Classical diffusion in media with weak disorder

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Renormalization-group analysis of the effective-field theory is used to show that weak disorder with long-range correlation influences substantially the diffusion coefficient and the mobility of a randomly walking particle in a two-dimensional system. It is shown that the effective temperature is also renormalized in models with nonpotential random forces, which leads to modification of the Einstein relation and of the Kubo formula. The formalism developed in the paper is used to calculate the excess-current-noise spectral density and the mesoscopic fluctuations of the kinetic coefficients. Possible applications to systems with hopping conduction are discussed.

INTRODUCTION

Hopping conduction by electrons in semiconductors, or exciton migration in molecular crystals, are examples of incoherent propagation of quasiparticles. In an ordered lattice, this motion is known to reduce to classical diffusion in the continual limit. A basic problem is that of the influence of weak disorder in the medium on the character of the diffusive motion. Thus, weak disorder is influential for coherent propagation of particles (quantum diffusion) in systems with dimensionality $d \leqslant 2$ (Refs. 1 and 2), owing to quantum interference in multiple scattering. There is, of course, no such mechanism for incoherent propagation (classical diffusion). At the same time, classical diffusion, in contrast to quantum, is sensitive to the type of disorder.

Weak disorder with long-range correlations was found to influence substantially the character of classical diffusion. Such a situation was observed by Derrida and Luck[3] in an analysis of a hopping model with asymmetric hopping probability ($W_{\mathbf{rr}'} \neq W_{\mathbf{r}'\mathbf{r}}$, where $W_{\mathbf{rr}'}$ is the probability of hopping from site $\mathbf{r}'$ to site $\mathbf{r}$):

$$\partial_t P_{\mathbf{r}} = \sum_{\mathbf{r}'} (W_{\mathbf{rr}'} P_{\mathbf{r}'} - W_{\mathbf{r}'\mathbf{r}} P_{\mathbf{r}}). \qquad (1)$$

It was shown that a weak random asymmetry leads at $d = 2$ to logarithmic divergences (at long times or low frequencies) in the perturbation-theory series for the diffusion and mobility coefficients.

In the continual limit, Eq. (1) goes over into the Fokker-Planck (FP) equation

$$[\partial/\partial t + \nabla(\mathbf{v} - D_0 \nabla)] P(\mathbf{r}, t) = 0, \qquad (2)$$

where $P(\mathbf{r}, t)$ is the distribution function of the randomly walking particle. Weak disorder in the symmetric hopping model ($W_{\mathbf{rr}'} = W_{\mathbf{r}'\mathbf{r}}$) reduces to random spatial fluctuations of the diffusion coefficient D_0 in (2), but these turn out to be insignificant for all d. A random asymmetry in the hopping probability gives rise to a random stationary velocity field $v_\alpha(\mathbf{r})$ in the FP equation (2) (α is a vector index) with zero mean value and with a correlator

$$\langle v_\alpha(\mathbf{r}) v_\beta(\mathbf{r}') \rangle = \gamma_0 \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}'). \qquad (3)$$

The presence of a random field v influences substantially the asymptotic behavior of the kinetic coefficients.

An equivalent formulation of the problem (2), (3) is given by the Langevin equation, which describes random walks of a particle under the influence of thermal noise $\boldsymbol{\eta}(t)$ in a random drift field $\mathbf{v}(\mathbf{r})$:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}) + \boldsymbol{\eta}(t), \qquad (4)$$

where $\boldsymbol{\eta}(t)$ is Gaussian white noise, $\bar{\boldsymbol{\eta}} = 0$,

$$\overline{\eta_\alpha(t) \eta_\beta(t')} = 2D_0 \delta_{\alpha\beta} \delta(t - t'). \qquad (5)$$

The model described by Eqs. (2) and (3) or by Eqs. (3)–(5) was considered in a number of recent papers.[4–8]

The exact solution obtained by Sinai[4] for the one-dimensional model demonstrates the strongly subdiffusive character of the random walks. A renormalization-group analysis carried out in Refs. 5–8 within the framework of a field-theoretical approach has shown that the weak disorder (3) is significant also at $d = 2$ (upper critical dimensionality). It was shown that the asymptotics of the diffusion coefficient[7] and of the mobility[5] are of the form

$$D(t) \propto D_0 (1 + \text{const}/\ln t), \qquad (6a)$$

$$\mu(t) \propto \mu_0 / \ln t. \qquad (6b)$$

Classical diffusion in disordered media is of interest, in part, because disorder in classical systems is considered as a possible universal source of the excess ($1/f$) current noise. The model [(3)–(5)] was considered recently just in this context.[9]

The correlator (3) of the random forces (velocities) has the simplest tensor structure. This may seem to be the most natural form of disorder in the models (4) and (5). This, however, is not so, as indicated even by the patent violation of the Einstein relation $D_0 = \mu_0 T$ between the macroscopic diffusion coefficient and the mobility (6). The point is that a random field with a correlator (3) is nonpotential, and owing to the presence of undamped stationary currents there is no complete thermodynamic equilibrium in a system with disorder of this type. By the same token, the feasibility of direct realization of the disorder (3) in lattice hopping

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systems (1) is doubtful. On the other hand, the most natural realization of the model (4), (5) with undamped stationary flows—diffusion in an incompressible liquid—is likewise not described by correlator (3).

It is thus of interest to consider two natural limiting situations: models with potential and solenoidal random field $\mathbf{v}(\mathbf{r})$. We proceed now to describe these models.

§1. DESCRIPTION OF MODEL. BASIC RESULTS

We consider models in which the disorder is described by a stationary random field $\mathbf{v}(\mathbf{r})$ in the Fokker-Planck equation (2) or in the Langevin equation (4), with the natural vector constraints imposed on the field $\mathbf{v}$:

$$\partial_\alpha v_\alpha(\mathbf{r})=0, \text{ solenoidal field (model II)}, \tag{7a}$$

$$\partial_\alpha v_\beta(\mathbf{r})-\partial_\beta v_\alpha(\mathbf{r})=0, \text{ potential field (model III)}. \tag{7b}$$

[The random field $\mathbf{v}$ in model III can obviously be regarded as a gradient of a random potential: $v_\alpha(\mathbf{r}) = -\partial_\alpha u(\mathbf{r})$.] The model with random field unrelated to the correlator (3) will be designated model I. In all the models, the random field $v(r)$ is assumed Gaussian with correlators $\langle \mathbf{v}\rangle = 0$ and

$$\langle v_\alpha(\mathbf{r}) v_\beta(\mathbf{r}')\rangle=\gamma_0 F_{\alpha\beta}(\mathbf{r}-\mathbf{r}'), \tag{8}$$

where the Fourier component $F_{\alpha\beta}$ of the correlator is of the form

$$F_{\alpha\beta}(\mathbf{k}) = \begin{cases} \delta_{\alpha\beta}, & \text{isotropic disorder (model I)}, \quad (8a) \\ \delta_{\alpha\beta} - k_\alpha k_\beta/k^2, & \\ \quad \text{transverse disorder (model II)}, & (8b) \\ k_\alpha k_\beta/k^2, & \\ \quad \text{longitudinal disorder (model III)}. & (8c) \end{cases}$$

A correct definition of the models implies the required ultraviolet cutoff in the correlators.

In models II and III the correlator in r-space contains a long-range part $\propto 1/r^2$ (at $d=2$). Model I, the local form of the correlator (3) notwithstanding, is also essentially "long-range." The point is that the essential disorder, as will be made clear, is one for which $F_{\alpha\beta}(k) \neq 0$ in the long-wave limit $k \to 0$. In this sense, all three models are analogous.

