The effect of commensurability in non-linear electron-sound interaction

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We study the generation of combination harmonics of two longitudinal sound waves propagating in the same direction in conductors in a constant magnetic field, for the case when the wavelengths of all waves are much smaller than the electron mean free path while the frequencies are less than the collision frequency. We prove that resonance (commensurability effect) may occur in this situation and manifest itself in a sharp dependence of the non-linear modulus that determines the difference-frequency generation on the ratio of the frequencies of the principal waves, provided this ratio is close to a rational number. The effect of the commensurability is a resonance on, generally speaking, arbitrary non-extremal Fermi-surface sections whose position depends on the frequencies of the principal waves and on the magnetic field. The width of the resonance is determined by the collision frequency on the resonant sections, averaged over a cyclotron period. The commensurability effect can be observed also for non-degenerate carriers.

It is well known that a sound wave in a conductor in a constant magnetic field **H** interacts resonantly with electrons which traverse, over a cyclotron period T, a path which is a multiple of the wavelength in the direction of the sound propagation[1]

$$T(p_{1z}^{(n)})q_{1z}\bar{v}_z(p_{1z}^{(n)})=2\pi n, \tag{1}$$

where p_z is the component of the quasi-momentum **p** in the direction of the magnetic field **H**, $\bar{v}_z(p_z)$ is the velocity on the Fermi surface along the magnetic field averaged over a cyclotron period, q_{1z} is the component of the sound wave vector $\mathbf{q}_1$ in the direction of **H**, and $n=0, \pm 1, \pm 2,..., \pm n_{\max}$. We assume that there are no open orbits, and moreover that $\omega_1^{-1}\gg\tau\gg T$, where ω_1 is the frequency of the wave and τ the relaxation time.

If there is yet another wave with frequency ω_2 and wavevector $\mathbf{q}_2\|\mathbf{q}_1$, it interacts resonantly with electrons for which the condition

$$T(p_{2z}^{(m)})q_{2z}\bar{v}_z(p_{2z}^{(m)})=2\pi m \tag{2}$$

is satisfied, with $m=0, \pm 1, \pm 2,..., \pm m_{\max}$.

Generally speaking, each of the waves interacts resonantly with its own group of electrons, except for Fermi-surface-section electrons with $\bar{v}_z=0$, which always interact resonantly with both waves. When $\omega_1/\omega_2=n/m$ (n and m being integers which have no comon divisor and $|n|\leqslant n_{\max}$, $|m|\leqslant m_{\max}$) there are additional sections with electrons which are at resonance with both waves, namely the sections with $p_z=p_{1z}^{(ln)}=p_{2z}^{(lm)}$, where $l=\pm 1, \pm 2,..., \pm l_{\max}$, $l_{\max}=n_{\max}/|n|=m_{\max}/|m|$.

Thus, if the ratio $\omega_1/\omega_2\to n/m$ when the frequencies are changed one must expect to observe resonance peculiarities in the generation of combination harmonics of the sound. It will be shown below that these peculiarities (commensurability effect) indeed may occur. The commensurability effect is a resonance on non-extremal orbits such that, in general,

$$\frac{\partial}{\partial p_z}(T\bar{v}_z)=\left|\frac{c}{eH}\right|\frac{\partial^2 S}{\partial p_z^2}\neq 0,$$

where e is the electron charge, c the velocity of light, and S the area of the Fermi-surface intersection with the plane $p_z=\text{const}$. We shall here restrict ourselves to considering just the resonance on non-extremal orbits.

1. We thus consider a conductor in a magnetic field in which two longitudinal sound waves propagate in the same direction. The lattice displacement vector $\mathbf{u}(\mathbf{r},t)$ is written in the form

$$\mathbf{u}(\mathbf{r},t)=\mathbf{u}_1(\mathbf{r})\exp\{i\mathbf{q}_1\mathbf{r}-i\omega_1 t\}+\mathbf{u}_2(\mathbf{r})\exp\{i\mathbf{q}_2\mathbf{r}-i\omega_2 t\}+\text{c.c.}, \tag{3}$$

where $\mathbf{u}_1(\mathbf{r})$ and $\mathbf{u}_2(\mathbf{r})$ are slowly varying wave amplitudes.

To describe the interaction of the electrons with longitudinal sound we shall, as in Ref. 2, where the generation of combination harmonics was considered in zero magnetic field, use the simplest model of a deformation potential in which the electron-lattice interaction energy $V=\Lambda\,\mathrm{div}\mathbf{u}$ and the elasticity equation can be written in the form

$$\rho_0\partial^2\mathbf{u}/\partial t^2=\lambda\Delta\mathbf{u}+\Lambda\nabla n, \tag{4}$$

where Λ is the deformation potential which we shall assume to be independent of the quasi-momentum, ρ_0 the crystal density, λ the elasticity modulus, and n the electron density.

