Solution of the problem of anomalies in supersymmetric gauge theories, and the operator expansion

A. I. Vaĭnshteĭn and M. A. Shifman

Institute of Theoretical and Experimental Physics
(Submitted 26 February 1986)
Zh. Eksp. Teor. Fiz. **91**, 723–744 (September 1986)

This article completes a series of papers on β-functions and the problem of anomalies in supersymmetric theories. Exact expressions for the β-functions are obtained in the framework of standard perturbation theory. The key observation is that the Wilson effective action $S_W(\mu)$ does not coincide with the sum $\Gamma(\mu)$ of the diagrams for the vacuum loops in an external field. The difference is due to infrared effects. The coefficient $1/g^2$ multiplying the operator W^2 in S_W is renormalized only at the one-loop level (the generalization of the nonrenormalization theorem for F-terms). This circumstance leads to a one-loop form of the anomalous operator equation for the supercurrent (a generalization of the Adler-Bardeen theorem). The exact Gell-Mann–Low function arises after the matrix element is taken. A quantity differing from $1/g^2$ by $\Sigma \ln Z_i$, where the factors Z_i describe the renormalization of the fields, appears in the observable amplitudes. (In this sense the Z-factor of the matter fields becomes observable.) The relationship with calculations of the instanton type is discussed.

1. INTRODUCTION

It is well known that perturbation-theory series in supersymmetric models[1] possess extraordinary properties. For example, for the F-terms loop corrections are completely absent (the so-called nonrenormalization theorems[2]), and the Gell-Mann–Low function in $N=2$ gauge theories is exhausted by one loop.[3] In the present paper we shall discuss the calculation of the effective action in $N=1$ gauge supersymmetric theories, i.e., in particular, the renormalization of the gauge coupling constant.

In the literature the term "effective action" is used in practice for two different quantities. One is obtained by calculating vacuum loops in external fields. The functional Γ of the external fields that is obtained in this way is often called the effective action, although its other name—the generator of one-particle-irreducible vertices—is more accurate. We shall adhere to the latter terminology. The other construction—the Wilson construction[4] of the effective action $S_W(\mu)$—differs solely in that in the vacuum loops only the contribution of virtual momenta $p>\mu$ is taken into account. The action $S_W(\mu)$ plays the role of the initial action with respect to low-frequency fields. We have introduced the subscript W in order to emphasize the difference between the two concepts. Thus, in the framework of the Wilson procedure we deal with a normal operator expansion. The effective action in the first sense is obtained by taking matrix elements of $\exp[iS_W(\mu)]$. We emphasize that the difference between the two definitions is connected with the contribution of the infrared region $p \lesssim \mu$ pertaining to the matrix elements.

In particular, in supersymmetric gauge theories the coefficient that multiplies the $W^\alpha W_\alpha$ structure and fixes the gauge constant g is different for Γ and S_W (starting from the two-loop diagrams). We shall show that the renormalization of the coefficient of W^2 in S_W has a one-loop character. The usual gauge constant is determined from Γ. The two-loop and higher contributions to this constant correspond, in the Wilson language, to the calculation of certain matrix elements.

It is clear that the behavior of the theory in the ultraviolet region can be studied conveniently in terms of S_W. In solving the problem of the anomalies, however, the use of S_W becomes not a question of convenience but a necessity, if the anomalous equations are written in operator form. A reflection of the fact that the coefficient of W^2 in S_W is renormalized at the one-loop level is the one-loop form of the anomaly in the supertrace. It is this observation which solves the problem of the higher orders.

The problems touched upon here have a rather long history, which we shall discuss briefly; this will make it possible to formulate the results in a more concrete form. The exact expression for the Gell-Mann–Low function $\beta(\alpha)$ in supersymmetric non-Abelian models with matter has already been known for several years:

$$\beta(\alpha)=-\frac{\alpha^2}{2\pi}\Big[3T(G)-\sum_i T(R_i)(1-\gamma_i)\Big]\Big(1-\frac{T(G)\alpha}{2\pi}\Big)^{-1}, \tag{1}$$

where the sum in the right-hand side is taken over all the matter multiplets, γ_i is the anomalous dimension of the ith matter superfield,

$$\gamma_i=-d\ln Z(\mu)/d\ln\mu=-C_2(R_i)\alpha/\pi+\dots,$$

$C_2(R_i)$ is the squared Casimir operator,

$$T^aT^a=C_2(R_i)I,$$

and the coefficients $T(R_i)$ determine the normalization of the generators:

$$\mathrm{Tr}(T^aT^b)=T(R_i)\delta^{ab}.$$

We have introduced $T(G) = T(adjoint)$. We recall that

$$T(R_i) = d(R_i)C_2(R_i)/d(adjoint).$$

In supersymmetric gluodynamics, in particular, the exact β-function is determined to be

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\frac{3T(G)}{1-[T(G)\alpha/2\pi]}. \tag{2}$$

The relations (1) and (2) were obtained in Refs. 5 in the framework of instanton calculus. From the very beginning, however, it was clear that a direct derivation from perturbation theory should also exist. The fact that the β-function can be written in a simple form [e.g., in supersymmetic gluodynamics it can be represented in the form of the geometric progression (2)] could not be accidental, of course, and answers were needed to the following obvious questions:

How can formulas of the type (1), (2) be obtained in perturbation theory?

What properties of the theory are responsible for the specific structure of the series in α for $\beta(\alpha)$? Are these properties also manifested in other quantities?

A partial answer, mainly to the first question, was given in Refs. 6 and 7, in which the results (1) and (2) were reproduced without the use of instantons. Here we shall change the emphasis from the calculational to the conceptual aspect.

Another line of investigation, which led to the need for the present work, is the famous problem of the anomalies in supersymmetric theories. About ten years ago Ferrara and Zumino drew attention to the fact that the classical supercurrent $S_{\mu\alpha}$ and the energy-momentum tensor $\Theta_{\mu\nu}$ are connected by a supersymmetry transformation.[8] In Ref. 8 a method was given for constructing a supermultiplet $J_{\alpha\dot\alpha}$ that incorporates $S_{\mu\alpha}$, $\Theta_{\mu\nu}$, and, in addition, the axial current a_μ.

All three objects a_μ, $S_{\mu\alpha}$, and $\Theta_{\mu\nu}$ are conserved classically, and, as is well known, have anomalies at the quantum level. Grisaru pointed out[9] that if a_μ, $S_{\mu\alpha}$, and $\Theta_{\mu\nu}$ appear in the same supermultiplet, this property should also be possessed by the corresponding anomalies. In Ref. 9 it was demonstrated that this is indeed the case at the one-loop level.

A problem arose in the two-loop graphs, the Adler-Bardeen theorem,[10] which establishes the one-loop character of $\partial_\mu a_\mu$, came into conflict with the many-loop expression

$$\Theta_{\mu\mu} = [\beta(\alpha)/4\alpha]G_{\mu\nu}{}^a G_{\mu\nu}{}^a \tag{3}$$

usually cited in the literature for the trace of the energy-momentum tensor. Many papers have been devoted to attempts to resolve the problem of the anomalies in supersymmetric theories. Our far-from-complete list of references includes more than ten articles.[11-22] One of the first detailed investigations was undertaken by Piguet and Sibold.[11] Unfortunately, despite individual achievements, complete understanding was not attained.

We shall explain our principal assertions about the β-function and anomaly using the example of an Abelian theory—supersymmetric quantum electrodynamics (SQED). The action of the model can be written in the form

$$S_W = \frac{1}{4e^2(\mu)}\int d^4x\, d^2\theta W^2 + \frac{Z(\mu)}{4}\int d^4x\, d^4\theta(\bar T e^V T + \bar U e^{-V}U), \tag{4}$$

where W is the supergeneralization of the stress tensor,

$$\begin{aligned} W_\alpha(x_L,\theta) = {}^1/{}_8\bar D^2 D_\alpha V = i\lambda_\alpha(x_L) - \theta_\alpha D(x_L) \\ - i\theta^\beta F_{\alpha\beta}(x_L) + \theta^2\partial_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}(x_L), \\ (x_L)_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} - 2i\theta_\alpha\bar\theta_{\dot\alpha}, \end{aligned} \tag{5}$$

and $T(x_L,\theta)$ and $U(x_L,\theta)$ are chiral matter superfields with charges $+1$ and -1, respectively. The action (4) is to be understood in the Wilson sense, i.e., all the operators in the right-hand side are normalized at the point μ, and $1/e^2(\mu)$ and $Z(\mu)$ are the corresponding coefficient functions. The mass term $mTU|_F$ is omitted in (4), since we assume that a high normalization point $\mu \gg m$ has been chosen. The maximum value of μ is equal to the ultraviolet-cutoff parameter M_0. At this point the action (4) is the initial action of SQED, and the coefficients $1/e^2(M_0)$ and $Z(M_0)$ are bare parameters. The coefficient functions for arbitrary μ are determined by the diagrams of the perturbation theory constructed from the initial action, with the specific feature (the fundamental point for us) that the region of integration over the virtual momenta flowing in all the loops of the Feynman graphs is given by the condition $\mu < k < M_0$.

Suppose that we wish to find the amplitudes of physical processes with external momenta $p \sim \mu$. The central point is that they do not coincide directly with expressions appearing in the action (4). The appropriate quantity for determining them is Γ—the generator of one-particle-irreducible vertices. Although, in its form, Γ contains the same structures as the action (4), their meanings are different: In Γ they are c-number functions, while in S_W they are operators. Because of this there is a difference in the coefficients. We shall denote the coefficients in Γ by the same symbols but in square brackets: $1/[e^2(\mu)]$ and $[Z(\mu)]$. The normalization point (momentum) μ in Γ is understood as the momentum of the external fields.