Model II describes, for example, Brownian motion of a particle in an incompressible liquid with random stationary flows. Model II, as will be shown below, appears naturally when hopping conduction is described in a $2D$ medium with charge impurities that produce a three-dimensional Coulomb field. This refers to systems in which the free-carrier density is much lower than that of the charged impurities. The screening radius can then be large enough compared with the length of the elementary hop. This is essential, since the screening radius, just as the phase relaxation of the phase τ_φ in quantum diffusion, serves as the infrared-cutoff parameter in the logarithmically divergent expressions.

Models II and III were independently introduced by Fisher *et al.*[10] and by us.[11] In a different physical context (seepage of a turbid liquid through a porous filter) these models were considered earlier by Aronovitz and Nelson.[12]

A renormalization-group analysis has shown that, in contrast to model I, weak disorder leads in models II and III to a substantial change of the asymptotic form of the mean squared displacement[10–12]:

$$D(t) \equiv \frac{\langle r^2(t)\rangle}{t} \propto \begin{cases} \ln^{1/2}(t/\tau), & \text{model II}, \\ (t/\tau)^{-g_0/2}, & \text{model III}. \end{cases} \tag{9}$$

Here

$$g_0=\gamma_0/(4\pi D_0^2) \tag{10}$$

at $d=2$ is a dimensionless parameter of the (weak) disorder. Expressions (9) are valid in the asymptotic region $g_0 \ln(t/\tau) \gg 1$, where τ is the ultraviolet-cutoff parameter and has, in lattice realizations, the meaning of the elementary hop.

No less interesting is the question of the response of a system to an external field in the presence of weak disorder. In model II, the disorder does not influence the effective mobility.[12] In model III, as will be shown, the mobility variation is similar to that of the diffusion coefficient. Thus,

$$\mu(t) \propto \begin{cases} 1, & \text{model II}, \\ (t/\tau)^{-g_0/2}, & \text{model III}, \end{cases} \tag{11}$$

where $\mu\mathbf{E} = \langle \mathbf{r}(t)\rangle/t$ and E is the external field. The low-frequency asymptotic form of the conductivity is also given by Eq. (11) with $t \to \omega^{-1}$.

Comparison of Eqs. (9) and (11) shows that the Einstein relation $D = \mu T$ is violated in model II, just as in model I [Eq. (6)]. It turns out that the fluctuation-dissipation theorem (FDT) is also violated in these models. We shall discuss the causes of the violation and show that the relations in questions are valid if modified. Namely, besides the renormalization of the kinetic temperatures it is necessary to take into account also the temperature renormalization, which is determined independently from the macroscopic Boltzmann distribution. We shall show that in model III the temperature is not renormalized because the model is potential, so that disorder does not alter the Einstein relation and the FDT.

The foregoing results pertained to physical quantities averaged over an ensemble of realizations of a random field. They are meaningful only if the relative fluctuations of the physical quantities from sample to sample are small. Using the formalism developed for the calculation of the correlator, we have shown that the relative fluctuations are indeed small in terms of the weak-disorder parameter g_0 [Eq. (10)]. It is important that the fluctuations are considerably smaller than the disorder-governed corrections to the kinetic coefficients, with values $\sim g_0 \ln(t/\tau) \gg g_0$. Nonetheless, in view of the long-range correlations of the random field, these fluctuations are anomalously large: they do not decrease with increasing size of the system, in contrast to the ordinary thermodynamic fluctuations ($\propto V^{-1/2}$). The ensuing situation is thus similar to that recently established in the quantum-diffusion problem.[13–15]

Marinari *et al.*[9] and Fisher[7] have recently speculated whether diffusion in a random field, described by model I

[Eqs. (3)–(5)] can be a universal source of the $1/f$ noise. The argument advanced in Ref. 9 was based on the subdiffusive behavior revealed by the exact solution of the one-dimensional model,[4] and also on certain numerical results that attested to the possibility of such a behavior also at $d = 2$. The analytic result (6a) obtained for weak disorder did not confirm the subdiffusive-behavior hypothesis at $d = 2$. To clarify the situation, we calculate here directly the current fluctuations having different diffusive behavior (5) in models I–III. An excess noise was found for models I and III, and the expression for its spectral density coincides in the linear (ohmic) regime with the known empirical Hooge formula (see the reviews[16–18]). At sufficiently low frequencies density coincides in the linear (ohmic) regime with the known empirical Hooge formula (see the reviews[16–18]). At sufficiently low frequencies, however, a nonlinear regime sets in; the $1/\omega$ dependence in the spectral density of noise is then saturated, as is also the frequency dependence of the conductivity.

The plan of the article is the following. In §2, using the representation of Green's functions in a path-integral form, we average over the realizations of the random field **v**. A renormalization group (RG) analysis of the ensuing effective theory of the field yields in §3 expressions for the renormalized diffusion and mobility coefficients. Part of the content of §§2 and 3 was described in a brief communication.[11] In §4 is described a formalism for the calculation of the correlators of the quantities of interest to us. In §5 we analyze the Einstein relation and the temperature renormalization. In §6 we prove the Kubo formula (FDT) in the disorder model III and show how it is modified in models I and II. In §7 is discussed the feasibility of self-averaging of the kinetic coefficients, as well as mesoscopic effects. In §8 is calculated the current pair correlator in the presence of an external field, and an expression is obtained for the spectral density of the excess noise. In §9 we trace the connection between the considered models, on the one hand, and the lattice models of hopping conduction, on the other. A qualitative interpretation of the results is proposed in the Conclusion.

§2. DERIVATION OF THE EFFECTIVE FUNCTIONAL

All the physical quantities of the system are expressed in terms of the Green's functions of the FP equation (2). We consider the motion of a particle in an external field **E**. This calls for replacing the random velocity $\mathbf{v}(\mathbf{r})$ in Eq. (2) by

$$\mathbf{V}(\mathbf{r}) = \mu_0\mathbf{E} + \mathbf{v}(\mathbf{r}).$$

We begin with consideration of the averaged Green's function $\langle G\rangle$ in terms of which the effective diffusion coefficient and the mobility are directly expressed. In the absence of disorder, the Green's function G_0 is the usual diffusive propagator

$$G_{0\omega}(k) = (-i\omega + i\mu_0\mathbf{kE} + D_0k^2)^{-1}. \qquad (12)$$

To calculate the Green's function in a disordered medium we can develop a perturbation theory in terms of the random field **v**, starting from the FP equation for the Green's function:

$$\hat{G}^{-1}G \equiv \hat{G}_0^{-1}G + \partial_\alpha(v_\alpha G) = \hat{I}. \qquad (13)$$

Averaging over the realizations of a random field $\mathbf{v}(\mathbf{r})$ leads to diagrams similar to those of the usual crossover technique.[19] The corresponding technique for the problem of classical diffusion in a disordered medium was developed by Dreĭzin and Dykhne.[20] In the theory considered, the internal lines correspond to diffusion propagators (12) (they arise in the known quantum-diffusion problem only after separation of the diagrams corresponding to slow motions, viz., diffusions and cooperons.[21,22]).

In view of the long-range character of the disorder correlations (8) all the diagrams diverge as $\omega \to 0$ in the case $d \leqslant 2$, and the diagrams with intersections have no additional small factors. Thus, in the considered classical-diffusion models, just as in the problem of quantum diffusion, as $\omega \to 0$ there is no small perturbation-theory parameter (at $d \leqslant 2$).

A convenient way of studying the low-frequency properties in this situation is to represent the Green's function as a path integral, followed by a RG analysis of the corresponding effective theory of the field.