Everything that was said in Ref. 2 about the applicability of this model for considering the generation of combination harmonics remains valid also in our case.

As the density, n depends non-linearly on the lattice displacement vector $\mathbf{u}(\mathbf{r},t)$, the initial waves (3) generate waves with combination frequencies $\mathbf{u}^{(\pm)}(\mathbf{r},t)$:

$$\mathbf{u}^{(\pm)}(\mathbf{r},t)=\mathbf{u}^{(\pm)}(\mathbf{r},t)\exp\{i(\mathbf{q}_1\pm\mathbf{q}_2)\mathbf{r}-i(\omega_1\pm\omega_2)t\}+\text{c.c.} \tag{5}$$

To fix the ideas we shall assume that $\omega_1>\omega_2>0$.

Assuming that the synchronism conditions are satisfied (the sound dispersion law is linear to high degree of accuracy) we can obtain by the usual methods equations for the slowly varying amplitudes $u^{(\pm)}(\mathbf{r})$:

$$\begin{aligned}\partial u^{(+)}(\eta)/\partial\eta&=\alpha^{(+)}u_1(\eta)u_2(\eta)-\gamma^{(+)}u^{(+)}(\eta),\\ \partial u^{(-)}(\eta)/\partial\eta&=\alpha^{(-)}u_1(\eta)u_2^*(\eta)-\gamma^{(-)}u^{(-)}(\eta),\end{aligned} \tag{6}$$

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where $\gamma^{(+)}$ and $\gamma^{(-)}$ are the linear damping coefficients for sound at, respectively, the sum and the difference frequencies, and η is the coordinate along the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$,

$$\alpha^{(\pm)}=\pm(2\bar{\lambda})^{-1}\Lambda^3\{\chi^{(2)}(\mathbf{q}_1,\omega_1;\pm\mathbf{q}_2,\pm\omega_2)+\chi^{(2)}(\pm\mathbf{q}_2,\pm\omega_2;\mathbf{q}_1,\omega_1)\}q_1q_2, \qquad (7)$$

$\bar{\lambda}$ is the renormalized elasticity modulus, $\bar{\lambda}=\rho_0 s$, where s is the true sound speed.

The non-linear susceptibility $\chi^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)$ determines the correction $\delta n^{(2)}$ to the electron density,

$$\delta n^{(2)}=[\chi^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)+\chi^{(2)}(\mathbf{q}_2,\omega_2;\mathbf{q}_1,\omega_1)]V_1V_2\exp\{i(\mathbf{q}_1+\mathbf{q}_2)r-i(\omega_1+\omega_2)t\}+\text{c.c.}, \qquad (8)$$

necessitated by the action of the two waves ($V=V_1\times\exp\{i\mathbf{q}_1\mathbf{r}-i\omega_1 t\}+V_2\exp\{i\mathbf{q}_2\mathbf{r}-i\omega_2 t\}+\text{c.c.}$).

2. We now turn to an evaluation of the non-linear susceptibility $\chi^{(2)}$. We write down the Boltzmann kinetic equation for the electron distribution function f:

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{dt_1}{dt}\frac{\partial f}{\partial t_1}-(\nabla V)_z\frac{\partial f}{\partial p_z}-(\mathbf{v}\nabla V)\frac{\partial f}{\partial \varepsilon}=I(f), \qquad (9)$$

where t_1 is the time of motion along the trajectory in the constant magnetic field $\mathbf{H}$, ε the energy, $I(f)$ the collision integral, and V the potential energy

$$V=V_1\exp\{i\mathbf{q}_1\mathbf{r}-i\omega_1 t\}+V_2\exp\{i\mathbf{q}_2\mathbf{r}-i\omega_2 t\}+\text{c.c.} \qquad (10)$$

So far we shall not assume that the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in (10) are parallel. The signs of ω_1 and ω_2 can be arbitrary. We write the derivative dt_1/dt in (9) in the form

$$dt_1/dt=1-\nabla V\partial t_1/\partial \mathbf{p}. \qquad (11)$$

We can use for the collision integral in our case the following approximation

$$I(f)=-[f-f_0(\varepsilon+V)]/\tau, \qquad (12)$$

where f_0 is the equilibrium distribution function and $\tau=\tau(\mathbf{p})$ the drift relaxation time.

