg D$.

Let us emphasize again that we cannot replace $\tau(\varepsilon)$ in Eq. (8) or (9) by the phonon relaxation time[3]

$$t_{\rm pr}(\varepsilon)=\tau_{\rm pr}\left(\frac{2\pi kT}{\varepsilon}\right)^4 \tag{13}$$

[for the meaning of $\tau_{\rm pr}$, see the formula (1)], since because of the rapid ($\propto\varepsilon^{-4}$) increase of $\tau(\varepsilon)$ with decreasing energy, these integrals diverge.

3. THE SIZE EFFECT AT LOW PRESSURES

In the temperature range (0.6–1.4 °K) of interest to us, the following processes are, as we have already indicated, responsible for the establishment of equilibrium in the phonon gas at low pressures: the decay of a phonon into two [this process establishes equilibrium among the phonons with a given momentum direction $\mathbf{j} = \mathbf{p}/p$; its relaxation time is $t_{1\to 2}(\varepsilon)$] and also the phonon-roton scattering [it isotropizes the phonon distribution function; its relaxation time is given by the expression (13)]. The remaining relaxation processes can be ignored, since their characteristic times are much longer than the relaxation times $\tau_{1\to 2}$ and $\tau_{\rm pr}$. Considering the time $\tau_{1\to 2}$ to be instantaneous in comparison with $\tau_{\rm pr}$, we can ignore the deviation of the phonon distribution function from the Bose function

$$n=(e^{\varepsilon\beta}-1)^{-1},$$

in which, however, the reciprocal temperature β is a function of $\mathbf{j}$ and z (but does not depend on ε!). To first order in $\partial T/\partial x$

$$\beta=\beta_0+\beta_1(\mathbf{j},\ z),\qquad n_1=n_0'\varepsilon\beta_1, \tag{14}$$

$\beta_0 = 1/kT_0$, where T_0 is the equilibrium He-II temperature.

To determine the function β_1, we must multiply the linearized kinetic equation (2), in which the relaxation times $\tau(\varepsilon)$ should be taken to be equal to $t_{\rm pr}(\varepsilon)$, by the phonon energy, and integrate over the modulus of the momentum. Taking (14) into account, we have

$$c\cos\theta\frac{\partial\beta_1}{\partial z}+\frac{1}{\tau_{\rm pr}}\beta_1=\sin\theta\cos\varphi c\left(1-\frac{ST}{\rho_n c^2}\right)\frac{1}{kT^2}\frac{\partial T}{\partial x}. \tag{15}$$

The boundary conditions for this equation naturally coincide with the boundary conditions for n_1 [see (3)]. Computing the function β_1 (and, with its aid, n_1), we obtain for the averaged thermal conductivity coefficient the expression

$$\langle\varkappa_{\rm p}\rangle=\frac{1}{3}C_{\rm p}cL_{\rm pr}\left(1-\frac{ST}{\rho_n c^2}\right)^2 J\left(\frac{c\tau_{\rm pr}}{D}\right), \tag{16}$$

where

$$J(\xi)=\frac{3}{2}\int_0^{\pi/2}\left[1-\xi\left(1-\exp\left\{-\frac{1}{\xi\cos\theta}\right\}\right)\cos\theta\right]\sin^3\theta\,d\theta. \tag{17}$$

From these expressions we immediately obtain the limiting values of the averaged thermal conductivity coefficient:

$$\langle\varkappa_{\rm p}\rangle\approx\begin{cases}\varkappa_{\rm p}\left(1-\dfrac{3}{8}\dfrac{L_{\rm pr}}{D}\right), & L_{\rm pr}\ll D\\[2ex] \dfrac{3}{4}\varkappa_{\rm p}\dfrac{D}{L_{\rm pr}}\left(\ln\dfrac{L_{\rm pr}}{D}+0{,}42\right), & L_{\rm pr}\gg D.\end{cases} \tag{18}$$

The dependence of the averaged thermal conductivity coefficient on the dimensions of the capillary in this case is identical to the plate-thickness dependence of the electrical conductivity,[15] with, of course, $I_{\rm pr}$ replaced by the electron mean free path l_e. The reason for this lies in the absence of dispersion, i.e., the fact that β_1 does not depend on the energy (analogous to the degeneracy of the electron gas in a metal).