We represent the Green's function in the form

$$G_\omega(\mathbf{r}, \mathbf{r}') = -\frac{i}{Z}\frac{\delta^2}{\delta\bar{h}(\mathbf{r})\delta h(\mathbf{r}')}\int \mathcal{D}\bar{\varphi}\mathcal{D}\varphi e^{iS[\bar{h}, h]}\Big|_{\bar{h}=h=0}, \qquad (14)$$

where

$$S[\bar{h}, h] = S + \delta S,$$

$$S = \int \{\bar{\varphi}[i\omega + \partial_\alpha(D_0\partial_\alpha)]\varphi - \bar{\varphi}\partial_\alpha(V_\alpha\varphi)\}d\mathbf{r}, \qquad (15)$$

$$\delta S = \int \{\bar{h}\varphi + \bar{\varphi}h\}d\mathbf{r}. \qquad (16)$$

The normalization factor Z in (14) is determined by a path integral of $\exp(iS)$ over the complex fields $\bar{\varphi}(\mathbf{r})$ and $\varphi(\mathbf{r})$. A formal proof (without analysis of the convergence of path integrals with non-Hermitian operators) that Eq. (14) is a Green's function of Eq. (13) can be easily obtained by using the following shifts of the integration variables: $\varphi \to \varphi + Gh$, $\bar{\varphi} \to \bar{\varphi} + \bar{h}G$. That the representation (14) is valid in the perturbative sense can be easily verified by expanding it in powers of the random field **v**. (The ensuing Gaussian integrals with action $S_0 \equiv S(\mathbf{v} = 0)$ are, of course, well defined.) This expansion is an exact duplicate of the direct expansion of Eq. (13) in powers of **v**.

The averaging over the disorder of $\mathbf{v}(\mathbf{r})$ is by the standard replica[1)] method[23] and reduces to replacing the action (15) by $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\text{int}}$, where

$$\mathcal{S}_0 = \int \bar{\varphi}(i\omega - \mu\mathbf{E}\partial + D\partial^2)\varphi d\mathbf{r}, \qquad (17)$$

$$\mathcal{S}_{int} = \frac{i\gamma}{2}\int (\partial_\alpha\bar{\varphi}\varphi)_{\mathbf{r}}F_{\alpha\beta}(\mathbf{r}-\mathbf{r}')(\partial_\beta\bar{\varphi}\varphi)_{\mathbf{r}'}\, d\mathbf{r}\, d\mathbf{r}'. \qquad (18)$$

Here $F_{\alpha\beta}$ is the correlator of the random fields (8), $\bar{\varphi}$ and φ are N-component fields in the replica space (we must put $N = 0$ in the final results); summation over replica indices is implied in expressions of type $\bar{\varphi}\varphi$. The bare values of the renormalized parameters D, μ, and γ are respectively D_0, μ_0, and γ_0. It is easy to verify that the expansion in powers of γ_0 duplicates (at $N = 0$) the diagram series of the crossover technique for the initial equation (13).

§3. RENORMALIZATION OF THE KINETIC COEFFICIENTS

A scale analysis of the effective functional (17), (18) shows that the interaction $\mathscr{S}_{\rm int}$ is substantial at $d \leqslant 2$. The renormalization of the functional in the upper critical dimensionality $d = 2$ is by the usual method[24] of integrating over the "fast" components of the fields $\bar{\varphi}$ and φ whose Fourier components differ from zero at $\lambda k_0 < k < k_0$, where k_0 is the ultraviolet-cutoff parameter and λ is a scale factor, $0 < \lambda < 1$. The frequency ω is not renormalized here in view of the conservation of the total probability:

$$-i\omega \int G_\omega(\mathbf{r}, \mathbf{r}')\, d\mathbf{r} = 1. \tag{19}$$

The RG equations for the renormalized parameters D, γ, and μ of the functional are derived by expanding over the loops. The effective expansion parameter is the dimensionless (at $d = 2$) charge

$$g = \gamma/(4\pi D^2). \tag{20}$$

Weak disorder corresponds to the small unrenormalized value (10) of this charge: $g_0 \ll 1$.

At $d = 2$ we obtain in the two-loop approximation the following RG equations ($N = 0$):

$$d \ln D/d\xi = \alpha g - 2(1-\alpha^2) g^2, \tag{21a}$$

$$d \ln \gamma/d\xi = -(1-\alpha) g - 2(1-\alpha^2) g^2, \tag{21b}$$

where $\xi = \ln \lambda^{-1}$ is the logarithmic RG variable, and

$$\alpha = \begin{cases} 0, & \text{model I}, \\ 1, & \text{model II}, \\ -1, & \text{model III}. \end{cases} \tag{22}$$

The calculations in the two-loop approximation were carried out by the method of dimensional regularization[25] in the form proposed in Ref. 26. Some details of the derivation of Eqs. (21) are given in the Appendix.

Equation (20) for the charge is obtained from the obvious combination of Eqs. (21a and b)

$$d \ln g/d\xi = -(1+\alpha) g + 2(1-\alpha^2) g^2. \tag{23}$$

Equations (21) and (23) were reported by us in a brief communication.[11] Similar equations in the one-loop approximation were derived in Refs. 10 and 12.

It can be seen from (34) that in models I and II the small nonrenormalized charge is decreased by the RG transformation (the "zero-charge" situation), so that the theory developed is asymptotically correct. In model III we have $dg/d\xi = 0$ accurate to two loops, so that the question of the exact $g(\xi)$ dependence remains open in this model.

The RG equations for the effective mobility μ in models I and III are of the form (the one-loop approximation suffices here)

$$d \ln \mu/d\xi = -g. \tag{24}$$

In model II the mobility is not renormalized, this being a simple consequence of the tensor structure of the correlator (8b).

Solving (21)–(24), we obtain the following expressions for the renormalized parameters D and μ:

$$D(\xi) = \begin{cases} D_0 \exp[-2g_0^2\xi/(1+g_0\xi)], & \text{model I}, \\ D_0(1+2g_0\xi)^{1/2}, & \text{model II}, \\ D_0 \exp(-g_0\xi), & \text{model III}, \end{cases} \tag{25}$$

$$\mu(\xi) = \begin{cases} \mu_0 (1+g_0\xi)^{-1}, & \text{model I}, \\ \mu_0, & \text{model II}, \\ \mu_0 \exp(-g_0\xi), & \text{model III}. \end{cases} \tag{26}$$

The physical quantities are calculated in the RG model with quadratic action, in which the unrenormalized parameters must be replaced by their renormalized values. The logarithmic RG parameter must then be set equal to $\xi = \ln(Dk_0^2/\omega)^{1/2}$ [or $\xi = \ln(t/\tau)^{1/2}$, where $\tau = 1/(Dk_0^2)$]. In the limit $g_0 \ln(t/\tau) \gg 1$ we arrive at the asymptotic expressions (9) and (11) of the Introduction, as well as at (6) for model I. It follows from the structure of the quadratic action (17) that in the presence of a constant external field the infrared-cutoff parameter in $\xi = \ln(k_0/k_{\rm ir})$ is

$$k_{\rm ir} = \max\{\mu_0 E/D_0,\ (\omega/D_0)^{1/2},\ r_0^{-1}\}, \tag{27}$$

where r_0 is the interaction-screening radius. In weak external fields

$$E < (\omega D_0)^{1/2}/\mu_0 \tag{28}$$

the kinetic coefficients do not depend on the external field (linear regime). The dependence of the mobility μ on $\xi \sim \ln \omega^{-1} \sim \ln t$ (at distances shorter than r_0) means in fact that the drift velocity in the linear regime depends on the duration of the drift. In the opposite case of sufficiently low frequencies (long times), the kinetic coefficients cease to depend on $\ln \omega$, but turn out to be functions of the applied external field. Of course, even in the static limit the nonlinearity does not manifest itself in such arbitrarily weak fields, since the field-independent infrared-cutoff parameter is the screening radius or the dimension of the system.

It can be seen from (25) and (26) that the Einstein relation does not hold in models I and II. It will be shown that the Kubo formula (FDT) is likewise incorrect in these models. We shall show that these relations are satisfied in a modified form. This requires a formalism that permits calculation of correlators of various physical quantities, a formalism described in the next section.