As will be shown below, the relation between $\alpha(\mu)$ and $[\alpha(\mu)]$ has the form[1)]

$$2\pi/[\alpha(\mu)] = 2\pi/\alpha(\mu) - 2\ln[Z(\mu)], \tag{6}$$

where $\alpha = e^2/4\pi$. The term with $\ln Z$ arises upon calculation of the photon matrix element of the operator $\int d^4\theta\{\bar T \exp(V)T + \bar U\exp(-V)U\}$ in (4). This matrix element is fixed by the so-called Konishi anomaly.[23-25] The observable quantity is $[\alpha(\mu)]$. The fact that it depends explicitly on the Z-factor is a new and unexpected element. We note that formula (6) also applies to the bare quantities (with $\mu = M_0$). Therefore, the following two models are physically equivalent: In the first model, the coefficient of W^2 in S_W is equal to $(4\pi\alpha_0)^{-1}$, and the coefficient of $\bar T T + \bar U U$ is equal to Z_0; In the second model, these coefficients are equal to $(4\pi[\alpha_0])^{-1}$ and 1, respectively. The quantity $\alpha(\mu)$ is renormalized only at the one-loop level, as already noted above; i.e.,

$$2\pi/\alpha(\mu) = 2\pi/\alpha_0 + 2\ln(M_0/\mu). \tag{7}$$

Combining (6) and (7), we obtain

$$2\pi/[\alpha(\mu)]=2\pi/[\alpha_0]+2\ln(M_0/\mu)-2\ln([Z(\mu)]/[Z_0]). \quad (8)$$

Apart from the one-loop logarithm, the dependence on μ enters only through the Z-factor. Differentiation with respect to $\ln\mu$ gives the β-function for the observable constant $[\alpha(\mu)]$:

$$\beta(\alpha)=(\alpha^2\pi)[1-\gamma(\alpha)], \quad (9)$$

where $\gamma(\alpha)$ is the anomalous dimension of the matter superfield:

$$\gamma=-d\ln[Z]/d\ln\mu=\alpha/\pi+\ldots \quad (10)$$

We now describe the equations for the anomalies in SQED. The supercurrent $J_{\alpha\dot\alpha}$ in this model has the form

$$J_{\alpha\dot\alpha}=-e^{-2}W_\alpha\bar W_{\dot\alpha}+Z\{{}^1/_6(D_\alpha(e^VS))e^{-V}\bar D_{\dot\alpha}(e^V\bar S) -{}^1/_6Se^VD_\alpha[e^{-V}\bar D_{\dot\alpha}(e^VS)]-{}^1/_6S\bar D_{\dot\alpha}(e^VD_\alpha\bar S) +(S\to T,\ V\to -V)\}, \quad (11)$$

and its supertrace is equal to

$$\bar D^{\dot\alpha}J_{\alpha\dot\alpha}={}^1/_{24}D_\alpha[(1/2\pi^2)W^2-\gamma Z\bar D^2(\bar Te^VT+\bar Ue^{-V}U)]. \quad (12)$$

The coefficients in the right-hand side are obtained by differentiating the action S_W [see (4)] with respect to the cutoff parameter M_0. The coefficient of W^2 in the operator relation (12) is exclusively one-loop. The equality (12) is the supergeneralization of the Adler-Bardeen theorem for the axial current.

The one-loop result for the coefficient of W^2 is general for all supersymmetric gauge theories. It is worth noting that in the right-hand side of (12) the second term, while formally equal to zero by the equations of motion, cannot be omitted. In fact, as was discussed above, it was the photon matrix element of precisely this operator that led to the difference between S_W and Γ. The calculation of the matrix element of $\bar T\exp(V)T+\bar U\exp(-V)U$ can be performed by making use of the Konishi anomaly[23-25]:

$$Z\bar D^2(\bar Te^VT+\bar Ue^{-V}U)=(1/2\pi^2)W^2. \quad (13)$$

By virtue of (13), in the photon matrix element of the anomaly (12) the β-function is recovered.

We turn now to non-Abelian theories. Since effects associated with matter are interpreted in essentially the same way as in electrodynamics, we shall concentrate on a purely gauge model with the action

$$S=\frac{1}{2g^2}\mathrm{Tr}\int d^2\theta\, d^4x\, W^2, \qquad W_\alpha=\frac{1}{8}\bar D^2(e^{-V}D_\alpha e^V). \quad (14)$$

Since non-Abelian fields are sources for each other, it follows that, even in the absence of matter, rescaling of the fields changes the magnitude of the charge in the Wilson action just as a change of scale of the matter fields did in electrodynamics. Specifically, when we change from fields V to fields ηV, i.e.,

$$W_\alpha\to{}^1/_8\bar D^2(e^{-\eta V}D_\alpha e^{\eta V}), \quad (15)$$

an action equivalent to (14) will be obtained, if, simultaneously with (15), we also replace the charge $g^2\to g_\eta^2$ using the formula

$$8\pi^2/g^2\to 8\pi^2/g_\eta^2=8\pi^2/g^2-2T(G)\ln\eta. \quad (16)$$

The charges in the Wilson action and in Γ will coincide if we normalize the kinetic term of the field V to unity. With this normalization the matrix element of the operator W^2 is obtained by simply replacing W_α by the external field. The normalization to unity implies that $\eta=g_\eta$. This means that the observable charge $[g^2]$ is connected with the charge in the Wilson action by the relation

$$8\pi^2/g^2=8\pi^2/[g^2]+T(G)\ln[g^2]. \quad (17)$$

Since the Wilson coupling constant g^2 is renormalized only at the one-loop level:

$$8\pi^2/g^2=8\pi^2/g_0^2-3T(G)\ln(M_0/\mu), \quad (18)$$

by differentiating (17) with respect to $\ln\mu$ we obtain the β-function (2) for the charge $[g^2]$.

We note that an explicit calculation of the two-loop contribution to the effective action in supersymmetric gauge theories has been undertaken recently in some very interesting and stimulating papers.[26] The technique of covariant supergraphs and, for the regularization, supersymmetric dimensional reduction were used. From the explicit calculations of Ref. 26 it can be seen that the two-loop part of the β-function arises upon expansion of an infrared indeterminacy of the form p^2/p^2, where p is the momentum of the external field. In fact, the integral over one of the loops is completely determined by the region of virtual momenta of order p. In our terminology this loop must be interpreted as the result of taking the matrix element of the operator W^2.

We shall say a few words about the anomaly for the supercurrent in the non-Abelian theory (14). The corresponding equation has the following form:

$$\bar D^{\dot\alpha}J_{\alpha\dot\alpha}=-\frac{T(G)}{16\pi^2}D_\alpha\,\mathrm{Tr}(W^\gamma W_\gamma), \quad (19)$$

$$J_{\alpha\dot\alpha}=-(2/g^2)\mathrm{Tr}(W_\alpha e^{-V}\bar W_{\dot\alpha}e^V), \quad (20)$$

where W_α is defined in (14). The absence of higher orders in g^2 in (19) corresponds exactly to the one-loop law (18) for the renormalization of g^2 in the Wilson action. As in electrodynamics, the β-function arises when the matrix element of the operator W^2 is taken.

The discussion is organized as follows. In Sec. 2 we consider the simple (nonsupersymmetric) example of the electrodynamics of a scalar field. For this example we demonstrate the difference between S_W and Γ by analyzing the renormalization of the charge in the two-loop approximation. The third section is devoted to supersymmetric electrodynamics. Features of non-Abelian models are discussed in Sec. 4. A comparison with calculations of the instanton type is given in Sec. 5. In the brief Conclusion we summarize the main results.

2. ELECTRODYNAMICS OF A SCALAR FIELD

In this section we discuss an instructive example—scalar electrodynamics. This model makes it possible to elucidate in a simplified situation some aspects of the results pertaining to supersymmetric models as formulated above. A concrete calculation of the two-loop β-function with the emphasis on the points that we shall need is described in detail in Ref. 27; here we shall concentrate on the interpretation of this calculation in the framework of the operator expansion.

We write the initial Lagrangian in the form

$$\mathcal{L}=-(1/4e_0^2)F_{\mu\nu}F_{\mu\nu}+(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi, \qquad (21)$$

where φ is a complex scalar field and $\mathcal{D}_\mu = \partial_\mu - iA_\mu$. Below we consider the construction, in the two-loop approximation, of both the Wilson effective action and the functional Γ that determines the irreducible vertex functions. The external-field method will be used.

We shall clarify how one introduces the normalization point μ in the functionals S_W and Γ. In both cases μ is the momentum of the external field (it is assumed that the mass of the field φ is negligible in comparison with μ). As we shall see, in one of the loop integrals a well defined part is built up from the infrared region of virtual momenta $k\sim\mu$. This part should be included in $\Gamma(\mu)$, but excluded from $S_W(\mu)$. As regards the ultraviolet cutoff, in our procedure only one-logarithm integrals arise, and these can be cut off in a stepwise manner from above at $k=M_0$. In principle, one can keep in mind that the theory is regularized by the introduction of Pauli-Villars fields (partners to the field φ) plus higher derivatives for the vector field A_μ.

In one loop the problem of calculating $S_W(\mu)$ is, of course, trivial. The result reduces to the following:

$$S_W=\int d^4x\left\{-\frac{1}{32\pi^2}\left(\frac{2\pi}{\alpha_0}+\frac{1}{3}\ln\frac{M_0}{\mu}\right)F_{\mu\nu}F_{\mu\nu} +Z\left(\frac{M_0}{\mu}\right)(\mathcal{D}_\mu\varphi)^*\mathcal{D}_\mu\varphi\right\}, \qquad (22)$$

where the factor

$$Z\left(\frac{M_0}{\mu}\right)=1-\left(1+\frac{\xi}{2}\right)\frac{\alpha}{\pi}\ln\frac{M_0}{\mu} \qquad (23)$$

depends on the gauge of the photon field, the propagator of which is

$$\mathcal{D}_{\mu\nu}=e^2(-g_{\mu\nu}+\xi k_\mu k_\nu/k^2)/k^2.$$

In this approximation $\Gamma(\mu)$ coincides in form with $S_W(\mu)$, since the photon matrix element of $(Z-1)\mathcal{D}_\mu\varphi^*\mathcal{D}_\mu\varphi$ must be taken into account only in the two-loop approximation.