There arises the question: If we allow for the finiteness of the relaxation time $\tau_{1\to 2}$, can we, in the case when $D \lesssim L_{\rm pr}$, neglect the phonon-roton scattering at all? In other words, it is not possible for the scattering on the rough surface to serve as the "final" relaxation mechanism in the computation of the coefficient of thermal conductivity of He II in a thin plane-parallel capillary? The answer is no. Indeed, writing the linearized kinetic equation with the collision integral $I_{1\to 2}\{n\}$ (in, for example, the form proposed by Callaway[17]), and taking account of the fact that the energy in the three-phonon process is conserved, i.e.,

$$\int I_{1\to 2}\{n\}\varepsilon p^2\,dp=0,$$

we immediately obtain for β_1 an equation that does not contain a relaxation term [cf. (15)]. It is clear that in this case $\beta_1 \propto (\cos\theta)^{-1}$, and the expression for $\langle\varkappa_{\rm p}\rangle$ diverges because of the phonons that do not collide with the surface of the capillary, with the result that allowance for the phonon-roton collisions is necessary. Furthermore, it can be shown [true, with the aid of simplifications that are not too well justified, namely, the introduction of the averaged times $\tau_{1\to 2}$ and $\tau_{\rm pr}$ instead of the energy dependent times $t_{1\to 2}(\varepsilon)$ and $t_{\rm pr}(\varepsilon)$] that the three-phonon relaxation time does not enter into the capillary-thickness dependence of $\langle\varkappa_{\rm p}\rangle$. This, it seems to us, shows that (18) correctly describes the size effect in He II at low pressures.

4. THE SIZE EFFECT AT HIGH PRESSURES

For $P > 18$ atm (according to some data, for $P > 15$ atm) the three-phonon processes are forbidden. We shall base our analysis on the assumption that the fastest relaxation process is the four-phonon small-angle scattering process with characteristic time $\tau_{2\to 2} \ll \tau_{\rm pr}$. This case is apparently realized at pressures $P \gtrsim 18$ atm and temperatures ~ 0.6 °K. It is interesting that the replacement of the instantaneous three-phonon process by instantaneous small-angle four-phonon process by instantaneous small-angle four-phonon scattering leads to a significant (roughly a factor of 10) change in the coefficient of thermal conductivity of an infinite volume of helium, although the three- or four-phonon relaxation time does not itself enter into the expression for the kinetic coefficient. As in the preceding section, the phonon-roton scattering serves as the "final" relaxation mechanism.

Let us consider the size effect in this situation. The distribution function can then be represented in the form

$$n=\{\exp\left[(\alpha+\varepsilon)\beta\right]-1\}^{-1},$$

since the small-angle four-phonon scattering process establishes a Bose-Einstein distribution with a nonzero chemical potential α and a temperature $1/\beta$ that depend on the direction $\mathbf{j}$ of motion of the phonons. The linear (in $\partial T/\partial x$) correction to the equilibrium phonon distribution function is now determined by the functions α and β_1 [cf. (14)]:

$$n_1=n_0'(\alpha\beta_0+\varepsilon\beta_1), \tag{19}$$

while the $\mathbf{j}$- and z-dependent (but ε-independent) functions α and β_1 should be determined from the kinetic equation (2), with $I\{n_i\} = I_{\rm pr}\{n_1\}$. Using for $I_{\rm pr}\{n_1\}$ the τ approximation with the relaxation time (13), and integrating (2) twice (once directly over the moduli of the momenta and a second time after it has been multiplied by the phonon energy), we obtain a system of two equations for the two functions α and β_1. Its solution [with zero boundary conditions for the phonons leaving the walls of the capillary[2)]; cf. (3)] has the form

$$\alpha^{\pm} = \sin\theta\cos\varphi c\tau_{\rm pr}\left(1-\frac{ST}{\rho_n c^2}\right)k\frac{\partial T}{\partial x}\left\{a_1\exp\left(-\frac{z\pm d}{L_{\rm pr}^{(1)}\cos\theta}\right)\right.$$

$$\left.+a_2\exp\left(-\frac{z\pm d}{L_{\rm pr}^{(2)}\cos\theta}\right)+a_3\right\}, \qquad (20)$$

$$\beta^{\pm} = \sin\theta\cos\varphi c\tau_{\rm pr}\left(1-\frac{ST}{\rho_n c^2}\right)\frac{1}{kT^2}\frac{\partial T}{\partial x}\left\{b_1\exp\left(-\frac{z\pm d}{L_{\rm pr}^{(1)}\cos\theta}\right)\right.$$

$$\left.+b_2\exp\left(-\frac{z\pm d}{L_{\rm pr}^{(2)}\cos\theta}\right)+b_3\right\}.$$

The plus sign (+) corresponds to $\cos\theta>0$; the minus sign (−), to $\cos\theta<0$. The numbers