§4. EQUATIONS OF STOCHASTIC HYDRODYNAMICS

The FP equation (2), which is equivalent to the Langevin equation (4), (5), determines the one-particle distribution function averaged over the "thermal" noise $\eta(t)$ (5).

To calculate the many-time correlators it is convenient to use the stochastic-hydrodynamic equations for the fluctuating (not yet averaged over the thermal noise) particle density

$$\rho(x) = \sum_a \delta(\mathbf{r} - \mathbf{r}_a(t)) \tag{29a}$$

and for the fluctuating current density:

$$\mathbf{j}(x)=\sum_a \dot{\mathbf{r}}_a\delta(\mathbf{r}-\mathbf{r}_a(t)). \quad (29b)$$

Here $\mathbf{r}_a(t)$ is the trajectory of the particle numbered a, whose motion is described by the Langevin equation (4). Differentiating (29a) with respect to time and using (4) and (5), we get

$$[G^{-1}\rho](x)=-\partial_\alpha\xi_\alpha(x), \quad (30)$$

where G^{-1} is the Fokker-Planck operator (13) and $x\equiv(r, t)$. This equation is in fact the continuity equation for the density (29a) and for the current (29b). The current density $\mathbf{j}$ can be expressed here directly in terms of the particle density ρ and the random force ξ:

$$j_\alpha(x)=(V_\alpha-D_0\partial_\alpha)\rho(x)+\xi_\alpha(x). \quad (31)$$

It can be verified that the random force $\xi(\mathbf{r}, t)$ due to the thermal noise $\eta(t)$ (5) has the following correlators: $\bar{\xi}=0$ and

$$\overline{\xi_\alpha(x)\xi_\beta(x')}=2D_0\overline{\rho(x)}\delta(x-x')\delta_{\alpha\beta}. \quad (32)$$

Here $\overline{\rho(x)}$ is the particle density averaged over the thermal noise and satisfying obviously the FP equation (2). This density can be easily expressed in terms of the Green's function of Eq. (13). Assuming that at the time $t_0\to-\infty$ the particles were uniformly distributed in space with density ρ_0 (meaning that the disorder and the external field were turned on adiabatically), we get

$$\overline{\rho(x)}=\rho_0\lim_{t_0\to-\infty}\int G(x,x_0)\,d\mathbf{r}_0. \quad (33)$$

In the absence of a time-dependent external field, the density $\overline{\rho(x)}\equiv\overline{\rho(\mathbf{r})}$ is independent of time:

$$\overline{\rho(\mathbf{r})}=\rho_0\lim_{\omega\to 0}\left[-i\omega\int G_\omega(\mathbf{r},\mathbf{r}_0)\,d\mathbf{r}_0\right]. \quad (34)$$

In the transition to (34) we used the relation

$$\lim_{t_0\to-\infty}(-i\omega+0)^{-1}\exp\{-i\omega(t-t_0)\}=2\pi\delta(\omega).$$

For the current density $\bar{\mathbf{j}}$ averaged over the thermal noise we obviously get from (31)

$$\overline{j_\alpha(x)}=(V_\alpha(x)-D_0\partial_\alpha)\overline{\rho(x)}. \quad (35)$$

Equation (30) determines the fluctuating density $\rho(x)$. Taking (33) into account, we obtain for this density

$$\rho(x)=\overline{\rho(x)}+\int\partial_\alpha'G(x,x')\xi_\alpha(x')\,dx' \quad (36)$$

($dx'\equiv d\mathbf{r}'dt'$; $\partial'_\alpha\equiv\partial/\partial r'_\alpha$). With (32) and (33) taken now into account, all density correlators can be easily expressed in terms of the Green's function of Eq. (13), i.e., reduced to a form convenient for averaging over the realizations. The corresponding expressions for the fluctuating current will be given below.

§5. EINSTEIN'S RELATION

Einstein's relation $D_0=\mu_0T$ is in fact a definition of the unrenormalized temperature T [which characterizes the power of the thermal noise (5)]. Since this relation does not hold in models I and II in the presence of a random field $\mathbf{v}(\mathbf{r})$, it is natural to ask which quantity serves as the effective physical temperature in the systems under consideration.

In the absence of disorder, the temperature $T=D_0/\mu_0$ determines the stationary Boltzmann distribution of the density of the randomly walking particles in an infinitesimal external field with potential U: $\bar{\rho}=\rho_0\exp(-U/T)$. By analogy with this equation, it is natural to introduce a temperature defined by the macroscopic Boltzmann distribution:

$$(1/T^*)\delta(\mathbf{r}-\mathbf{r}')=-\rho_0^{-1}\langle\delta\overline{\rho(\mathbf{r})}/\delta U(\mathbf{r}')\rangle|_{U=0}. \quad (37)$$

In models I and II the temperature T^* depends substantially on the disorder parameter. It is easy to verify by using the ordinary (crossover) perturbation theory that in models I and II there exist for this quantity logarithmic corrections similar to the corrections for the diffusion or mobility. This means that the effective temperature is renormalized, i.e., its physical value depends on the scale of the system (or on the frequency, time, screening, etc., i.e., on any parameter that "cuts off" the logarithmic RG variable). Using Eq. (34) for $\bar{\rho}$ and the functional representation (14)–(16) for the Green's function, we obtain the temperature (37) in the form

$$\frac{1}{T^*}\delta(\mathbf{r}-\mathbf{r}')$$
$$=\mu_0\lim_{\omega\to 0}(-i\omega)\int d\mathbf{r}_0\langle\langle\bar{\varphi}(\mathbf{r}_0)\varphi(\mathbf{r})\partial_\alpha'[\partial_\alpha'\bar{\varphi}(\mathbf{r}')\varphi(\mathbf{r}')]\rangle\rangle, \quad (38)$$

where $\langle\langle\ldots\rangle\rangle$ denotes functional averaging with weight $\exp(iS)$ (15) and averaging over the realizations of the disorder. After averaging over the disorder we obtain for (38) a functional representation with a dual set of replica indices. It is convenient to renormalize the pre-exponential factor in this representation by adding to the action a vertex proportional to the corresponding source M_α (to the matrix in the replica space):

$$\int\partial_\alpha\bar{\boldsymbol{\varphi}}\mathbf{M}_\alpha\boldsymbol{\varphi}\,dr.$$

In the upshot we obtain (at $N=0$) in the one-loop approximation (which determines completely the asymptotes of the considered models I and II) the following RG equation for the temperature:

$$d\ln T^*/d\xi=g. \quad (39)$$

It is easy to verify that the temperature T^* defined by this equation ensures satisfaction of the Einstein relation in the modified form

$$D(\xi)=\mu(\xi)T^*(\xi). \quad (40)$$

Note that the definition (37) of the effective temperature allows us to represent (40) in the natural form

$$\sigma(\mathbf{r},\mathbf{r}')=-D\langle\delta\bar{\rho}(r)/\delta U(r')\rangle|_{U=0}$$

(where $\sigma=\rho_0\mu$ is the conductivity); this is perfectly analogous, e.g., to the form in which the Einstein relation is written for the problem of quantum diffusion of interacting electrons.[27]

In model III, Einstein's relation holds in the usual form with an unrenormalized temperature $T = D_0/\mu_0$. The nonrenormalizability of the temperature T^* (37) in model III is due to the potential character of the disorder. Indeed, it is precisely because of the potential character that the density of randomly walking particles, which is the stationary solution of the FP equation (2), takes the form

$$\bar{\rho}(\mathbf{r}) = \rho_0 \exp[-(u(\mathbf{r}) + U(\mathbf{r}))/T]. \tag{41}$$

[Here $-\partial_\alpha u(\mathbf{r}) = v_\alpha(\mathbf{r})$, see (7b).] Since the averaging of (41) over the realizations of the disorder does not affect the Boltzmann $\exp(-U/T)$ dependence of the density on the external field, the coefficient of proportionality of the macroscopic diffusion coefficient to the mobility is still the usual (unrenormalizable) temperature T.