Solving the kinetic equation by an iteration method we easily get an expression for $\chi^{(2)}$. We can write it in the form

$$\chi^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)=\chi_0^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)+\chi_r^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2), \qquad (13)$$

where

$$\chi_0^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)=\frac{1}{2}\frac{2}{(2\pi)^3}\left|\frac{eH}{c}\right|\int d\varepsilon\,dt_1\,dp_z\left[1+i\omega\exp\{-i\varphi(t_1,\mathbf{q},\omega)\}\int_{t_1}^{t_1+T}dt_2\exp\{i\varphi(t_2,\mathbf{q},\omega)\}\right]\times R(\mathbf{q},\omega)\frac{\partial^2 f_0}{\partial\varepsilon^2}, \qquad (14)$$

$$R(\mathbf{q},\omega)=\{\exp[iT(q_z\bar{v}_z-\omega-i\overline{\tau^{-1}})]-1\}^{-1},$$

$$\varphi(t_1,\mathbf{q},\omega)=\int_0^{t_1}dt'(\mathbf{q}\mathbf{v}(t')-\omega-i\tau^{-1}),$$

$$\overline{\tau^{-1}}=\frac{1}{T}\int_0^T dt\,\tau^{-1}(t),\qquad \mathbf{q}=\mathbf{q}_1+\mathbf{q}_2,\qquad \omega=\omega_1+\omega_2.$$

For $\chi_r^{(2)}$ we have

$$\chi_r^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)=-\frac{2}{(2\pi)^3}\left|\frac{eH}{c}\right|\int dt_1\,dp_z\,d\varepsilon\times R(\mathbf{q},\omega)\exp\{-i\varphi(t_1,\mathbf{q},\omega)\}\int_{t_1}^{t_1+T}dt_2\exp\{i\varphi(t_2,\mathbf{q},\omega)\}\times\left\{q_{2z}\frac{\partial}{\partial p_z}+\mathbf{q}_2\mathbf{v}\frac{\partial}{\partial\varepsilon}+\mathbf{q}_2\frac{\partial t_2}{\partial \mathbf{p}}\frac{\partial}{\partial t_2}\right\}\times\exp\{-i\varphi(t_2,\mathbf{q}_1,\omega_1)\}R(\mathbf{q}_1,\omega_1)\int_{t_2}^{t_2+T}dt_3\times\omega_1\exp\{i\varphi(t_3,\mathbf{q}_1,\omega_1)\}\frac{\partial f_0}{\partial\varepsilon}. \qquad (15)$$

We restrict ourselves here to the case when $q_1r_H\sim q_2r_H\sim qr_H\sim 1$, where r_H is the characteristic Larmor radius and, moreover, $(\mathbf{q}_1\mathbf{H})\sim(\mathbf{q}_2\mathbf{H})\lesssim 1$.

In accordance with what was said above we shall assume that $\omega_1\sim\omega_2\ll\tau^{-1}\ll\Omega$, where Ω is the cyclotron frequency, $\Omega=|eH/m^*c|$.

In that situation we get from (14)

$$\chi_0^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)=\chi_0^{(2)}=(1/2)\partial^2 n_0/2\partial\varepsilon_F^2, \qquad (16)$$

where n_0 is the equilbrium electron density and ε_F the Fermi energy.

To evaluate $\chi_r^{(2)}$ we must recognize that the resonance factors $R(\mathbf{q},\omega)$, $R(\mathbf{q}_1,\omega_1)$ in (15) have poles in the variable p_z close to the real axis (at distances much less than the characteristic Fermi momentum p_F). If $q_zq_{1z}>0$ the poles of both factors $R(\mathbf{q},\omega)$ and $R(\mathbf{q}_1,\omega_1)$ lie on the same side of the real axis and one checks easily from (15) that one can neglect the quantity $\chi_r^{(2)}$ in comparison with $\chi_0^{(2)}$ (because of the parameter $\omega_{1,2}/\Omega$). When $q_zq_{1z}<0$ the poles of the factors $R(\mathbf{q},\omega)$ and $R(\mathbf{q}_1,\omega_1)$ lie on different sides of the real axis. Each of these factors has one or more fixed poles close to the points $\bar{v}_z(p_z)=0$ and a sequence of fixed poles the positions of which depend on the z-components of the wavevectors. When the ratio q_z/q_{1z} tends to a rational number, a subset of the poles of the function $R(\mathbf{q},\omega)$ may approach a subset of the poles of the function $R(\mathbf{q}_1,\omega_1)$. In that case the main contribution to $\chi_r^{(2)}$ is caused by pairs of poles of $R(\mathbf{q},\omega)$ and $R(\mathbf{q}_1,\omega_1)$ which lie close to one another but on different sides of the real axis.