We turn now to the two-loop analysis. In the two-loop approximation the coefficient of F^2 in S_W is determined by the graph of Fig. 1. We separate out the integration over the virtual photon and leave this integration to the end of the calculation. Before this last integration the calculation of $S_W(\mu)$ is equivalent to the calculation of the photon polarization operator $\Pi_{\mu\nu}$ in the one-loop approximation (Fig. 2). To be more precise, we need to calculate only one term in

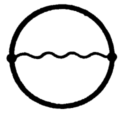

FIG. 1. Two-loop contribution to $S_W(\mu)$ in scalar electrodynamics. The thick line is the propagator of the scalar particle in the external field; the wavy line is the photon propagator.

the operator expansion for $\Pi_{\mu\nu}$—namely, the term with $F_{\alpha\beta}F_{\alpha\beta}$. The coefficient of this term is finite and well defined. Then the last integration over the photon momentum k gives a logarithmic integral of the type $\int d^4k k^{-4}$, which can be cut off from above at $k=M_0$ and from below at $k=\mu$ (for more detail, see Ref. 27). Specifically, in the x-representation

$$\mathcal{L}^{(2)}_{g\Phi\Phi}=\frac{1}{2}\int d^4x\, i\mathcal{D}_{\mu\nu}(x)\,\Pi^{(F^2)}_{\mu\nu}(x), \qquad (24)$$

where[2)]

$$\Pi_{\mu\nu}=i\langle T\{J_\mu(x)J_\nu(0)\}\rangle,$$
$$J_\mu=i\varphi^*\overleftrightarrow{\mathcal{D}}_\mu\varphi=i[\varphi^*\mathcal{D}_\mu\varphi-(\mathcal{D}_\mu\varphi^*)\varphi], \qquad (25)$$

and the superscript (F^2) indicates that in $\Pi_{\mu\nu}$ we must retain only the operator F^2; the region of integration over x in (24) is given by the inequality $M_0^{-1}<|x|<\mu^{-1}$. In formula (24) $\mathcal{D}_{\mu\nu}$ is the free photon propagator:

$$\mathcal{D}_{\mu\nu}=e^2\left(-g_{\mu\nu}+\xi\frac{\partial_\mu\partial_\nu}{\partial^2}\right)\frac{i}{4\pi^2x^2}$$
$$=e^2\left[-g_{\mu\nu}+2\xi\left(g_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2}\right)\right]\frac{i}{4\pi^2x^2}. \qquad (26)$$

The operator expansion for $\Pi_{\mu\nu}$ was constructed in Ref. 27, in which an explicit expression was found for the propagator $G(x,0)$ of the scalar particle in the external field:

$$G(x,0)=\left\langle x\left|\frac{1}{\mathcal{P}^2}\right|0\right\rangle=\frac{i}{4\pi^2x^2}+\frac{i}{512\pi^2}x^2F^2 \qquad (27)$$

(in the Fock-Schwinger gauge for $A_{\mu,\mathrm{ext}}$, i.e., $x_\mu A_{\mu,\mathrm{ext}}=0$). It is important that in the construction of $S_W(\mu)$ we must keep only the singular term in the propagator (27). It is this term which corresponds to large ($p\sim k$; see Fig. 2) virtual momenta in the loop of Fig. 2. The nonsingular term [in momentum space this is proportional to $\partial^2/\partial p_\mu\partial p_\nu\delta^{(4)}(p)$] represents the infrared effect and, in the diagram of Fig. 2, corresponds to virtual momenta p of the order of the momentum μ of the external field. In the expression (27) the limit of zero μ has been taken.

In other words, the second region ($p\sim\nu$) bears no rela-

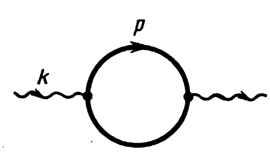

FIG. 2. By cutting the photon line in Fig. 1, we arrive at the photon polarization operator $\Pi_{\mu\nu}$. We are interested in the coefficient of $F_{\alpha\beta}F_{\alpha\beta}$ in the operator expansion for $\Pi_{\mu\nu}$.

tion to the coefficient of the Wilson operator expansion and will be taken into account in the calculation of the matrix element; see below.

We give here the formula (25) from Ref. 27 (the quantity that we shall need is denoted in Ref. 27 by $\Pi^{(1)}_{mv}$):

$$\Pi^{(F^2)}_{\mu\nu} = -(i/192\pi^4 x^4)(x^2 g_{\mu\nu} - x_\mu x_\nu) F^2(0). \qquad (28)$$

We draw attention to the fact that the result (28) is not transverse. A reflection of this fact will be the dependence of $S_W(\mu)$ on the gauge parameter ξ. Combining (24), (26), and (28), we obtain

$$S_W(\mu) = \int d^4x \left\{ -\frac{1}{32\pi^2}\left[\frac{2\pi}{\alpha_0} + \frac{1}{3}\ln\frac{M_0}{\mu} + \frac{\alpha_0}{2\pi}\left(1-\frac{\xi}{2}\right)\ln\frac{M_0}{\mu}\right] F_{\alpha\beta}F_{\alpha\beta} + Z\left(\frac{M_0}{\mu}\right)\mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi \right\}. \qquad (29)$$

The situation is, at first sight, paradoxical. In fact, the two-loop coefficient of $F_{\alpha\beta}F_{\alpha\beta}$ in $S_W(\mu)$ depends on the gauge! The corresponding constant, of course, cannot be observable. How can we reconcile (29) with the well known expression for the renormalization of α in scalar electrodynamics (an expression which, of course, does not contain a gauge parameter)?

The answer should be clear to the reader from Sec. 1. The coefficient of the operator F^2 in $S_W(\mu)$ in reality does not coincide with the observable charge. In determining $1/[\alpha]$ it must be taken into account that a nonzero contribution to the amplitudes (in the given case, in an external photon field) is given by the matrix element of the operator $\mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi$ over the external field.

Formally, the operator $\int d^4x \mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi$ is equal to zero by the equations of motion. We can convince ourselves, however, that in an external gauge field the following anomaly relation holds:

$$\langle \mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi\rangle = (1/64\pi^2) F_{\alpha\beta}F_{\alpha\beta}, \qquad (30)$$

which follows from formula (27) for the propagator $G(x,0)$.

The exact propagator $G(x,0)$, of course, satisfies the equations of motion $-\mathscr{D}^2 G(x,0) = \delta^4(x)$. However, in the calculations we divide this propagator into two parts—a part singular in x, which contributes to the coefficient of the operator expansion, and a part regular in x, which can be interpreted as a matrix element. In fact,

$$\langle \mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi\rangle = \lim_{x\to 0}(-i\mathscr{D}_\mu^2 G^{reg}(x,0)) = (1/64\pi^2)F_{\alpha\beta}F_{\alpha\beta},$$

where G^{reg} is the second term in the right-hand side of (27) (all the results pertain to the Fock-Schwinger gauge for the external field; see Ref. 27).

If we now return to the action (27), we can determine, by going over from S_W to Γ, what must be identified with the observable constant. Taking the matrix element over the external photon field for the F^2 structure, we obtain

$$\Gamma = \int d^4x \left\{ -\frac{1}{32\pi^2}\left[\frac{2\pi}{\alpha_0} + \frac{1}{3}\ln\frac{M_0}{\mu} + \frac{\alpha}{2\pi}\left(1-\frac{\xi}{2}\right)\ln\frac{M_0}{\mu}\right] + \frac{1}{64\pi^2}\left[Z\left(\frac{M_0}{\mu}\right) - 1\right]\right\} F^2 + \dots \qquad (31)$$

Invoking (23), we can convince ourselves that in the sum the dependence on the gauge drops out, as we should expect, and

$$\frac{2\pi}{[\alpha]} = \frac{2\pi}{\alpha_0} + \frac{1}{3}\ln\frac{M_0}{\mu} + \frac{\alpha}{\pi}\ln\frac{M_0}{\mu} + O(\alpha^2) \qquad (32)$$

in complete agreement with the known result.

Here it is appropriate to make a few remarks. In going over from S_W to Γ, as is usual in perturbation theory we calculated the matrix element only of that part of S_W which can be interpreted as a perturbation with respect to the chosen operator basis, i.e., only of the part $(Z-1)\int d^4x \mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi$.

We note also that we have used a certain specific procedure of ultraviolet and infrared regularization, which, of course, is not obligatory. In the literature Feynman diagrams are often calculated using, e.g., dimensional regularization both for the ultraviolet and for the infrared region. In fact, all these calculations, irrespective of the form of the infrared regularization, pertain to Γ and give the correct answer (including the infrared region) for the observable charge.

In the example under consideration (scalar electrodynamics), the separation into an ultraviolet and an infrared contribution (i.e., into OPE coefficients and matrix elements) is not unique. In particular, the anomaly relation (30) is not connected with the well known conformal anomaly; the coefficient in the right-hand side of (30) depends on the procedure adopted. Of course, if we had adopted some specific calculational scheme, then in the framework of this scheme both the coefficient of F^2 and $\langle \mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi\rangle$ are fixed in a fully determinate manner, but "transfer" between them can occur if we go over to another scheme. Confirmation of this is given by SQED (see Sec. 3), in which both a component (nonsuperfield) treatment and an analysis in terms of superfields are possible. The scalar particles appear in the matter sector of SQED. The spinor fields that also appear in SQED do not, in a component treatment, have an anomaly analogous to (30), $\langle\bar\psi\mathscr{D}\psi\rangle = 0$, i.e., spinor electrodynamics does not have an infrared part in the charge renormalization. In the framework of a superfield analysis the relation (30) is replaced by the Konishi anomaly (13).

3. SUPERSYMMETRIC QUANTUM ELECTRODYNAMICS

The Wilson action in SQED has the form

$$S_W = \frac{1}{4e^2}\int d^4x\, d^2\theta\, W^2 + \frac{Z}{4}\int d^4x\, d^4\theta(\bar T e^V T + \bar U e^{-V} U). \qquad (33)$$

Using the same general approach as in the preceding section, we show here that going over from S_W to Γ gives formula (6) for the observable charge. In addition, we prove the general theorem that the renormalization of the coefficient of the operator W^2 is exhausted by one loop. A conceptually similar statement about the one-loop character of the renormalizations was made in Refs. 5. Since the parameters e^2 and Z in (33) are not observable and depend on the method of quantization, to ensure the validity of both assertions it is important to use the superfield formalism. In other words, it is important that one descends from the mass shell in an explicitly supersymmetric manner.

The analysis of the relationship between S_W and Γ is carried out in the framework of the same program as in the preceding section. We must find the propagator of the matter superfield in the external field, and separate it into a singular and a nonsingular part. The singular part must be used in the calculation of the coefficient $1/e^2$, and the nonsingular part then fixes the value of the matrix element of the operator

$$\int d^4x\, d^4\theta\, (\bar{T}e^V T+\bar{U}e^{-V}U) = -\frac{1}{2}\int d^4x\, d^2\theta\, \bar{D}^2(\bar{T}e^V T+\bar{U}e^{-V}U). \tag{34}$$

In this problem the program is easier to carry out because there is no second or subsequent loop for $1/e^2$, and consequently no transfer between different terms in S_W. In other words, in the framework of the superfield formalism the separation into coefficients of the operator expansion and matrix elements becomes unique. One manifestation of this situation is the fact that the calculation of the matrix element of (34) can be formulated as the Konishi anomaly relation (13).

Here there is a direct analogy with the Adler anomaly in the axial current[10]:

$$\partial_\mu a_\mu=\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=(1/8\pi^2)F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{35}$$

As is well known, this anomaly has two aspects. On the one hand, it can be exhibited as an infrared effect by considering the transition of the axial current to two photons:

$$a_\mu \propto (q_\mu/q^2)F_{\rho\nu}\tilde{F}_{\rho\nu},$$

where Q is the momentum of the axial current. It is immediately clear from this formula that the fermion loop is saturated by small virtual momenta of order q. Multiplying by q_μ, we obtain (35). This result, which emphasizes the infrared character of the effect, corresponds to the calculation of the matrix element of $(\mathscr{D}_\mu\varphi^*\mathscr{D}_\mu\varphi)$ in Sec. 2 by separation of the nonsingular part of the propagator in the external field.