§6. FLUCTUATION-DISSIPATION THEOREM (FDT)

We show in this section that not only the Einstein relation but also the Kubo formula (FDT) is violated in models with disorder (8a, b). We prove the FDT for the model with disorder (8c) and clarify the physical causes of its violation in models I and II.

In the absence of disorder, the FDT is of the form

$$\overline{j_\alpha(\mathbf{r},\ \omega) j_\beta(\mathbf{r}',\ -\omega)} = 2T \operatorname{Re} \sigma_{\alpha\beta}(\omega) \delta(\mathbf{r} - \mathbf{r}'). \tag{42}$$

We calculate the spectral density of the correlator in (42) for a disordered system using for the fluctuating current density an expression that follows directly from (31), (33), and (36):

$$j_\alpha(x) = \overline{j_\alpha(x)} + \int \partial_t \mathscr{P}_{\alpha\beta}(x, x') \xi_\beta(x') dx', \tag{43}$$

where, by definition,

$$\partial_t \mathscr{P}_{\alpha\beta}(x, x') = (V_\alpha(x) - D_0 \partial_\alpha) \partial_\beta' G(x, x') + \delta_{\alpha\beta} \delta(x - x'). \tag{44}$$

Here $\mathscr{P}_{\alpha\beta}(x, x')$ is the Green's function of the differential equation

$$[\delta_{\alpha\gamma} \partial_t + (V_\alpha(x) - D_0 \partial_\alpha) \partial_\gamma] \mathscr{P}_{\gamma\beta}(x, x') = \delta_{\alpha\beta} \delta(x - x'). \tag{45}$$

Using now expression (32) and recognizing that in the absence of an external field $\bar{j}$ [Eq. (35)] is independent of time, we obtain the left-hand side of (42) in the form

$$\overline{j_\alpha(\mathbf{r}, \omega) j_\beta(\mathbf{r}', -\omega)} = 2D_0 \omega^2 \int \mathscr{P}_\omega^{\alpha\gamma}(\mathbf{r}, \mathbf{r}_0) \mathscr{P}_{-\omega}^{\beta\gamma}(\mathbf{r}', \mathbf{r}_0) \overline{\rho(\mathbf{r}_0)} d\mathbf{r}_0, \tag{46}$$

which is valid for any realization of the random field.

An expression for the conductivity $\sigma_{\alpha\beta}(\omega)$ can be easily obtained by using Eq. (35) for $\overline{\mathbf{j}(\mathbf{r}, t)}$:

$$\sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \omega) = \left.\frac{\delta \overline{j_\alpha}(\mathbf{r}, \omega)}{\delta E_\beta(\mathbf{r}', \omega)}\right|_{\mathbf{E}=0} = -i\omega \mu_0 \mathscr{P}_\omega^{\alpha\beta}(\mathbf{r}, \mathbf{r}') \overline{\rho(\mathbf{r}')}, \tag{47}$$

where it is convenient to express the right-hand side of (47) in terms of the Green's function of Eq. (45).

We now define the quantity

$$\Delta_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \omega) = \overline{j_\alpha(\mathbf{r}, \omega) j_\beta(\mathbf{r}', -\omega)} - T[\sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \omega) + \sigma_{\beta\alpha}(\mathbf{r}', \mathbf{r}; -\omega)]. \tag{48}$$

We apply the operator of Eq. (45) to the right-hand side of (48) and set the initial condition $\Delta \equiv 0$ in the ordered system by virtue of (42). We then obtain for Δ, with allowance for (46) and (47), the expression

$$\begin{aligned}\Delta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2; \omega) &= -i\omega \int \{\mathscr{P}_{-\omega}^{\alpha\mu}(\mathbf{r}_1, \mathbf{r}') \partial_\mu' \mathscr{P}_\omega^{\beta\nu}(\mathbf{r}_2, \mathbf{r}') \overline{j_\nu(\mathbf{r}')} - \text{H.c.}\} d\mathbf{r}' \\ &+ (-i\omega) \int [\partial_\nu' v_\mu(\mathbf{r}') - \partial_\mu' v_\nu(\mathbf{r}')] \\ &\times \mathscr{P}_{-\omega}^{\alpha\nu}(\mathbf{r}_1, \mathbf{r}') \mathscr{P}_\omega^{\beta\mu}(\mathbf{r}_2, \mathbf{r}') \overline{\rho(\mathbf{r}')} d\mathbf{r}'.\end{aligned} \tag{49}$$

Here $\bar{\rho}$ and $\bar{\mathbf{j}}$ are given by (15) and (21b) in the absence of an external field.

For a potential random field (model III) the second term of (49) vanishes [by virtue of (7b)]. Substituting the solution for $\bar{\rho}$ in the model III (41) in expression (35) for the current $\bar{\mathbf{j}}$, we verify that $\mathbf{j} = 0$, as it should in the potential model, i.e., the first term of (49) vanishes. Thus, in model III we have $\Delta = 0$ (even prior to averaging over the disorder realizations), and this proves the validity of the FDT in model III.

On the contrary, in disorder models I and II, the FDT in its usual formulation (42) turns out to be incorrect. In fact, even in first-order perturbation theory in g_0 (10) the quantity $\langle \Delta \rangle$ (49) differs from zero. It is easy to verify, however, that an FDT in modified form holds for all the considered disorder models. It is necessary to replace in (42) (averaged over the disorder), just as in the Einstein relation, the unrenormalized temperature T by the effective temperature T^* (39).

§7. MESOSCOPIC FLUCTUATIONS OF KINETIC COEFFICIENTS

So far, we have dealt with calculations of physical quantities averaged over an ensemble of realizations of a random field $\mathbf{v}(\mathbf{r})$. An important question is whether these averages are of significance for an individual sample. To answer this question we must calculate the fluctuations of the physical quantities from sample to sample. We consider by way of example the fluctuations of the conductivity $\sigma_{\alpha\beta}$:

$$\sigma_{\alpha\beta} = L^{-2} \int d\mathbf{r}\, d\mathbf{r}'\, \delta \bar{j}_\alpha(\mathbf{r}) / \delta E_\beta(\mathbf{r}'). \tag{50}$$

In contrast to (47), it is convenient here to use (35) and (34) to express $\sigma_{\alpha\beta}$ in terms of the Green's functions (13) of the Fokker-Planck equation:

$$\sigma_{\alpha\beta} = \sigma_0 \delta_{\alpha\beta} - i\omega L^{-2} \sigma_0 \int v_\alpha(\mathbf{r}) \partial_\beta' G_\omega(\mathbf{r}, \mathbf{r}') \times G_\omega(\mathbf{r}', \mathbf{r}'') d\mathbf{r}\, d\mathbf{r}'\, d\mathbf{r}'' |_{\omega=0}. \tag{51}$$

We omit here the diffusive contribution to the current ($\propto D_0 \partial_\alpha \ldots$), since it vanishes under cyclic boundary conditions. The relative value of the conductivity fluctuations in different realizations is determined by the correlator

$$\mathscr{K}_{\alpha\beta,\gamma\delta} = \langle \delta\sigma_{\alpha\beta} \delta\sigma_{\gamma\delta} \rangle / \sigma_0^2, \tag{52}$$

where $\delta\sigma_{\alpha\beta} = \sigma_{\alpha\beta} - \langle \sigma_{\alpha\beta} \rangle$. In the lowest nonvanishing order of perturbation theory we get

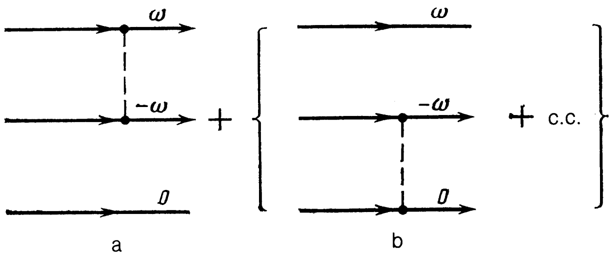

FIG. 1. Diagrams for the "current-current" correlator: the solid lines correspond to diffusion propagators in the external field, and the dashed lines to the correlator (3); the vertices contain gradients that act on the included propagators.