Thus, let $kq_z\approx nq_{1z}$, where k and n are integers which do not have a common divisor, such that

$$|2\pi k/q_{1z}|,\quad |2\pi n/q_z|<|(T\bar{v}_z)_{max}|. \qquad (17)$$

In that case we can get from (15) after some transformations

$$\chi_r^{(2)}(\mathbf{q}_1,\omega_1;\mathbf{q}_2,\omega_2)=4\pi i\theta(-q_z q_{1z})\frac{|q_z|q_{2z}\omega_1}{q_{1z}}\sum_{l=0}^{l_{max}}\left(1-\frac{1}{2}\delta_{0l}\right)\mathrm{sign}\left(\frac{\partial}{\partial p_z}(T\bar{v}_z)\right)$$
$$\times\frac{m^*(p_z)}{\Omega^2(p_z)}\left\{\frac{J^*_{nl}(\mathbf{q},p_z)J_{(n-k)l}(\mathbf{q}_2,p_z)J_{kl}(\mathbf{q}_1,p_z)}{\left[2\pi kl\left(q_z/q_{1z}-\frac{n}{k}\right)+iT\overline{\tau^{-1}}q_{2z}/q_{1z}\right]^2}+\text{c.c.}\right\}\Bigg|_{p_z=p_{1z}^{(lk)},\ \varepsilon=\varepsilon_F} \quad (18)$$

where $\theta(-q_z q_{1z})$ is the theta function and δ_{0l} the Kronecker symbol,

$$J_n(\mathbf{q},p_z)=\frac{1}{T}\int_0^T dt\exp\left\{i\int_0^t dt'\,\mathbf{q}(\mathbf{v}(t',p_z)-\bar{\mathbf{v}}(p_z))+in\Omega(p_z)t\right\},\quad (19)$$

$\bar{\mathbf{v}}(p_z)$ is the velocity averaged over a cyclotron period and the quasi-momenta $p_{1z}^{(lk)}$ are determined from the condition

$$T(p_{1z}^{(lk)})\bar{v}_z(p_{1z}^{(lk)})q_{1z}=2\pi kl,\quad (20)$$

l is a positive integer and l_{max} a positive integer such that

$$l_{max}<|q_{1z}(T\bar{v}_z)_{max}/2\pi k|<(l_{max}+1).\quad (21)$$

Generally speaking, $T\bar{v}_z$ can be a non-monotonic function of p_z. Therefore there are some values of the product kl in (20) to which several values of the quasi-momentum p_z may correspond. It is clear that we must take into account in (18) all solutions of Eq. (20). We note that if the electrons of the sections $p_z=p_{1z}^{(kl)}$ are at resonance with both waves the electrons of the cross-sections $p_z=-p_{1z}^{(kl)}$ are also at resonance (the dispersion law $\varepsilon(p)$ is by assumption an even function of the quasi-momentum p). The contribution of the sections $p_z=-p_{1z}^{(lk)}$ to (18) can be expressed in terms of the contribution of the sections $p_z=p_{1z}^{(lk)}$. It is described by the second terms in the braces in Eq. (18).

In obtaining Eq. (18) we assumed that the difference $|(q_z/q_{1z})-(n/k)|\ll 1$. If, however, the ratio of q_z and q_{1z} is such that this difference is not small for any values of n and k satisfying the inequality (17), the main contribution to $\chi_r^{(2)}$ comes from electrons with small $\bar{v}_z$. This contribution is described by the term (or terms) with $l=0$ in (18). As in this situation one can neglect terms with $l\neq 0$ in (18), it is clear that Eq. (18) is valid for any ratio of q_z and q_{1z}.

3. We now evaluate the coefficients $\alpha^{(\pm)}$ in Eq. (6) which determine the generation of combination sound frequencies. From (17), (13), and (18) we get

$$\alpha^{(\pm)}=(2\lambda)^{-1}\Lambda^3 q_1 q_2\partial^2 n_0/\partial\varepsilon_F^2,\quad (22)$$