On the other hand, since we are concerned with a divergence, the anomaly relation (35) can be obtained as a result of ultraviolet (e.g., Pauli-Villars) regularization of a_μ.

In the language of the spectral current in an external field, the two-aspect character of the anomalies implies that the number of levels arriving at the origin is equal to the number of levels intersecting the ultraviolet cutoff.

The operator $D^2[\bar{T}\exp(V)T+\bar{U}\exp(-V)U]$ of interest to us is the direct supergeneralization of $\partial_\mu a_\mu$. The matrix element of this operator is fixed uniquely by the Konishi anomaly (13). It follows from this that in first order in $Z-1$ we can obtain from (33) an expression for Γ:

$$\Gamma=\left[\frac{1}{4e^2}-\frac{1}{16\pi^2}(Z-1)\right]\int d^4x\, d^2\theta W^2 + \text{matter terms}. \tag{36}$$

As already noted, for $1/e^2$ we have the one-loop law

$$8\pi^2/e^2=8\pi^2/e_0^2+2\ln(M_0/\mu). \tag{37}$$

If we confine ourselves to first order in $Z-1$, the result (36) can be represented as the replacement of the ultraviolet parameter M_0 under the logarithm by M_0/Z. This circumstance, of course, is not accidental, and is in one-to-one correspondence with the Konishi-anomaly derivation based on ultraviolet regularization.

In fact, we shall introduce explicitly the Pauli-Villars regulators T_R, U_R for matter; i.e., we add to the action (33) the regulator part

$$S^{reg}=\frac{Z}{4}\int d^4x\, d^4\theta\,(\bar{T}_R e^V T_R+\bar{U}_R e^{-V}U_R) + \left(\frac{M_0}{2}\int d^4x\, d^2\theta\, T_R U_R + \text{h.c.}\right). \tag{38}$$

When the regulators are taken into account the naive equations of motion are satisfied. Therefore, we an use these equations in the perturbation proportional to $Z-1$. Then the perturbation is written in the form

$$\Delta S=-(Z-1)\left(\frac{M_0}{2}\int d^4x\, d^2\theta\, T_R U_R+\text{h.c.}\right). \tag{39}$$

After the matrix element is taken, the extra term (39) reproduces the term proportional to $Z-1$ in (36). Thus, from the term $O(Z-1)$ we have established that the mass term of the regulator in (38) does not contain Z. After this it is fairly obvious that the summation of all orders in $Z-1$ in the matrix element of $\exp(i\Delta S)$ over the gauge fields is equivalent to the replacement $M_0\to M_0/Z$. Consequently, the observable charge, appearing in Γ, is given by

$$8\pi^2/[e^2]=8\pi^2/e_0^2+2\ln(M_0/Z_\mu). \tag{40}$$

The relation (40) is the final result for SQED. If we differentiate it with respect to $\ln\mu$, we obtain the β-function given in formula (9).

As regards Eq. (12) for the anomaly in the supercurrent, it is obtained by differentiation of the action (33) with respect to $\ln M_0$. Here it should be borne in mind that for $1/e^2$ the one-loop law (37) holds, and the Z-factor depends on the ratio M_0/μ.

It remains for us to prove the theorem stated above concerning the absence of higher loops in $1/e^2$. In fact, we merely reformulate slightly the arguments of Refs. 5. Thus, we shall assume for definiteness that we are calculating the two-loop coefficient of the operator W^2 in the effective action. For this, in the external-field method we must consider the graph of Fig. 3. We note that in the Abelian case the superfield V does not interact with the background field. The given diagram can be regularized as in Sec. 2. Namely, by cutting the line V we obtain a sub-block that is finite both in the ultraviolet and in the infrared region. The last integration over the virtual momentum of the V propagator is cut off in the ultraviolet region at $k=M_0$, and in the infrared region at $k=\mu$. This ultraviolet cutoff can be introduced via higher derivatives of the field V. Having in mind an n-loop diagram, we can also formulate a general regularization procedure:

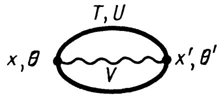

FIG. 3. Two-loop contribution to $\Gamma(\mu)$ in SQED. The thick line is the propagator of the matter superfield in the external gauge field; the wavy line is the propagator of the gauge superfield.

the introduction into the Lagrangian of Pauli-Villars fields with mass M_0 in combination with higher derivatives of V. For us, only the fact of the existence of superfield regularization in four-dimensional space-time is important.

In the background-field technique the expression for the diagram of Fig. 3 has the form

$$\Delta S \propto \int d^8 z_1\, d^8 z_2 \mathscr{D}(z_1, z_2) G(z_1, z_2) G(z_2, z_1), \tag{41}$$

where $z = (x, \theta, \bar{\theta})$, and $\mathscr{D}(z_1, z_2)$ and $G(z_1, z_2)$ are the Green's functions of the vector superfield and the covariant-chiral superfield in the external field. An operator representation of these propagators can be found in Refs. 28 and 26:

$$\mathscr{D}(z_1, z_2) = \langle z_1 | [\nabla_\mu^2 - iW^\alpha \nabla_\alpha + i\bar{W}^{\dot{\alpha}} \nabla_{\dot{\alpha}}]^{-1} | z_2 \rangle,$$
$$G(z_1, z_2) = \langle z_1 | -\nabla^\alpha \nabla_\alpha (\nabla_\mu^2 - iW^\alpha \nabla_\alpha - {}^1\!/_2 i [\nabla^\alpha W_\alpha])^{-1} \bar{\nabla}_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}} | z_2 \rangle. \tag{42}$$

It is of fundamental importance that in formulas (42) the external field V does not appear explicitly, but appears only through W_α and the covariant derivatives. Thus, the method is explicitly gauge-invariant with respect to the external field. In particular, under gauge transformations of the external field the propagator $\mathscr{D}$ is not changed, while the propagator G is changed as follows:

$$G(z_1, z_2) \to \exp[iK(z_1)] G(z_1, z_2) \exp[-iK(z_2)], \tag{43}$$

where $K(z)$ is a real superfield of general form. It is obvious that the integrand in (41) is invariant under the transformation (43). In the non-Abelian case K is a matrix in color space, and the analog of the expression (41) contains a color trace. The gauge invariance of the integrand in (41) implies that it is expressed entirely in terms of the quantity $W_\alpha^{\rm ext}$ and its covariant derivatives (the superscript ext indicates the c-number external field). From (41), after the integration over z_2, we obtain

$$\Delta S = \int d^4 x\, d^2\theta\, d^2\bar{\theta}\, f(x, \theta, \bar{\theta}), \tag{41'}$$

where f does not depend explicitly on x, θ, or $\bar{\theta}$, and is a function of $W_\alpha^{\rm ext}$. If $f(x, \theta, \bar{\theta})$ is expressed locally in terms of $W_\alpha^{\rm ext}(x, \theta)$ (and its derivatives), it is obvious that a structure of the form $\int d^4 x d^2\theta W^2(x, \theta)$ cannot be obtained. On the other hand, in the specific two-loop calculations of Ref. 26 this structure did arise. The reason is that the function f in Ref. 26 was expressed nonlocally in terms of W, e.g.,

$$f = W^\alpha \frac{D_\beta D^\beta}{\Box} W_\alpha.$$

Here an infrared singularity is present. If the momentum of the external field W is equal to p, a $1/p^2$ pole could arise only from the region of virtual momenta of order p. In accordance with this, as explained above, this region need not be included in the coefficients of the OPE for S_W, but is taken into account when the matrix element is taken. This completes the proof for the graph of Fig. 3.

The argument presented above does not apply to a one-loop graph with a chiral superfield inside it. In fact, for the proof given above the presence of a quantum interaction vertex accompanied by integration over $d^2\theta d^2\bar{\theta}$ was essential. The "superfluous" $d^2\bar{\theta}$ can then no longer be eliminated. The one-loop supergraph in the external-field technique cannot be written in the form (41). The proof that the two-loop diagram for S_W is equal to zero can be generalized without difficulty to all the higher-order loops and to non-Abelian theories. It is the generalization of the theorem concerning the nonrenormalizability of the F-terms.[2]

We shall mention one of the interesting indirect consequences of the analysis performed. We refer to the anomalous dimension of the operator $\bar{T} \exp(V) T + \bar{U} \exp(-V) U$. One of the renormalization-invariants is the operator appearing in the right-hand side of Eqs. (12) for the supercurrent anomaly. Another renormalization-invariant combination

$$Z\bar{D}^2(\bar{T}e^V T + \bar{U}e^{-V} U) - (1/2\pi^2) W^2, \tag{44}$$

appears in the Konishi equation (13). The renormalization invariance of (44) can be demonstrated by considering the increment of S_W upon change of Z_0. From the fact that the operator (44) and the right-hand side of (12) are independent of the normalization point it follows that this property is also possessed by the operator

$$(1-\gamma) Z(\bar{T}e^V T + \bar{U}e^{-V} U). \tag{45}$$

4. NON-ABELIAN GAUGE THEORIES

The principal assertions for this case have been formulated in the Introduction. Here we shall elucidate the derivation of the results for the example of supersymmetric gluodynamics. The inclusion of matter does not require special analysis, since it does not differ from the example of SQED considered above.

The difference from SQED consists in the fact that the gauge fields are sources for each other, and therefore the matrix element $\langle W^2 \rangle$ of the operator W^2 does not reduce to a c-number function $W_{\rm ext}^2$. For this matrix element the formula

$$\langle W^2 \rangle = \frac{\beta(\alpha)}{\beta_{1\,\rm loop}(\alpha)} W_{ext}^2 = \left[1 + \frac{T(G)\alpha}{2\pi} + \dots\right] W_{ext}^2 \tag{46}$$

is valid, where $\beta(\alpha)$ and $\beta_{1\,\rm loop}(\alpha)$ are the exact and one-loop β-functions, respectively. The fact that $\langle \beta_{1\,\rm loop}(\alpha) W^2 \rangle$ reduces precisely to $\beta(\alpha) W_{\rm ext}^2$ is obvious from the renormalization invariance of these quantities. Thus, the right-hand side of (46) can be regarded as the definition of the Gell-Mann–Low function. Our aim is the constructive calculation of $\beta(\alpha)$; the $(l+1)$-loop coefficient in $\beta(\alpha)$ is fixed by the l-loop coefficient in the matrix element (46).