$$\mathscr{K}=(\gamma_0/L)^2\int\frac{d^2k}{(2\pi)^2}\frac{k_\beta k_\delta k_\mu k_\nu}{(D_0k^2)^4}\{F_{\alpha\gamma}(\mathbf{k})F_{\mu\nu}(\mathbf{k})+F_{\alpha\nu}(\mathbf{k})F_{\gamma\mu}(\mathbf{k})\}.$$

The long-range character of the disorder correlations (8) ($F_{\alpha\beta}(\mathbf{k})\neq 0$ as $k\to 0$) leads to a divergence in $\mathscr{K}$:

$$\mathscr{K}_{\alpha\beta,\gamma\delta}=g_0{}^2S_{\alpha\beta\gamma\delta}L^{-2}\int d^2k/k^4, \quad (53)$$

where

$$S_{\alpha\beta\gamma\delta}=\begin{cases}\frac{1}{2}(5\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}), & \text{model I,}\\ \delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}, & \text{model III.}\end{cases} \quad (54)$$

In model II the conductivity, which is not subject to corrections in any realization of the disorder, has naturally no fluctuations. (At the same time, fluctuations of, e.g., the diffusion coefficient are present in all three models and also contain a diverging part analogous to (53).)

The expression (53) for the relative fluctuations of the conductivity coincides, after a suitable replacement of the disorder parameter, with the result recently obtained in the quantum-diffusion problem[13–15] (in the case of model III, even the tensor structure (54) coincides). The diverging integral in (53) is cut off at the reciprocal dimension L^{-1} of the system, so that the relative value of the fluctuations is independent of the system size. This picture, of course, is valid only for "mesoscopic" distances, i.e., distances that are large compared with the length of the elementary hop (or of the mean free path), but small compared with the screening radius in the considered classical systems (or compared with the diffusion length $(D\tau_\varphi)^{1/2}$, where τ_φ is the phase loss time, in the quantum-diffusion problem).

We emphasize that according to (53) the relative fluctuation $\delta\sigma/\sigma$ is small in terms of the weak-disorder parameter g_0 (10). It is this which justifies the consideration of physical quantities averaged over the disorder realizations.

§8. CURRENTS CORRELATOR IN AN EXTERNAL FIELD. EXCESS NOISE

The formalism developed allows us to calculate the correlators of the currents in an external field. This makes possible a direct verification, in the weak-disorder region, of the hypothesis advanced in Ref. 9, that random walks in media with disorder (8a) can serve (at $d=2$) as a universal source of the excess $1/f$ noise (see, e.g., the reviews[16,18]). The spectral density of the currents correlator

$$\mathscr{P}(\omega)=\overline{\langle I(\omega)I(-\omega)\rangle}=L^{-2}\int\overline{\langle j_x(\mathbf{r},\omega)j_x(\mathbf{r}',-\omega)\rangle}d^2r\,d^2r' \quad (55)$$

is calculated with the aid of Eq. (46), where the constant uniform external field $\mathbf{E}=(E_x,0)$ is included in the Green's function. We emphasize that an important role in the averaging $\langle\ldots\rangle$ over the realizations of the random field $\mathbf{v}(\mathbf{r})$ is played by the coordinate dependence (34) of $\overline{\rho(\mathbf{r})}$ due to this field.

In the absence of disorder, $\mathscr{P}(\omega)$ (55) is independent of the external field and reduces to the usual Nyquist noise: $\mathscr{P}_0(\omega)=2D_0\rho_0=2T\sigma_0$. The disorder gives rise to an excess noise $\delta\mathscr{P}_I(\omega)$ (that depends on the flowing current). The essential features of the excess noise are already manifested in first-order perturbation theory in g_0 (10). Figure 1 shows diagrams for $\delta\mathscr{P}_I(\omega)$. We call attention to the fact that a diagram with three lines corresponds to the pair correlator (55). The "extra" line with zero frequency is due to the presence of $\overline{\rho(\mathbf{r})}$ (34) in the correlator (56). Diagrams of type 1b take into account scattering by the disorder-generated density fluctuations. In the diagrams of type 1a, where the zero-frequency line is isolated, its presence reduces to multiplying the remaining parts of the diagrams by ρ_0.

The calculations yield in models I and III the following expressions for the excess noise $\delta\mathscr{P}_I(\omega)$ (at $d=2$):

$$\delta\mathscr{P}_I(\omega)/\mathscr{P}_0(\omega)=3g_0\varkappa\,\mathrm{arctg}\,(1/\varkappa)+{}^1/_2\alpha g_0\times[1-\varkappa^{-1}\,\mathrm{arctg}\,\varkappa-{}^1/_2\ln(1+\varkappa^2)], \quad (56)$$

where $\varkappa\equiv(\mu_0E)^2/(4D_0|\omega|)$, and α is a coefficient that depends on the model (22).

In weak fields (28), corresponding to the linear regime, the ratio of the excess-noise spectral density to the squared current is the same for models I and III:

$$\delta\mathscr{P}_I(\omega)/\langle\bar{I}\rangle^2=(3\pi g_0/4)N^{-1}|\omega|^{-1}. \quad (57)$$

The entire excess noise is due here to static fluctuations of the density $\overline{\rho(\mathbf{r})}$. In model II, where $\overline{\rho(\mathbf{r})}$ is constant, there is no excess noise in the linear-response approximation.

Equation (57) is an accurate replica of Hoog's[15,17] empirical formula for $1/f$ noise ($N\equiv L^2\rho_0$ is the number of carriers), Hoog's constant being expressed in terms of the microscopic parameters of the model and equal to $2\pi g_0/4$. In the region of weak external fields (28) and weak disorder (10), however, the obtained excess noise is small compared with Nyquist's equilibrium noise. In "strong" fields (in the region of non-ohmic conductivity), the excess noise (56) saturates and ceases to depend on either the external field or the frequency. Thus, only some tendency towards $1/f$ noise is observed for weak disorder in the considered models I and III. Further analysis of this problem calls for consideration of the strong-disorder region or possibly for modification of the models.

§9. CONNECTION WITH LATTICE HOPPING MODELS

The models of classical diffusion in disordered media, considered above, can be regarded as a continual limit of lattice hopping models. The latter are widely used to describe hopping conduction in impurity semiconductors, inversion layers, ionic crystals,[28,29] and in the study of diffusion of excitons in molecular crystals.[30]

The distribution function $P_{\mathbf{r}}(t)$ of a particle that wanders over lattice sites obeys to the kinetic equation (1). In a disordered lattice, the probability $W_{\mathbf{rr}'}$ of hopping from site $\mathbf{r}'$ to site $\mathbf{r}$ takes the form

$$W_{\mathbf{rr}'}=W^0_{\mathbf{r}-\mathbf{r}'}+\delta W_{\mathbf{rr}'}, \tag{58}$$

where $W^0_{\mathbf{rr}'}$ is the hopping probability in a regular lattice, and the random function $\delta W_{\mathbf{rr}'}$ describes the weak disorder.