$$\alpha^{(-)}=-(2\bar{\lambda})^{-1}\Lambda^3 q_1 q_2\left\{\frac{\partial^2 n_0}{\partial\varepsilon_F^2}+4\pi i\left(\frac{\omega_1-\omega_2}{\omega_2}\right)\omega_2 q_{1z}\sum_{l=0}^{l_{max}}\left(1-\frac{1}{2}\delta_{0l}\right)\mathrm{sign}\left(\frac{\partial}{\partial p_z}(T\bar{v}_z)\right)\frac{m^*(p_z)}{\Omega^2(p_z)}\right.$$
$$\left.\times\left[\frac{J^*_{(k-m)l}(\mathbf{q}_1-\mathbf{q}_2,p_z)J_{-ml}(\mathbf{q}_1,p_z)J_{kl}(-\mathbf{q}_2,p_z)}{[2\pi kl(\omega_1/\omega_2-m/k)+iT\overline{\tau^{-1}}\omega_1/\omega_2]^2}+\text{c.c.}\right]\Bigg|_{p_z=p_{1z}^{(ml)}}\right\},\quad (23)$$

where m and k are positive integers ($k<|q_{2z}(T\bar{v}_z)_{max}/2\pi|$, $m<|q_{1z}(T\bar{v}_z)_{max}/2\pi|$) for which $|\omega_1/\omega_2-m/k|$ is a minimum. Here l_{max} is given by the obvious condition

$$l_{max}<|q_{1z}(T\bar{v}_z)_{max}/2\pi m|<(l_{max}+1).$$

(In (22), (23), as in section 1, $\mathbf{q}_1\uparrow\uparrow\mathbf{q}_2$, $q_1>q_2>0$, $\omega_1>\omega_2>0$.) It follows from Eqs. (22) and (23) that the coefficient $\alpha^{(+)}$ is determined by the static response of the electron system to the inhomogeneous field; this response is independent of the magnetic field and is produced by the electrons from the whole of the Fermi surface, while the electrons which interact resonantly with the two sound waves contribute to the coefficient $\alpha^{(-)}$ as in the case when there is no magnetic field.[2]

Bearing in mind that in our situation $J_k(\mathbf{q}_1-\mathbf{q}_2,p_z)\sim J_k(\mathbf{q}_{1,2},p_z)\sim 1$ for $k\sim 1$ (and also for $k=0$), we get from (22) and (23), for any ratio of ω_1 to ω_2,

$$|\alpha^{(-)}/\alpha^{(+)}|\sim 1+(s/v_F)(q_2 v_F\tau)^2,\quad (24)$$

where v_F is the Fermi velocity.

It is clear from (24) that the ratio $|\alpha^{(-)}/\alpha^{(+)}|$ can be very large. In order of magnitude it is the same as in zero magnetic field.[2] The contribution from the resonant particles to $\alpha^{(-)}$ dominates if

$$(s/v_F)(q_2 v_F\tau)^2\gg 1.\quad (25)$$

As an example we consider the case when in (23) $l_{max}=1$, $k=1$, $m=3$, and we assume that $T\bar{v}_z$ is a monotonic function of p_z. The resonance contribution to $\alpha^{(-)}$ will then come from electrons with $p_z\approx 0$ ($\bar{v}_z(p_z=0)=0$) and from electrons from the non-extremal sections $p_z=\pm p_{1z}^{(3)}$. If $|(\omega_1/\omega_2)-3|\gg\overline{\tau^{-1}}/\Omega$ the contribution from electrons $p_z\approx 0$ may be larger than the contribution from the sections $p_z=\pm p_{1z}^{(3)}$. When $|\omega_1/\omega_2-3|\lesssim\overline{\tau^{-1}}/\Omega$ they are, in general, of the same order of magnitude.

We see thus that in this case there is resonance (commensurability effect) manifesting itself in a sharp dependence of the non-linear modulus $\alpha^{(-)}$, meaning also of the amplitude of the sound with the differences frequency, on the ratio ω_1/ω_2 at $\omega_1/\omega_2 \approx 3$, $(s/v_F)(q_2 v_F \tau)^2 \gtrsim 1$. The ratio of $\alpha^{(-)}$ at resonance to $\alpha^{(-)}$ far from resonance is, generally speaking, of the order unity, provided that $(s/v_F)(q_2 v_F \tau)^2 \gtrsim 1$. The width of the resonance is determined by the reciprocal relaxation time $\overline{\tau^{-1}}$ on the non-extremal sections.

The commensurability effect does not manifest itself in the non-linear modulus $\alpha^{(+)}$ and hence in the sound wave with the sum frequency. One can check from Eq. (18) that if the principal waves propagate in opposite directions the wave undergoing resonance caused by the commensurability effect will be the one with the sum frequency and not with the difference frequency.

[1]É. A. Kaner, V. G. Peschanskiĭ, and I. A. Privorotskiĭ, Zh. Eksp. Teor. Fiz. **40**, 214 (1961).

[2]A. P. Kopasov, Fiz. Nizk. Temp. **12**, 403 (1986) [Sov. J. Low Temp. Phys. **12**, No. 4 (1986)].

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