First of all we recall that the Wilson action in supersymmetric gluodynamics is given exactly by the one-loop expression

$$S_W(\mu)=\frac{1}{2}\left[\frac{1}{g_0^2}-\frac{3T(G)}{8\pi^2}\ln\frac{M_0}{\mu}\right]\int d^4x\,d^2\theta\,\mathrm{Tr}\,W^\alpha W_\alpha. \qquad (47)$$

This fact was proved in the preceding section. At the one-loop level the expression for $\Gamma(\mu)$ obviously has the same form. The next loops in $\Gamma(\mu)$ are obtained by calculating the matrix elements of S_W [to be more precise, the relationship $\exp(i\Gamma)=\langle\exp(iS_W)\rangle$ holds]. We shall establish the connection between $\Gamma(\mu)$ and $S_W(\mu)$ at the two-loop level. In this approximation,

$$\Gamma(\mu)=\frac{1}{2g_0^2}\int d^4x\,d^2\theta\,\mathrm{Tr}\,W_{ext}^2$$
$$-\frac{1}{2}\frac{3T(G)}{8\pi^2}\ln\frac{M_0}{\mu}\int d^4x\,d^2\theta\langle\mathrm{Tr}\,W^2\rangle, \qquad (48)$$

where $\langle\mathrm{Tr}\,W^2\rangle$ is the matrix element of the operator $\mathrm{Tr}\,W^2$. In the passage from (47) to (48) the matrix element is calculated entirely from the correction to the bare action. Therefore, it is sufficient to find $\langle\mathrm{Tr}\,W^2\rangle$ to order $O(\alpha)$.

The one-loop part of the result (46) can be extracted from Ref. 22 [formulas (A21) and (A22)]. The main complication in the calculation of the matrix element $\langle W^2\rangle$ is the necessity of infrared regularization. Specifically, in Ref. 22 dimensional reduction was used for this purpose. We, however, should like to give here another derivation, pertaining directly to four-dimensional space. The physical transparency of this derivation will help us afterwards in our discussion of the relationship with previous analyses.

Our treatment will refer to the G-component of the superfield W^2; this component has the form

$$\mathrm{Tr}\,W^2|_G={}^1/_4(G_{\mu\nu}^a\tilde{G}_{\mu\nu}^a-2\partial_\mu a_\mu), \qquad (49)$$

where $a_\mu=-\lambda^a\sigma_\mu\bar{\lambda}^a$ is the axial gluino current; $\lambda_\alpha^a\,(\alpha=1,2)$ is the Weyl spinor. In terms of the Majorana spinor $\lambda_\alpha^a\,(\alpha=1,\ldots,4)$ the same current is written as $a_\mu=\lambda^a\gamma_\mu\gamma_5\lambda^a/2$.

An important result is that not only the fermionic part but also the bosonic part of $W^2|_G$ can be represented as a total derivative:

$$G_{\mu\nu}^a\tilde{G}_{\mu\nu}^a=\partial_\mu K_\mu,$$
$$K_\mu=2\varepsilon_{\mu\nu\gamma\delta}(A_\nu^a\partial_\gamma A_\delta^a+{}^1/_3 f^{abc}A_\nu^aA_\gamma^bA_\delta^c). \qquad (50)$$

Therefore, as will be shown below, the calculation of the matrix element of $G\tilde{G}$ can be formulated in terms of a certain anomaly, just as can be done for $\partial_\mu a_\mu$. To be more precise, both a_μ and K_μ have infrared poles of the type $(q_\mu/q^2)G\tilde{G}$, the coefficient of which is fixed uniquely. This assertion pertains to the following kinematics: The two gluons in the final state have momenta k_1 and k_2, with $k_1^2=k_2^2=0$ and $q=k_1+k_2\neq 0$. The existence of the infrared pole in a_μ is a well known fact[29,30] reflecting the existence of the axial anomaly. Apparently, the analogous pole in K_μ has not been discussed in the literature.

In the calculation of the matrix element of K_μ we shall use the external-field formalism:

$$A_\mu^a=A_\mu^{a\,ext}+a_\mu^a, \qquad (51)$$

where $A_\mu^{a\,\mathrm{ext}}$ and a_μ^a are the external field and quantum field. In the one-loop approximation we need that part of the current K_μ which is quadratic in the quantum field a_μ^a:

$$K_\mu^{(2)}=2\varepsilon_{\mu\nu\gamma\delta}a_\nu^a\mathscr{D}_\gamma a_\delta^a. \qquad (52)$$

The matrix element of interest to us is obtained from (52) by substituting the Green's function of the quantum field:

$$\langle K_\mu\rangle=-2g^2\varepsilon_{\mu\nu\gamma\delta}\,\mathrm{Tr}_{color}\langle x|\mathscr{P}_\gamma\left[\frac{1}{\mathscr{P}^2-2G}\right]_{\delta\nu}|x\rangle. \qquad (53)$$

Here all the quantities are matrices in color space:

$$(\mathscr{P}_\gamma)^{ab}=i(\delta^{ab}\partial_\gamma+f^{acb}A_\gamma^c),\qquad [\mathscr{P}_\mu\mathscr{P}_\nu]^{ab}=-G_{\mu\nu}^{ab}=-f^{acb}G_{\mu\nu}^c.$$

To simplify the formulas we have omitted the superscript ext for the external fields. The formalism used is explicitly gauge-invariant with respect to the external field. As regards the gauge of the quantum field, (53) implies the Feynman gauge: $\Delta\mathscr{L}=-1/2(\mathscr{D}_\mu a_\mu)^2$. The dependence of the answer on the gauge of the quantum field is discussed below.

We shall expand the propagator (53) in powers of $\mathscr{P}^{-2}G$. The term of zeroth order in G drops out because of the contraction with $\varepsilon_{\mu\nu\gamma\delta}$. In the second-order term

$$\mathscr{P}_\gamma\mathscr{P}^{-2}G_{\delta\rho}\mathscr{P}^{-2}G_{\rho\nu}\mathscr{P}^{-2}$$

the intensities G are contracted on one index and cannot give the unique structure $q_\mu G_{\alpha\beta}\tilde{G}_{\alpha\beta}$ that determines the longitudinal part of K_μ. In the end, there remains only the term linear in G:

$$\langle K_\mu\rangle=-8g^2\,\mathrm{Tr}_{color}\langle x|\mathscr{P}_\gamma\mathscr{P}^{-2}\tilde{G}_{\mu\gamma}\mathscr{P}^{-2}|x\rangle. \qquad (54)$$

Instead of calculating (54) directly, we can compare this expression with the matrix element of the spinor axial current a_μ, the anomaly of which is well known:

$$\langle a_\mu\rangle=-{}^1/_2 ig^2\,\mathrm{Tr}_{color}\,\mathrm{Tr}_{spin}\langle x|\gamma_\mu\gamma_5(\mathscr{P}_\alpha\gamma^\alpha)^{-1}|x\rangle$$
$$=-{}^1/_2 ig^2\,\mathrm{Tr}_{color}\,\mathrm{Tr}_{spin}\langle x|\gamma_\mu\gamma_5\mathscr{P}_\alpha\gamma^\alpha(\mathscr{P}^2-{}^1/_2G_{\alpha\beta}\sigma_{\alpha\beta})^{-1}|x\rangle.$$

As in the preceding case, upon expansion in σG there remains in the longitudinal part of $\langle a_\mu\rangle$ only the linear term

$$\langle a_\mu\rangle=-2g^2\,\mathrm{Tr}_{color}\langle x|\mathscr{P}_\gamma\mathscr{P}^{-2}\tilde{G}_{\mu\gamma}\mathscr{P}^{-2}|x\rangle. \qquad (55)$$

Comparison of (54) and (55) shows clearly that K_μ contains exactly the same pole $(q_\mu/q^2)G\tilde{G}$ as does a_μ, but with an extra factor 4. Since

$$\langle\partial_\mu a_\mu\rangle=(\alpha/4\pi)T(G)\,(G_{\mu\nu}^a\tilde{G}_{\mu\nu}^a)^{ext}, \qquad (56)$$

for $\langle G\tilde{G}\rangle$ we obtain

$$\langle G\tilde{G}\rangle=\langle\partial_\mu K_\mu\rangle=(G_{\mu\nu}^a\tilde{G}_{\mu\nu}^a)^{ext}[1+T(G)\alpha/\pi], \qquad (57)$$

where we have added 1 from the classical part.

From (56) and (57) for the G-component of W^2 [see (49)] we have

$$\langle \mathrm{Tr}\, W^2|_G\rangle = W^2|_G^{ext}[1+T(G)\alpha/\pi - T(G)\alpha/2\pi]. \qquad (58)$$

Bearing in mind the supersymmetry, we arrive at the conclusion that the one-loop part of the relation (46) has been reproduced.

It remains to demonstrate that the quantum field is independent of the gauge. In an arbitrary gauge the propagator of the field a_μ^a has the form

$$\mathscr{D}_{\mu\nu}(x,\ y) = \langle x|[\mathscr{P}^2 g_{\mu\nu} - 2G_{\mu\nu} - \xi\mathscr{P}_\mu\mathscr{P}_\nu]^{-1}|y\rangle. \qquad (59)$$

It is not difficult to verify the following operator relation:

$$\mathscr{P}_\mu(\mathscr{P}^2 g_{\mu\nu} - 2G_{\mu\nu}) = \mathscr{P}^2\mathscr{P}_\nu + i\mathscr{D}_\gamma G_{\gamma\nu}. \qquad (60)$$

If we assume, as is required in the external-field method, that the external field satisfies the equation of motion $\mathscr{D}_\gamma G_{\gamma\nu} = 0$, then the second term in the right-hand side of (60) drops out and the propagator $\mathscr{D}_{\mu\nu}(x,y)$ can be written as follows:

$$\mathscr{D}_{\mu\nu}(x,y) = \langle x|\left(\frac{1}{\mathscr{P}^2 - 2G}\right)_{\mu\nu} + \frac{\xi}{1-\xi}\mathscr{P}_\mu\frac{1}{\mathscr{P}^4}\mathscr{P}_\nu|y\rangle.$$

We now return to the matrix element of K_μ [see (53)]. The ξ-dependent part is equal to

$$\Delta_\xi\langle K_\mu\rangle = 2g^2\frac{\xi}{1-\xi}\tilde{G}_{\mu\nu}(x)\langle x|\frac{1}{\mathscr{P}^4}\mathscr{P}_\nu|x\rangle. \qquad (61)$$

Formally, by virtue of the gauge invariance with respect to the external field, $\langle x|\mathscr{P}^{-4}\mathscr{P}_\nu|x\rangle$ can be expressed entirely in terms of $\mathscr{D}_\gamma G_{\gamma\nu}$, which is equal to zero. However, in the kinematics under consideration, $k_1^2 = k_2^2 = 0$, the expression $\langle x|\mathscr{P}^{-4}\mathscr{P}_\nu|x\rangle$ is not defined in the infrared region (it contains $1/k^2$). For the regularization one can give an infrared-regularizing mass m to the quantum field, i.e., $\mathscr{P}^2 \to \mathscr{P}^2 - m^2$ with $m^2 \ll q^2 \equiv (k_1 + k_2)^2$. The same device is also used (see Ref. 29) in the fermionic triangle. After the introduction of the mass m as an infrared regulator, expression (61) vanishes in actuality, not just in the formal sense.