$a_1\approx-59.5,\quad a_2\approx1.42.\quad a_3\approx58.0,$

$b_1\approx7.07,\quad b_2\approx-0.835,\quad b_3\approx-6.23$

are the results of an approximate computation of the integrals containing the Bose function. The quantities

$$L_{\rm pr}^{(1)}\approx30\tau_{\rm pr}c,\qquad L_{\rm pr}^{(2)}\approx0.7\tau_{\rm pr}c,$$

have the meaning of phonon mean free paths. They have the same temperature dependence, which is explained by the fact that we took into account only one relaxation process: the scattering of the phonons on the rotons. Note that the condition $L_{\rm pr}^{(1)}\gg L_{\rm pr}^{(2)}$ ($L_{\rm pr}^{(1)}/L_{\rm pr}^{(2)}\approx50$) enables us to consider the intermediate asymptotic form in the capillary thickness D.

Using (19) and (20), we can easily compute the value of the capillary-thickness-averaged thermal conductivity coefficient $\langle\varkappa_{\rm p}\rangle$. It formally coincides with (16), but the function $J(c\tau_{\rm pr}/D)$ has a significantly more complicated form than (17):

$$J\left(\frac{c\tau_{\rm pr}}{D}\right)=\frac{45}{8\pi^4}\int_0^{\infty}\int_0^{\pi/2}\frac{x^3e^x\sin^3\theta}{(e^x-1)^2}\left\{(a_1+b_1x)\frac{L_{\rm pr}^{(1)}\cos\theta}{D}\right.$$

$$\times\left[1-\exp\left(-\frac{D}{L_{\rm pr}^{(1)}\cos\theta}\right)\right]+(a_2+b_2x)\frac{L_{\rm pr}^{(2)}\cos\theta}{D}$$

$$\left.\times\left[1-\exp\left(-\frac{D}{L_{\rm pr}^{(2)}\cos\theta}\right)\right]+(a_3+b_3x)\right\}dx\,d\theta. \qquad (21)$$

The integral J is easily evaluated in the limiting cases.

A. Thick capillary ($D\gg L_{\rm pr}^{(1)}\gg L_{\rm pr}^{(2)}$):

$J\approx9.87-3.54L_{\rm p}^{(1)}/D.$

As the capillary thickness tends to infinity, the integral J approaches the limiting value $J_\infty\approx9.87$. This factor shows that replacing the three-phonon process ($\tau_{1\to2}=0$) by the four-phonon process ($\tau_{2\to2}=0$) causes the thermal conductivity coefficient value to increase by roughly a factor of 10 (as we have already noted). The factor J_∞ appears in the Khalatnikov formula[3] as well if we set $\tau_{2\to3}=\infty$ in it.

B. Intermediate case ($L_{\rm pr}^{(1)}\gg D\gg L_{\rm pr}^{(2)}$):

$J=1.22+7.07(D/L_{\rm pr}^{(1)})\ln(L_{\rm pr}^{(1)}/D).$

C. Narrow capillary ($L_{\rm pr}^{(1)}\gg L_{\rm pr}^{(2)}\gg D$):

$$J\approx7.07\frac{D}{L_{\rm pr}^{(1)}}\ln\frac{L_{\rm pr}^{(1)}}{D}+0.91\frac{D}{L_{\rm pr}^{(2)}}\ln\frac{L_{\rm pr}^{(2)}}{D}.$$

We have not discarded the term with $L_{\rm pr}^{(1)}$, since there is a relatively large coefficient standing in front of it.

At pressures of about 24 atm the shortest relaxation times are[11] $t_{\rm pr}$ and t. Since both of these processes lead to the total relaxation of the phonon gas, we can use the τ approximation, assuming that

$$\tau^{-1}(\varepsilon)=t_{\rm pr}^{-1}+t^{-1},\qquad t^{-1}=\frac{p_0^2N_{\rm r}}{5\rho\mu c^2}\frac{\omega^4\tau_{\rm r}}{(1+\omega^2\tau_{\rm r}^2)},\qquad \varepsilon=\hbar\omega, \qquad (22)$$

where $N_{\rm r}$ is the number of rotons in a unit volume and $\tau_{\rm r}$ is a parameter governed by the roton-roton interaction (see Ref. 11), and compute $\langle\varkappa_{\rm p}\rangle$ with the aid of the formulas (5) and (6). But the capillary-thickness-dependence of the averaged thermal conductivity coefficient requires further treatment, since for a wide capillary the energy dependence $\tau=\tau(\varepsilon)$, (22), leads to the divergence of the correction term. At the same time L can be computed from the formula (9).