In the continual limit, Eq. (1) goes over into the FP equation (2). The correlator of the random velocities $\mathbf{v}$ that enter in (2) is expressed in terms of the correlator of the probabilities δW as follows:

$$\langle v_\alpha(\mathbf{r})v_\beta(\mathbf{r}')\rangle=\sum_{\rho,\rho'}(r-\rho)_\alpha(r'-\rho')_\beta\langle\delta W_{\mathbf{r}\rho}\delta W_{\mathbf{r}'\rho'}\rangle. \tag{59}$$

On going from the lattice equation (1) to the FP equation (2) we have neglected the higher gradients of $P_{\mathbf{r}}$, which are insignificant (in the weak-disorder region) for the analysis of the long-wave properties of the model. In this sense, there is no significance to the possible deviation of the distribution of δW (or of $\mathbf{v}$) from Gaussian. Allowance for this deviation would lead to the appearance of higher gradients in the effective functional $\mathscr{S}_{\rm int}$ (18). It is easy to verify that the spatial fluctuations of the diffusion coefficient $D(\mathbf{r})$ are also inessential. In this case

$$D_0=\frac{1}{2}\sum_{\mathbf{r}'}(\mathbf{r}-\mathbf{r}')^2W^0_{\mathbf{r}-\mathbf{r}'}. \tag{60}$$

The simplest asymmetric ($\delta W_{\mathbf{rr}'}\neq\delta W_{\mathbf{r}'\mathbf{r}}$) hopping model, in which the quantities $\delta W_{\mathbf{rr}'}$ fluctuate independently if they do not pertain to the same bond, goes over in the continual limit into model I.[3,5]

A crystal with magnetic impurities was suggested[10] as a possible realization of model II. A natural continual realization of model II constitutes random walks of a particle in an incompressible liquid with random stationary vortices ("whirlpools")[11] or in an incompressible liquid that seeps through a porous filter.[12] The most natural and interesting from the standpoint of application to lattice systems is model III. It describes the long-wave limit of a hopping system, in which the random asymmetry is due to a chaotic field $\mathbf{E}(\mathbf{r})$ of charged impurities. We shall demonstrate this, confining ourselves for simplicity to allowance for hops only between neighboring sites. We have then

$$W_{\mathbf{r},\,\mathbf{r}+\mathbf{a}}=W_0\exp[-e\mathbf{E}(\mathbf{r})\mathbf{a}/2T]. \tag{61}$$

In the approximation linear in the field ($\delta W\propto E$) the random-velocities correlator is proportional, when account is taken of (66), to the correlator $\langle\mathbf{EE}\rangle$. The random field of charged impurities in a $2D$ system is represented in the form

$$\mathbf{E}(\mathbf{k})=-2\pi i\mathbf{k}(\varepsilon k)^{-1}\sum_i e_i\exp i\mathbf{k}\mathbf{r}_i,$$

where $e_i=\pm e$ are the charges of impurities with random coordinates $\{\mathbf{r}_i\}$, and ε is the dielectric constant. (Of course, the field of an individual impurity is assumed three-dimensional: $\mathbf{E}\propto\mathbf{r}/r^3$.) Averaging over the impurity positions, we obtain

$$\langle E_\alpha(\mathbf{k})E_\beta(\mathbf{k}')\rangle=(2\pi e)^2ck_\alpha k_\beta'(\varepsilon k)^{-2}(2\pi)^2\delta(\mathbf{k}+\mathbf{k}'), \tag{62}$$

where c is the two-dimensional impurity density. From this, using Eqs. (61) and (59), we obtain for the random-velocities correlations the expression (8c) (arriving thus at model III), where

$$\gamma_0=\frac{\pi^2e^4c}{\varepsilon^2T^2}\left(\frac{W_0za^2}{2}\right)^2 \tag{63}$$

(z is the coordination number in the lattice). The number in the parentheses can be easily seen to coincide in this case with the diffusion coefficient D_0. Consequently, the weak-disorder criterion (10) takes in the considered hopping model the form

$$g_0\equiv\pi e^4c/4T^2\varepsilon^2\ll1. \tag{64}$$

Expression (62) for the correlator is valid in the momentum region $k\gg r_0^{-1}$ (r_0 is the screening radius). At $k\ll r_0^{-1}$ the correlator (62) is proportional to $k_\alpha k_\beta$, so that in this momentum region the disorder-produced interaction becomes unimportant. This means that r_0^{-1} becomes the infrared-cutoff parameter of the theory [see (27)]. Of course, we are interested in situations in which the screening radius is macroscopically large. This is valid for systems in which the carrier density is much lower than the density of the charged impurities, as is the case, for example, when a charged particle is injected into a disordered dielectric,[29] in inversion layers (MIS structures),[29] or in weakly doped semiconductors with maximum or minimum degree of compensation K.[28] In the latter case the ratio of the screening radius to the length of the elementary hop is proportional to a large parameter, viz, $(1-K)^{2/3}$ for strong compensation $(1-K\ll1)$ and $K^{-1/2}$ for weak compensation $(K\ll1)$.[28] The static conductivity acquires then, by virtue of (26) and (27), an additional (non-analytic) dependence on K and a characteristic temperature dependence $\exp(-T^{-2})$ that follows from (64).

We emphasize that we are considering situations in which the hopping conduction is along charged impurities (donors), so that the question of electron "capture" by an individual impurity does not arise. In the case (64) of weak disorder (i.e., when the Coulomb-energy fluctuations due to the random arrangement of the impurities are small compared with the temperature) there is also no carrier capture by typical fluctuations of the impurity potential. Thus, the kinetic parameters of the system depend little on the specific impurity arrangement also at distances $r\lesssim r_0$, so that it is reasonable to consider quantities averaged over an ensemble. This is confirmed by the rigorous analysis in §7.

CONCLUSION

We have shown that weak disorder influences substantially (and in different ways) the classical diffusion and the mobility. Let us interpret these results qualitatively. Figure 2 shows a typical realization of the random field [(8), (4)] in each of the considered models. The particle drift along the force lines of the random field is constantly destroyed by the

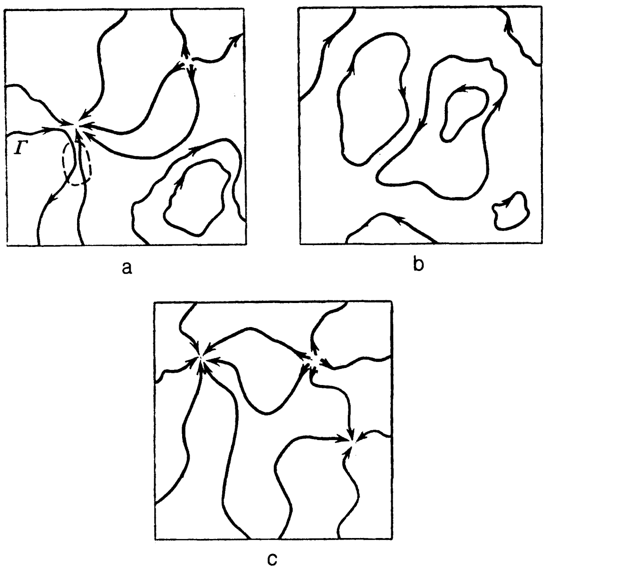

FIG. 2. Typical realizations of the random field $v(\mathbf{r})$: a—isotropic disorder (field without constraints, model I); b—transverse disorder (solenoidal field, model II); c—longitudinal disorder (potential field, model III). The line Γ (Fig. 2a) that diverts a particle from a trap is impossible in model III, since the circulation along the dashed contour differs from zero.

strong [if condition (10) is met] random "wind" $\eta(t)$ (4), which "blows away" a particle from one force line to another. However, the preserved tendency towards ballistic motion along the cycles leads in the case of a solenoidal field (model II, Fig. 2b) to a superdiffusive propagation: $\langle \mathbf{r}^2 \rangle / t \to \infty$ as $t \to \infty$. At the same time, the presence of cycles does not influence the directional motion of the particle, so that the disorder does not affect the mobility in model II.

A potential random field (model III) is characterized by an assembly of sources and sinks (Fig. 2c). In this case the randomly walking particles have a predominant tendency to avoid the sources and be held back by the sinks that act as traps. This leads to a subdiffusive behavior: $\langle \overline{\mathbf{r}^2} \rangle / t \to 0$ as $t \to \infty$, and the mobility is also decreased.