We shall make a few simple remarks connected with this result. The fixing of the infrared pole in K_μ is a unique procedure that does not depend on how one descends from the mass shell. In other words, the matrix element of $W^2|_G$ is bound to be the same in the component and superfield formalisms. Extension to the other components of W^2 implies the superfield formalism. We have checked that the answer (46) is obtained from Ref. 22, which uses the superfield formalism and supersymmetric dimensional reduction.

We note that the bosonic anomaly in $W^2|_G$ is twice as large as the fermionic anomaly and of the opposite sign. Effectively, this changes the sign of $\langle W^2|_G\rangle$ in comparison with the case when the bosonic anomaly is not taken into account (as is usually the case[13,16,19]). We shall return again to the discussion of this circumstance in Sec. 6.

We have not calculated explicitly the two-loop and higher terms in the relation (46). We give here an indirect argument that is analogous to the analysis given in SQED and will make it possible to make the generalization to all loops.

In SQED the second loop in Γ arose when the matrix element of the operator

$$^1/_4(Z-1)\int d^4x\, d^4\theta(\bar{T}e^V T + \bar{U}e^{-V}U)$$

was taken. This matrix element is saturated in the infrared region, but by virtue of the universal character of the anomaly the result for the second loop in $1/[g^2]$ can be formulated as the replacement of the regulator mass $M_0 \to M_0/Z$ in the one-loop logarithm. In this form this result is valid in general for all loops [see (37)–(40)].

For a non-Abelian theory an explicit procedure for obtaining (46) by means of ultraviolet regularization has not been constructed, but there is no doubt that this can be done. It then seems natural that a situation analogous to that in SQED will obtain. Namely, the effect of the higher loops in $1/[g^2]$ reduces to the replacement $M_0/\mu \to ([Z_0]/[Z(\mu)])^\Delta M_0/\mu$ in the one-loop logarithm, where Δ is a certain power. (Since the ultraviolet procedure has not been specified, we have not ruled out the possibility that $\Delta \neq 1$.) In the given case, obviously, $[Z(\mu)] = 1/[g^2(\mu)]$.

The two-loop result, which is obtained by substituting (46) into (48), can be represented in the form

$$\frac{1}{[g^2(\mu)]} = \frac{1}{[g_0^2]} - \frac{3T(G)}{8\pi^2}\ln\left\{\frac{M_0}{\mu}\left(\frac{[Z_0]}{[Z(\mu)]}\right)^{1/3}\right\}, \qquad (62)$$

$$\frac{[Z_0]}{[Z(\mu)]} = \frac{[g^2(\mu)]}{[g_0^2]} = 1 + \frac{3T(G)}{8\pi^2}g_0^2\ln\frac{M_0}{\mu} + O(g_0^4). \qquad (63)$$

If now we do not expand $[Z_0]/[Z(\mu)]$ in the gauge constant, the result (62) is exact. From this we immediately obtain the β-function (2).

The fact that the exponent Δ in the given case has been found to equal 1/3 can be explained naturally in terms of calculations of the instanton type (see Sec. 5).

5. COMPARISON WITH CALCULATIONS OF THE INSTANTON TYPE

In the whole investigation we have leaned heavily on the fact that the coefficient of $\int d^2\theta W^2$ in S_W was renormalized only in one loop. As already discussed, this fact generalizes the well known theorem concerning the nonrenormalizability of the F-terms.[2] Below we shall give a somewhat nonstandard proof of the theorem, from which it will be seen in which cases the theorem can be violated. At the same time, our arguments will show why one-loop renormalization is possible for the structure $\int d^2\theta W^2$.

The basic idea of the construction to be developed is as follows. For any supersymmetric field theory there are several (a minimum of four) supercharge generators, and it is possible to choose an external field that is invariant under the action of some of the supercharges. For this special external field certain structures in the action can vanish. The assertion concerning the nonrenormalizability will pertain to those structures which do not vanish upon substitution of the background field.

For example, in the Wess-Zumino model with the action

$$S^{WZ} = \frac{1}{4}\int d^4x\, d^4\theta\, \bar{\Phi}\Phi + \frac{1}{2}\int (d^4x\, d^2\theta\, \mathscr{W}(\Phi) + \mathrm{h.c.}), \quad (64)$$

where $\mathscr{W}$ is the superpotential,the appropriate external field is

$$\bar{\Phi}^{ext}=0, \quad \Phi^{ext}=C_1+C_2{}^{\alpha}\theta_{\alpha}+C_3\theta^2, \tag{65}$$

where the C_i are constants. The relation (65) implies that Φ and $\bar{\Phi}$ are interpreted as independent variables, not related by complex conjugation (a kind of analytic continuation). The x-independent chiral field (65) is not changed under the action of the supercharges $\bar{Q}_{\dot{\alpha}}$, i.e., under the transformations

$$\delta\theta_{\alpha}=0, \quad \delta\bar{\theta}_{\dot{\alpha}}=\bar{\varepsilon}_{\dot{\alpha}}, \quad \delta x_{\alpha\dot{\alpha}}=2i\theta_{\alpha}\bar{\varepsilon}_{\dot{\alpha}}.$$

Consequently, in the quantum problem for the deviations $\Phi-\Phi_{\rm ext}$ we have exact symmetry under the transformations that can be generated by $\bar{Q}_{\dot{\alpha}}$. In the terminology of quantum states, we are concerned with boson-fermion degeneracy. There is enough of this degeneracy for the cancellation of all the quantum corrections to $\Gamma(\Phi_{\rm ext})$, i.e.,

$$\Gamma(\Phi_{ext})=S(\Phi_{ext}). \tag{66}$$

The situation is absolutely analogous to the calculation of corrections to the energy of the vacuum in empty space, i.e., with $\Phi_{\rm ext}=0$. The formula (66) implies that the second term in (64) is nonrenormalizable. The first term vanishes in the external field (65), and its renormalization is not fixed.

What changes when we go over to gauge theories [e.g., to supersymmetric gluodynamics with the action (14)]?

The general reasoning remains as before. As in the Wess-Zumino model, we can take the external field to be purely chiral and independent of the coordinates. Moreover, it is sufficient that these conditions be fulfilled for the intensities W_{α} and $\bar{A}_{\dot{\alpha}}$ but not for the prepotential V (Γ depends only on gauge-invariant quantities). Another, more complicated variant (see Ref. 5), which is also suitable for our purposes, is the instanton solution of Ref. 31. Although in this case the field depends on x, nevertheless there is invariance of the external field (more precisely, of gauge-invariant structures of the type W^2) under the supertransformations that can be generated by $\bar{Q}_{\dot{\alpha}}$ (but not by Q_{α}). From this we might arrive at the conclusion that renormalizations are absent in the structure $\int d^2\theta W^2$—a conclusion analogous to that reached above for the F-terms in the Wess-Zumino model. Such a conclusion is correct in the Wess-Zumino model, but, as is well known, is incorrect in gauge models.

A point omitted in the proof is that in certain cases the fermion-boson symmetry can be broken, namely, when the action of $\bar{Q}_{\dot{\alpha}}$ on the state gives zero. In a somewhat different language, more customary for calculations in external fields, the effect consists in the appearance of zero modes.

As is well known, in an instanton example (we recall that here and below we are discussing supersymmetric gluodynamics) the number of modes of a vector field with eigenvalue $\lambda_n^2\neq 0$ is equal to [3)] $4-2=2$. At the same time, two fermionic modes with eigenvalue λ_n and a further two with eigenvalue $-\lambda_n$ are present.[32]

This relationship between the bosonic and fermionic modes is a consequence of the invariance under $\bar{Q}_{\dot{\alpha}}$ and holds in any field with this invariance. It is not difficult to see that it is precisely this balance (1:2) which ensures the cancellation of the quantum corrections. (For instantons the phenomenon was first discovered in Ref. 33.) In particular, the one-loop correction is proportional to

$$\sum_{bos}\frac{1}{2}\ln\lambda_n{}^2-\sum_{ferm}\frac{1}{2}\ln|\lambda_n|$$

and vanishes to the extent that for each bosonic level there are two fermionic levels. Cancellation in the next loops is ensured by the exact symmetry under $\bar{Q}_{\dot{\alpha}}$.

We now turn to the zero modes. The same symmetry under $\bar{Q}_{\dot{\alpha}}$ leads here to an "incorrect" ratio of the numbers of bosonic and fermionic zero modes, namely, 2:1. In fact, the zero modes of the vector field $A_{\alpha\dot{\alpha}}$ and the spinor field λ_{α} are essentially the same, and satisfy the equations

$$\mathscr{D}^{\alpha\dot{\alpha}}\lambda_{\alpha}=0, \quad \mathscr{D}^{\alpha\dot{\alpha}}A_{\alpha\dot{\beta}}=0. \tag{67}$$

In the second of these equations the dotted index of the field $A_{\alpha\dot{\beta}}$ is not affected; this fact is also a consequence of the invariance under $\bar{Q}_{\dot{\alpha}}$. From this it is obvious that there are twice as many bosonic zero modes as fermionic zero modes. The imbalance in the zero modes also leads to the result that the quantum effects are not cancelled completely, and $\Gamma(W^{\rm ext})\neq S(W^{\rm ext})$. To be more specific, the zero modes give the following correction to Γ:

$$(\Delta\Gamma)_{zm}=-\sum_{bos}\frac{1}{2}\ln\frac{M_0{}^2[Z_0]}{\mu^2[Z]}+\sum_{ferm}\frac{1}{2}\ln\frac{M_0[Z_0]}{\mu[Z]}, \tag{68}$$

where the sums are taken over the bosonic and fermionic zero modes. The factors Z_0/Z take into account the fact that the higher loops affect the normalization of the zero modes (and only this normalization). The corresponding renormalization coincides with that of the external field, since the coefficients of the expansion in the zero modes have the meaning of collective coordinates of the external field. On the other hand, by definition, the renormalization of the external field coincides with the renormalization of the charge, i.e.,

$$[Z_0]/[Z]=[g^2]/[g_0{}^2], \tag{69}$$

where $g_0=g(M_0)$, and $g=g(\mu)$.