Let us, without giving the temperature dependence of $\varkappa_{\rm p}=\langle\varkappa_{\rm p}\rangle$ [it is determined by the relation connecting the parameters contained in the formulas (22) and (7)], find out how the first correction to $\varkappa_{\rm p}$ depends on D in the case of a wide capillary. The factor attached to $1/D$ contains, in accordance with the formula (6), the integral (23)

$$J=\int_0^{\infty}\frac{\varepsilon^4e^{\varepsilon/kT}\tau^2(\varepsilon)\,d\varepsilon}{(e^{\varepsilon/kT}-1)^2}$$

$$\times\int_0^{\pi/2}\left\{1-\exp\left]-\frac{D}{c\tau(\varepsilon)\cos\theta}\right]\right\}\cos\theta\sin^3\theta\,d\theta. \qquad (23)$$

If we neglect the exponential function, the energy integral will diverge, since according to (22), $\tau(\varepsilon)\propto\varepsilon^{-2}$ for $\varepsilon\to0$. This means that, at sufficiently large values of D the value of the integral is determined by the behavior of the integrand as $\varepsilon\to0$, where

$$\tau(\varepsilon)\approx t(\varepsilon)=t_0\left(\frac{kT}{\varepsilon}\right)^2,\qquad t_0=\frac{5\hbar^2\rho\mu c^2}{p_0^2N_{\rm p}\tau_{\rm p}(kT)^2}. \qquad (24)$$

Substituting (24) into the integral (23), and replacing ε by the dimensionless variable $x=(D/ct_0)^{1/2}(\varepsilon/kT)$, we obtain

$$J=t_0^2(kT)^5\left(\frac{ct_0}{D}\right)^{1/2}\int_0^{\infty}\frac{dx\exp[(ct_0/D)^{1/2}x]}{\{\exp[(ct_0/D)^{1/2}x]-1\}^2}$$

$$\times\int_0^{\pi/2}\{1-\exp(-x^2/\cos\theta)\}\cos\theta\sin^3\theta\,d\theta.$$

Since $ct_0\ll D$, we can expand the exponential functions. Then

$$J=t_0^2(kT)^5\left(\frac{D}{ct_0}\right)^{1/2}\int_0^\infty\frac{dx}{x^2}\int_0^{\pi/2}\left[1-\exp\left(-\frac{x^2}{\cos\theta}\right)\right]\cos\theta\sin^3\theta d\theta.$$

Now we can remove $\cos\theta$ from the exponential function. As a result

$$J={}^8/_{21}t_0^2(kT)^5(\pi D/ct_0)^{1/2}.$$

Thus,

$$\langle\varkappa_\mathrm{p}\rangle=\varkappa_\mathrm{p}\left[1-\frac{15\pi^{1/2}}{7\pi^4}\frac{ct_0}{L}\left(\frac{ct_0}{D}\right)^{1/2}\right]. \qquad (25)$$

Note that the dispersion of the relaxation time had a significant effect on the dependence of $\langle\varkappa_\mathrm{p}\rangle$ on D: the square root dependence is a consequence of the asymptotic formula (24).

To compute $\langle\varkappa_\mathrm{p}\rangle$ for $D\to 0$, we can use the formula (10), since the resulting integral does not diverge. From it we obtain the following characteristic thickness dependence of the averaged thermal conductivity coefficient:

$$\langle\varkappa_\mathrm{p}\rangle=\frac{1}{3}C_\mathrm{p}c\left(1-\frac{ST}{\rho_n c^2}\right)^2\frac{3}{4}D\ln\frac{L_0}{D}, \qquad (26)$$

the L_0 value being dependent on the relation between t and t_pr at $\varepsilon\sim 7kT$, where the function $\varepsilon^4 e^{\varepsilon/kT}(e^{\varepsilon/kT}-1)^{-2}$ has its maximum.

The size effect in helium at temperatures ranging from 0.6 to 1.4 °K can be observed at low pressures in capillaries of thickness D lying in the range from 10^{-5} to 10^{-2} cm and at high pressures in the case $10^{-5}\lesssim D\lesssim 10^{-1}$ cm. Unfortunately, we do not know of any experiments in which the thermal conductivity of He II was measured as a function of the capillary thickness, and with which we could compare the expressions obtained. Further, we wish to point out the desirability of experimental observation of the thermal conductivity coefficient ($\varkappa$) anomaly due to the change in sign of γ (see the Introduction). As far as we know, there has not been any experimental search for the anomaly.

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1) The diffusivity coefficient and its effect on the boundary conditions for the phonon distribution function have been investigated by Adamenko and Fuks.[14]

2) At the walls α and β_1 vanish independently. The vanishing of the sum $\varepsilon\beta_0+\varepsilon\beta_1$ (with $\alpha\neq 0$ and $\beta=0$) would make these quantities depend on ε, which would be contrary to our assumption.

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