In model I (Fig. 2a) there are sources and sinks, as well as cycles. The influence of the traps is weakened in this case by the existence of force lines of type Γ (Fig. 2a) that lead a particle away from a trap. The tendencies towards capture by "weakened" traps and towards ballistic motion along the cycles cancel each other, so that the disorder hardly affects the diffusion: $\langle \mathbf{r}^2 \rangle / t \to D$ as $t \to \infty$. The mobility, however, which is sensitive only to the presence of traps, decreases.

It is clear from the same qualitative picture that the growth, found above, of the effective temperature (and the associated violation of the Einstein relation) in the nonpotential models I and II is due to the presence of undamped currents in specific realizations of the disorder. Of course, the presence of stationary currents means that there is no complete thermodynamic equilibrium in these models. This is not surprising for nonpotential models. In natural realizations of model II as liquids infiltrating (forced through) porous filters[12] the presence of external action is obvious.

We note finally that in the potential model III, where there are no stationary currents, the temperature is not renormalized, nor is, accurate to two loops, the effective charge. It is not excluded that these facts are related and that there is no charge renormalization also in higher-order loops.

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APPENDIX

Derivation of the RG equations in the two-loop approximation

Here we consider a somewhat more general case, when the random-velocities correlator is given by

$$F_{\alpha\beta}(\mathbf{k}) = \gamma_\perp (\delta_{\alpha\beta} - k_\alpha k_\beta / k^2) + \gamma_\parallel k_\alpha k_\beta / k^2. \qquad \text{(A1)}$$

It can be easily seen that model I corresponds to $\gamma_\perp = \gamma_\parallel$, while models II and III are obtained from (A1) if $\gamma_\parallel = 0$ or $\gamma_\perp = 0$. A similar correlator, containing simultaneously a longitudinal and a transverse part, is encountered when the models (7) are considered in a magnetic field.[31]

The renormalized values of the coefficients $\gamma_\parallel$ and $\gamma_\perp$ are obtained by integration over the fast components φ_0 and $\bar{\varphi}_0$ of the fields φ and $\bar{\varphi}$. To separate the fast and slow variables, we introduce into the action the vertex

$$S_\lambda = i D_0 \lambda^2 \int \bar{\varphi}_0 \varphi_0 \, d\mathbf{r}, \qquad \text{(A2)}$$

which is needed for infrared regularization. To cut off the ultraviolet divergences we use the method of dimensional regularization in a space $d = 2 - \varepsilon$ ($\varepsilon > 0$).

By calculating (at the required accuracy) the diagrams a, b, c, and d of Fig. 3 we have for $\gamma_\perp$ and $\gamma_\parallel$

$$\gamma_\perp = \gamma_\perp^0 - \frac{1+\varepsilon}{2} I \frac{\gamma_\parallel^0 \gamma_\perp^0}{D_0^2} + \left[\frac{1+2\varepsilon}{8} \gamma_\parallel^0 + \frac{1}{8} \gamma_\perp^0 \right] I^2 \frac{\gamma_\parallel^0 \gamma_\perp^0}{D_0^4}, \qquad \text{(A3)}$$

$$\gamma_\parallel = \gamma_\parallel^0 + \left(\frac{1-\varepsilon}{2} \gamma_\perp^0 - \gamma_\parallel^0 \right) I \frac{\gamma_\parallel^0}{D_0^2} + \left[\frac{2+\varepsilon}{4} (\gamma_\parallel^0)^2 - \frac{1+2\varepsilon}{8} \gamma_\parallel^0 \gamma_\perp^0 - \frac{1-2\varepsilon}{8} (\gamma_\perp^0)^2 \right] I^2 \frac{\gamma_\parallel^0}{D_0^4} \qquad \text{(A4)}$$

where

$$I = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^2 + \lambda^2} = \frac{\lambda^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma\left(\frac{\varepsilon}{2} \right). \qquad \text{(A5)}$$

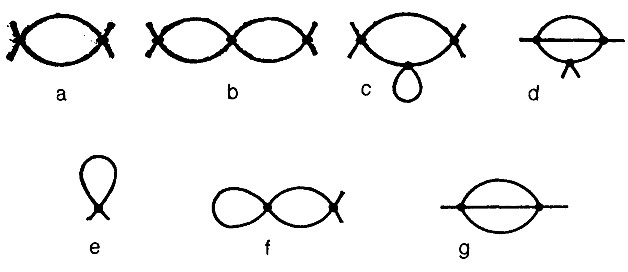

FIG. 3. One-loop corrections to γ and D (a, e); two-loop corrections to γ (b, c, d) and D (f, g).

Similarly, calculating diagrams e, f, and of Fig. 3 we obtain

$$D=D_0+\frac{2-\varepsilon}{4}I\frac{\gamma_\perp{}^0-\gamma_\parallel{}^0}{D_0}$$
$$+\left[\frac{1}{8}(\gamma_\parallel{}^0)^2-\frac{3\varepsilon}{8}\gamma_\parallel{}^0\gamma_\perp{}^0-\frac{1-\varepsilon}{8}(\gamma_\perp{}^0)^2\right]\frac{I^2}{D_0{}^3}. \qquad (A6)$$

To derive the RG equation we must differentiate (A3) and (A4) with respect to $d\xi = d(\ln \lambda^{-1})$, recognizing that $dI/d\xi \sim \varepsilon I$ (A5), and then express the unrenormalized values γ^0 and D^0 from (A3) and (A4) in terms of the renormalized γ and D, with allowance for only one-loop corrections (proportional to the first power of I). As a result, the singular terms proportional to $\varepsilon I^2 \sim 1/\varepsilon$ are canceled out in the expressions for $d\gamma/d\xi$ and $dD/d\xi$, and only the terms $\sim \varepsilon^2 I^2$ that are finite as $\varepsilon \to 0$ remain. In the limit as $\varepsilon \to 0$ we now have

$$d\ln D/d\xi=g_\perp-g_\parallel-2g_\perp g_\parallel, \qquad (A7)$$
$$d\ln g_\perp/d\xi=g_\parallel-2g_\perp+g_\parallel(3g_\perp-g_\parallel), \qquad (A8)$$
$$d\ln g_\parallel/d\xi=-g_\perp(1-g_\parallel-g_\perp), \qquad (A9)$$

where

$g_{\parallel(\perp)}=\gamma_{\parallel(\perp)}/(4\pi D^2)$.

Contributions to the mobility μ are made by diagrams e, f, and g of Fig. 3, but these contain the Green's functions in the external field E:

$G_0(\mathbf{k})=[D_0(k^2+\lambda^2)+i\mu\mathbf{kE}]^{-1}$.

The derivation of the RG equation for μ is similar to the derivation of (A7) and yields

$$d\ln \mu/d\xi=-g_\parallel(1+g_\perp). \qquad (A10)$$

In the particular cases $g_\perp = g_\parallel$, $g_\parallel = 0$, and $g_\perp = 0$, which correspond respectively to models I, II, and III, Eqs. (A7)–(A10) go over into (21) and (24).

Note added in proof (16 June 1986). It turns out that in the potential model III there is no renormalization of the charge g in all orders of the loop expansion; the supernormalizability of this model is closely related to the satisfaction of the Einstein relation. By the same token, expressions (25) and (26) for model III are exact (see V. E. Kravtsov, I. V. Lerner, and V. I. Yudson, Spectrosc. Inst. Preprint 3, 1986).

[1)]The functional integration in Refs. 6, 7, and 10 was over time-dependent fields. In this case $Z = 1$ (Ref. 22), so that the replica method was not needed for the averaging. In the RG approach, both procedures are equivalent: the "extra" dependence on the temporal variables or on the replica indices leads to vanishing of the unphysical diagrams with closed loops; the expressions for the remaining diagrams coincide.

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