We now rewrite the answer (68) for $\Delta\Gamma$ in terms of the number n_f of fermionic zero modes:

$$(\Delta\Gamma)_{zm}=-{}^3/_2 n_f[\ln(M_0/\mu)+{}^1/_3\ln([Z_0][Z])]. \tag{70}$$

Next, the coefficient n_f is fixed by the index theorem:

$$n_f=\frac{T(G)}{16\pi^2}\int d^4x\,G_{\mu\nu}{}^a\tilde{G}_{\mu\nu}{}^a=\frac{T(G)}{16\pi^2}\int d^4x\,d^2\theta\,\mathrm{Tr}\,W^{\alpha}W_{\alpha}. \tag{71}$$

The second equality (71) is guaranteed by the self-duality of the external field. Substituting (71) into (70) and comparing the result with the initial action

$$S=\frac{1}{4g_0{}^2}\left\{\int d^4x\,d^2\theta\,\mathrm{Tr}\,W^2+\int d^4x\,d^2\bar{\theta}\,\mathrm{Tr}\,\bar{W}^2\right\}, \tag{72}$$

we find the charge-renormalization law [compare with (17) and (18)]

$$\frac{1}{[g^2]}=\frac{1}{[g_0{}^2]}-\frac{3T(G)}{8\pi^2}\left(\ln\frac{M_0}{\mu}+\frac{1}{3}\ln\frac{[g^2]}{[g_0{}^2]}\right). \quad (73)$$

Here it is appropriate to compare this result with the analysis in the preceding section. First of all, we note that the exponent 1/3 in the expression $(M_0/\mu)(Z_0/Z)^{1/3}$ [compare with (62)] arose in a natural manner. In fact, the coefficient of $\ln(M_0/\mu)$ in (68) is equal to $-(n_b-\frac{1}{2}n_f)$, while $\ln(Z_0/Z)$ is multiplied by $-\frac{1}{2}(n_b-n_f)$, where n_b is the number of bosonic zero modes.

The latter factor $\frac{1}{2}(n_b-n_f)$ is in one-to-one correspondence with the calculation, given in the preceding section, of the matrix element of Tr W^2. The residues at the poles that occur in K_μ and a_μ are assumed to be n_b and n_f, respectively. In fact, we have discovered an index theorem for the zero modes of a non-Abelian vector field. The fact that $n_b=2n_f$ is manifested in perturbation theory in the fact that the bosonic anomaly gives a contribution to the matrix element of Tr W^2 that is twice as large as (and of the opposite sign to) that given by the fermionic anomaly.

The arguments given above do not use the explicit form of the background field. Everything has been reduced to the number of zero modes, which are fixed by index theorems. Therefore, besides the instanton example, an x-independent self-dual external field is also suitable for our purposes. However, for such a field the integral $\int d^4xG\tilde{G}$ that appears in the index theorem needs, strictly speaking, to be defined more fully. One possible way of doing this is to introduce a finite volume[34,35]—specifically, a torus L^4. The simplest self-dual field in this case (the toron) was discovered by 't Hooft.[34] The intensity of the toron field does not depend on the coordinates, and the topological charge and action are half those for the instanton [in the color group SU(2)]. Our general reasoning in this case reduces to the following: There are two zero fermionic modes (they are generated when one acts on the toron field with the supertransformations Q_α) and four bosonic modes (ordinary translations).

It is instructive to find the matrix element of the operator Tr W^2 in the toron field. Essentially, this was done in Ref. 36, in which the condensate $\langle\lambda^\alpha\lambda_\alpha\rangle_{\text{toron}}$ was determined. The result for the average value in the toron field reduces to

$$\langle \mathrm{Tr}\, W^2\rangle=-\langle \mathrm{Tr}\,\lambda^2\rangle=CL^{-3}[g(L)]^{-2}\exp(-4\pi^2/[g^2(L)]), \quad (74)$$

where L is the size of the box. We have given formula (74) in order to emphasize that this result is exact—there are no corrections in g_0^2 to it. It can be seen from (74) that the operator Tr W^2 (but not $[\beta(\alpha)/\alpha^2]$Tr W^2) is a renormalization invariant; it is the matrix element of precisely this operator that can be expressed in terms of observable quantities and does not depend on M_0.

6. CONCLUSION

This paper, we hope, completes our lengthy endeavors in the study of two related problems in supersymmetric gauge theories: ultraviolet renormalizations, and the structure of the supermultiplet of anomalies. We have clarified how exact relations for the β-functions arise in ordinary perturbation theory. The key finding is that the Wilson action $S_W(\mu)$ does not coincide with the sum $\Gamma(\mu)$ of vacuum loops in an external field, because of the presence of infrared effects in the diagrams. The widely known theorem concerning the nonrenormalizability of the F-terms is extended to the operator $\int d^2\theta W^2$ in S_W. The coefficient $1/g^2$ of this operator is not renormalized at the two-loop and higher levels.[4)] The first coefficient in the β-function for the observable coupling constant expresses the renormalization of $1/g^2$, while the second and all the subsequent coefficients reflect the infrared effects of taking the matrix element.

The observable gauge constant $1/[g^2]$ appears in Γ and differs from $1/g^2$ by $\Sigma_i C_i \ln Z_i$, where the Z_i are the factors describing the renormalization of the fields, and the C_i are numbers that appear when the matrix elements are calculated. It can be said that the Z-factors of the matter fields become observable.

It can be seen that standard perturbation theory is extremely ineffective for calculating the renormalization of the gauge constant $[g^2]$. Working with ordinary supergraphs, one has to take into account much that is superfluous (ghosts, etc.), i.e., much that later drops out of the answer in any case. In this sense, calculations of the instanton type, which reduce the problem to an essentially classical problem with a finite number of degrees of fredom (a few zero modes), are much more economical. The Z-factors encountered on this route obviously pertain to the external fields and not to the quantum fields.

In the instanton approach the geometrical meaning of $n_b-\frac{1}{2}n_f$ is entirely transparent—it is the one-loop coefficient in the β-function. Essentially, this implies that the first loop is determined by infrared effects. Unfortunately, in ordinary perturbation theory we have not found a way of reasoning that adequately reflects this phenomenon.

Evidently (at least, we hope that this is so), ordinary perturbation theory can be improved in such a way that, in calculations of the renormalization of W^2, one should not have to be concerned at all with quantum fields, ghosts, etc.

As regards the problem of the supermultiplet of anomalies, in our approach it is solved as follows. Since the renormalization of $1/g^2$ in S_W is exhausted by one loop, in the anomaly equation [see (19)] for the supercurrent $J_{\alpha\dot{\alpha}}$ in its operator form the coefficient of W^2 is purely one-loop. It is this statement which generalizes the Adler-Bardeen theorem to supersymmetric theories. What is unusual is the fact that the anomaly in the trace $\Theta_{\mu\mu}$ of the energy-momentum tensor is determined by the first loop.[5)] The usual expression for $\Theta_{\mu\mu}$, which is proportional to the exact β-function, is recovered after the average is taken in an external gauge field. Then the same β-function arises in the matrix element of $\partial_\mu a_\mu$ as well. Here we are concerned with another unusual aspect—taking the average of the operator $G\tilde{G}$ does not reduce at all to replacing the operator by a c-number external field. The effect can be formulated as a manifestation of an anomaly in the current K_μ [see (57)].

We now briefly discuss the connection between our results and what was already known in the literature. In the papers of Piguet and Sibold[11] a detailed investigation of the quantity Γ and the matrix elements of the supercurrent was undertaken. The analysis was carried out in terms of Ward indentities. It was concluded that a supersymmetric construction $J_{\alpha\dot{\alpha}}$ and a supermultiplet of anomalies are possible, but their results (and the authors fully perceived this) were not formulated in terms of operator equations. Although, in principle, the program of Piguet and Sibold was fully correct [and, in fact, from the relations given in their papers one can extract the formula (9) for SQED], in practice the construction was excessively complicated. Our advance is due to the fact that we have introduced the following extra elements:

a) the language of operator expansion (the difference between S_W and Γ);

b) the supersymmetric descent from the mass shell, which is important for the statement concerning the one-loop renormalization of $1/g^2$ in S_W. We emphasize that the supersymmetric descent from the mass shell is important for the analysis of the coefficients in S_W; in Γ the details of the supplementary definition are not so important.

A subsequent series of investigations[13–16] was initiated by the work of Jones.[12] In this series of investigations the efforts were concentrated around the following question: How can the Alder-Bardeen theorem for $\partial_\mu a_\mu$ and the existence of higher orders in the trace $\Theta_{\mu\mu}$ be reconciled? The program, which was most clearly formulated in Ref. 13, consisted in the following: Two different axial currents were introduced, one of which (a_μ^{AB}) appeared in the Adler-Bardeen relation, while the other (a_μ^{SS}) was a term of a supermultiplet. It was assumed that these currents differ by an ultraviolet subtractive constant.

As we now understand, the very formulation of the problem was incorrect and was to a considerable degree associated with an incorrect interpretation of the status of the anomalies. In fact, the initial premise $\Theta_{\mu\mu} = [\beta(\alpha)/4\alpha] G^a_{\mu\nu} G^a_{\mu\nu}$ does not hold in the operator form, and, consequently, the principal stumbling block is removed. The Gell-Mann–Low function appears in the right-hand side of the equality for $\Theta_{\mu\mu}$ only in the case when the right-hand side is taken in the sense of a matrix element. On the other hand, the original proof of the theorem of Ref. 10 for $\partial_\mu a_\mu$ is in fact an operator statement. In the derivation of the theorem in Ref. 10 a certain two-limit technique was used (for a recent discussion see Ref. 20), with two regulator masses M_{R1} and M_{R2} ($M_{R1} \gg M_{R2}$). In the framework of this technique, corrections of second and higher orders to $\partial_\mu a_\mu$ are absent. This statement, as such, pertained to amplitudes with external momenta p in the interval $M_{R1} \gg p \gg M_{R2}$, i.e., in our sense it had an operator character.[6)] In Ref. 20 we generalized the two-limit technique to the supersymmetric case and found that the situation with $\Theta_{\mu\mu}$ is exactly the same as that with $\partial_\mu a_\mu$, i.e., $\Theta_{\mu\mu}$ is exhausted by one loop in the two-limit sense. The answer, however, could not be regarded as final, since in fact we were interested in one-limit regularization (i.e., the passage to $p \ll M_{R2}$) and the constructive calculation of the β-function. In the passage to $p < M_{R2}$ the clash with supersymmetry would have arisen again, since it appeared that taking the matrix elements of $G\tilde{G}$ and G^2 should have given different answers. It was tacitly assumed that the matrix element of the operator $G\tilde{G}$ coincided with a c-number function $G\tilde{G}$, while for G^2 there was no such coincidence.

The postulate (borrowed from Ref. 10) that the operator $G\tilde{G}$ coincides with the corresponding matrix element lay at the basis of the proof of the "no go" theorem of Ref. 14, which excludes the existence of a supermultiplet of anomlies. The theorem, of course, is invalid, since $\langle G\tilde{G}\rangle \neq G\tilde{G}^{\rm ext}$. An attempt to circumvent the "no go" theorem of Ref. 14 was made by Kazakov[18] and by Jones *et al.*[19] These authors assumed that a change of $G\tilde{G}$ in a supersymmetric calculation in comparison with that in a nonsupersymmetric calculation (of the Adler-Bardeen type) will arise because of a subtractive constant in $G\tilde{G}$.

In Refs. 21 and 22, which made a big impression on us in the technical aspect and partly stimulated the present investigation, the problem of the supermultiplet of currents was solved by a constructive two-loop calculation with the use of supersymmetric dimensional reduction. The authors explicitly constructed expressions for a_μ^{AB} and a_μ^{SS}. For $d = 4 - \varepsilon (\varepsilon > 0)$, besides W^2 there exists another operator ($\hat{\hat{\Gamma}}^2$) that is gauge-invariant in respect of the external field, where Γ is the connection and the double hat, in accordance with Refs. 21 and 22, denotes the projection onto the "extra" ε dimensions. The answer obtained in Refs. 21 and 22 for the two-loop diagram reduced to the operator $(C/\varepsilon^2)\int d^4x d^4\theta \hat{\hat{\Gamma}}\hat{\hat{\Gamma}}$, and not to the operator $(C/\varepsilon)\int d^2\theta d^4x W^2$ that arises in the one-loop diagram. Furthermore, the authors used the fact that in supersymmetric dimensional reduction $\bar{\nabla}^2\hat{\hat{\Gamma}}\hat{\hat{\Gamma}} = -\varepsilon W^2$.

According to the approach developed here, the solution of the problem of anomalies in no way requires the introduction of two axial currents, two operators $G\tilde{G}$, etc. The two currents introduced in Ref. 22 differ, in fact, not by an ultraviolet constant but by an infrared-singular expression. As a manifestation of this, the current difference $a_\mu^{AB} - a_\mu^{SS}$ from Ref. 22 cannot be written in the limit $\varepsilon \to 0$.

In our language the situation can be explained easily: In essence, in Ref. 22 what was calculated by means of the procedure of supersymmetric dimensional reduction was the matrix element of $\int d^2\theta W^2$, and this matrix element is built up wholly in the infrared region. As regards the question of the different schemes for the operator $G\tilde{G}$, the main point lies not in the difference between the operators in the different schemes but in the difference between the operator and the matrix element. The latter is fixed uniquely.

Here it is appropriate to clarify to which scheme of renormalizations the results (1), (2), and (9) for the β-functions belong. Our definition is close to the moment scheme. We fix the gauge charge $[g^2(\mu)]$ at a certain external-field momentum $p \sim \mu$. Expressing $[g^2(\mu)]$ in terms of g_0, the bare charge, and the cutoff parameter M_0, we obtain an expression for g_0^2 in terms of M_0. No subtractions are made in the process. It is, apparently, the latter fact which explains the discrepancy between our three-loop coefficient in the Gell-Mann–Low function and that found in Ref. 41.

In conclusion we mention Ref. 7, in which two derivations of formula (1) in the framework of perturbation theory were given. One of the derivations was based on infrared regularization in a box of finite volume. Although the relation (7) from Ref. 7 for the connection between $1/[\alpha]$ and $1/\alpha_0$ is correct, the reasoning that was used in the derivation [see formula (6) in Ref. 7] was not strictly correct.

The authors are grateful to V. I. Zakharov, Ya. I. Kogan, D. I. Kazakov, A. Yu. Morozov, V. A. Novikov, A. V. Smilga, V. V. Sokolov, and V. L. Chernyak for useful discussions.

[1)] The Z-factors for the matter fields in S_W and $\Gamma(Z(\mu)$ and $[Z(\mu)]$, respectively) evidently coincide. This is certainly so if the Konishi anomaly is purely one-loop; see below. There exist different arguments in favor of the equality $Z(\mu) = [Z(\mu)]$ for matter, and below we shall frequently not distinguish Z and $[Z]$ for matter fields. Where necessary, it is easy to trace which of the Z-factors is appearing in any particular expression.

[2)] We omit the part of $\Pi_{\mu\nu}$ of the form $-2g_{\mu\nu}\langle\varphi^*\varphi\rangle$, i.e., diagrams of the "tadpole" type, which are not important for the analysis.

[3)] Strictly speaking, a vector field in a fixed (covariant) gauge has four modes. However, allowance for the determinant of the ghosts is equivalent to discarding two modes.

[4)] Another example to which the generalized theorem applies is the Fayet-Iliopoulos D-term $\int d^4\theta V$ in the Abelian theory. The absence of two-loop, three-loop, etc., renormalizations for this term was discovered in Ref. 37. In order to demonstrate the applicability of our proof, we rewrite this term in the form $\int d^4\theta V \sim \int d\theta^\alpha \bar{D}^2 D_\alpha V \sim \int d\theta^\alpha W_\alpha$. It is clear that it can be called an F-term in the same sense as $\int d^2\theta W^2$.

[5)] Here there is a certain analogy with the supermultiplet of anomalies in an external gravitational field.[38–40] As was shown in Ref. 40, a correct treatment of the contribution of the scalar fields requires modification of $\Theta_{\mu\nu}$, not a_μ.

[6)] In Ref. 10 arguments were given that further descent into the region $P \ll M_{R2}$, i.e., the taking of the matrix element of $G\tilde{G}$, does not change the coefficient. Unlike the first part of the theorem, these arguments did not have a general character.

[1] Yu. A. Gol'fand and E. P. Likhtman, Pis'ma Zh. Eksp. Teor. Fiz. **13**, 323 (1971) [JETP Lett. **13**, 452 (1971)]; D. V. Volkov and V. P. Akulov, Phys. Lett. **46B**, 109 (1973); J. Wess and B. Zumino, Nucl. Phys. **B70**, 39 (1974).

[2] B. Zumino, Nucl. Phys. **B89**, 535 (1975); P. C. West, Nucl. Phys. **B106**, 219 (1976); M. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. **B159**, 429 (1979).

[3] P. S. Howe, K. S. Stelle, and P. C. West, Phys. Lett. **124B**, 55 (1983).

[4] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).

[5] V. A. Novikov, M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Nucl. Phys. **B229**, 381 (1983); Phys. Lett. **166B**, 329 (1986).

[6] M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Phys. Lett. **166B**, 334 (1986).

[7] A. I. Vaĭnshteĭn, V. I. Zakharov, and M. A. Shifman, Yad. Fiz. **43**, 1602 (1986) [Sov. J. Nucl. Phys. **43**, (1986)].

[8] S. Ferrara and B. Zumino, Nucl. Phys. **B87**, 207 (1975).

[9] M. Grisaru, in: Recent Developments in Gravitation (Cargese Lectures 1979) (eds. M. Levy and S. Deser), Plenum Press, New York (1979), p. 130.

[10] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).

[11] T. E. Clark, O. Piguet, and K. Sibold, Nucl. Phys. **B143**, 445 (1978); Nucl. Phys. **B172**, 201 (1980); O. Piguet and K. Sibold, Nucl. Phys. **B196**, 428, 447 (1982).

[12] D. R. T. Jones, Phys. Lett. **123B**, 45 (1983).

[13] M. T. Grisaru and P. C. West, Nucl. Phys. **B254**, 249 (1985).

[14] A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 161 (1984) [JETP Lett. **40**, 920 (1984)].

[15] P. Breitenlohner, D. Maison, and K. S. Stelle, Phys. Lett. **134B**, 63 (1984).

[16] D. R. T. Jones and L. Mezincescu, Phys. Lett. **136B**, 242 (1984); Phys. Lett. **138B**, 293 (1984).

[17] D. Espriu, Preprint No. 57/84, University of Oxford (1984).

[18] D. I. Kazakov, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 272 (1985) [JETP Lett. **41**, 335 (1985)].

[19] D. R. T. Jones, L. Mezincescu, and P. West, Phys. Lett. **151B**, 219 (1985).

[20] V. A. Novikov, M. A. Shifman, A. I. Vaĭnshteĭn, and V. I. Zakharov, Phys. Lett. **157B**, 169 (1985).

[21] M. T. Grisaru, B. Milewski, and D. Zanon, Phys. Lett. **157B**, 174 (1985).

[22] M. T. Grisaru, B. Milewski, and D. Zanon, Nucl. Phys. **B266**, 589 (1986).

[23] K. Konishi, Phys. Lett. **135B**, 439 (1984).

[24] T. E. Clark, O. Piguet, and K. Sibold, Nucl. Phys. **B159**, 1 (1979).

[25] K. Konishi and K. Shizuya, Preprint No. IFUP-TH-5/85, University of Pisa (1985).

[26] M. T. Grisaru and D. Zanon, Nucl. Phys. **B252**, 578 (1985); M. T. Grisaru and B. Milewski, Phys. Lett. **155B**, 357 (1985).

[27] A. I. Vaĭnshteĭn and M. A. Shifman, Yad. Fiz. **44**, 498 (1986) [Sov. J. Nucl. Phys. **44** (1986)].

[28] M. T. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. **B159**, 429 (1979); S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, Superspace, Benjamin Cummings, Reading, Mass. (1983).

[29] A. D. Dolgov and V. I. Zakharov, Nucl. Phys. **B27**, 525 (1971).

[30] G. 't Hooft, in: Recent Developments in Gauge Theories (eds. G. 't Hooft *et al.*), Plenum Press, New York (1980), p. 241.

[31] A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. **59B**, 85 (1975).

[32] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976); L. S. Brown, R. D. Carlitz, D. B. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1978).

[33] A. D'Adda and P. DiVecchia, Phys. Lett. **73B**, 162 (1978).

[34] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).

[35] E. Witten, Nucl. Phys. **B202**, 253 (1982).

[36] E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984).

[37] W. Fischler, H. P. Nilles, J. Polchinski, S. Raby, and L. Susskind, Phys. Rev. Lett. **47**, 757 (1981).

[38] M. J. Duff and P. van Nieuwenhuizen, Phys. Lett. **94B**, 179 (1980); W. Siegel, Phys. Lett. **103B**, 107 (1981); M. J. Duff, in: Supergravity 81 (eds. S. Ferrara and J. G. Taylor), Cambridge University Press (1982), p. 112.

[39] R. É. Kallosh, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 509 (1983) (JETP Lett. **37**, 607 (1983)].

[40] M. T. Grisaru, N. K. Nielsen, W. Siegel, and D. Zanon, Nucl. Phys. **B247**, 157 (1984).

[41] L. V. Avdeev and O. V. Tarasov, Phys. Lett. **112B**, 356 (1982).

Translated by P. J. Shepherd