Dynamics and radiation emission by particles trapped by an electrostatic wave in a transverse magnetic field

G. M. Zaslavskiĭ, S. S. Moiseev, and A. A. Chernikov

Institute of Space Research, USSR Academy of Sciences
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We investigate a new type of radiation, produced when particles are accelerated in the field of a longitudinal wave in a transverse magnetic field. Equations are obtained for the intensity and frequency of the spontaneous emission, and it is shown that the relativistic Doppler effect can cause the emission frequency to exceed considerably the bounce frequency of the trapped particle oscillations. A dispersion equation that describes the instability of a relativistic electron beam accelerated along a longitudinal-wave front is derived and analyzed. The instability growth rate is found and the frequency and gain length of the stimulated coherent radiation are determined when the exciting waves interact coherently.

1. INTRODUCTION

Particles moving ahead of a wave front in a magnetic field become accelerated.[1] A potential wave $\varphi(x,t) = \varphi_0 \cos(k_0 x - \omega_0 t)$ of sufficiently large amplitude traps particles whose velocities are close to the phase velocity of the wave, and accelerates them in the wave-propagation direction. If the magnetic field is parallel to the wave front (is directed along the z axis), it deflects the particles and causes them to collide repeatedly with the wave; each such collision is accompanied by an increase of the velocity v_y. At large values of v_y the Lorentz force ev_yB_0/c exceeds the electrostatic force $-e\partial\varphi/\partial x$ and, in the nonrelativistic case, the particle leaves the potential well of the wave. It is shown in Ref. 2 that this acceleration mechanism causes nonlinear damping of plasma waves in a transverse magnetic field. In Refs. 3 and 4 a relativistic modification was considered for this acceleration mechanism that, if the field $E_0 = k_0\varphi_0$ exceeds B_0 sufficiently, makes possible trapping of the particles in a potential well, accompanied by an unlimited acceleration along the axis. Particle acceleration along the wave front is accompanied by oscillations of the trapped particles in a direction perpendicular to the wave front. These oscillations give rise to radiation of a new type, briefly described in Ref. 5.

This paper is devoted to a detailed analysis of this radiation. Equations are obtained for the intensity and frequency of the spontaneous emission and it is shown that its frequency can greatly exceed the oscillation frequency of the particles trapped in the potential well of the wave. The frequency multiplication is due to the relativistic Doppler effect. The influence of the collective effects on the emission is considered for the case when the accelerated electron beam has a sufficiently high density and small spread in energy and angle. The collective effects are caused by the interaction between the radiation modes and the longitudinal vibrations of the relativistic electron beam, and increase the radiation power greatly through coherence. Expressions are obtained for the frequencies of the excited waves, and estimates are presented of the temporal growth rates for the case of emission along the beam, when the wave vector of the excited electromagnetic wave is parallel to the beam velocity.

2. DYNAMICS OF TRAPPED PARTICLES IN A TRANSVERSE MAGNETIC FIELD

The Hamiltonian H describing the interaction of an electron with a traveling electrostatic wave in a transverse magnetic field with a vector potential $\mathbf{A} = B_0 x \mathbf{e}_y$ is given by

$$(H-e\varphi)^2 = m^2c^4 + c^2p_x^2 + c^2\left(P_y - \frac{e}{c}B_0x\right)^2, \tag{2.1}$$

where $\mathbf{P} = \mathbf{p} + e\mathbf{A}/c$ is the generalized momentum of the particle. The system of canonical equations of motion is

$$\begin{gathered}\dot{x} = c^2p_x\left[m^2c^4 + c^2p_x^2 + c^2\left(P_y - \frac{e}{c}B_0x\right)^2\right]^{-1/2},\\ \dot{y} = c^2\left(P_y - \frac{e}{c}B_0x\right)\left[m^2c^4 + c^2p_x^2 + c^2\left(P_y - \frac{e}{c}B_0x\right)^2\right]^{-1/2},\\ \dot{p}_x = ceB_0\left(P_y - \frac{e}{c}B_0x\right)\left[m^2c^4 + c^2p_x^2 + c^2\left(P_y - \frac{e}{c}B_0x\right)^2\right]^{-1/2}\\ -eE_0\sin(k_0x - \omega_0t), \quad \dot{P}_y = 0.\end{gathered} \tag{2.2}$$

Since the Hamiltonian (2.1) is independent of the coordinate y, the generalized-momentum component P_y is an integral of the motion. We therefore assume hereafter

$$p_y = -eB_0x/c. \tag{2.3}$$

The canonical equations of motion can then be written in the form

$$\begin{gathered}\dot{x} = c^2p_x(m^2c^4 + c^2p_x^2 + e^2B_0^2x^2)^{-1/2},\\ \dot{p}_x = -e^2B_0^2x(m^2c^4 + c^2p_x^2 + e^2B_0^2x^2)^{-1/2} - eE_0\sin(k_0x - \omega_0t).\end{gathered} \tag{2.4}$$

The motion along the y axis is given by

$$\dot{y} = -ceB_0x(m^2c^4 + c^2p_x^2 + e^2B_0^2x^2)^{-1/2}. \tag{2.5}$$

A particle trapped by an electrostatic-wave field moves along the x axis at an approximately constant velocity equal to the phase velocity of the wave

$$x_0(t) = (\omega_0/k_0)t, \tag{2.6}$$

and correspondingly $\dot{x}_0 = \omega_0/k_0$. We use the first equation of (2.4) to express the momentum p_x in terms of the velocity $\dot{x}$ and of the coordinate x of the particle:

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$$p_x/mc=(\dot{x}/c)\,(1+\omega_H{}^2x^2/c^2)^{1/2}(1-\dot{x}^2/c^2)^{-1/2}, \tag{2.7}$$

where $\omega_H = eB_0/mc$ is the nonrelativistic cyclotron frequency. Substituting (2.6) in (2.7) we obtain the time dependence of the momentum $p_x^{(0)}$:

$$p_x{}^{(0)}(t)/mc=\beta_{\rm ph}\gamma_{\rm ph}\,(1+\omega_H^2\beta_{\rm ph}{}^2t^2)^{1/2}, \tag{2.8}$$

where $\beta_{\rm ph} = \omega_0/k_0c$ and $\gamma_{\rm ph} = 1/(1-\beta^2_{\rm ph})^{1/2}$. It follows from the integral (2.3) of the motion that

$$p_y{}^{(0)}(t)/mc=-\omega_H\beta_{\rm ph}t. \tag{2.9}$$

The total particle energy $\mathscr{E} = (m^2c^4 + c^2p_x^2 + c^2p_y^2)^{1/2}$ increases with time as

$$\mathscr{E}/mc^2\equiv\gamma(t)=\gamma_{\rm ph}\,(1+\omega_H^2\beta_{\rm ph}{}^2t^2)^{1/2}. \tag{2.10}$$

Relations (2.8)–(2.10), which describes the averaging of the resonant particles in the transverse magnetic field, were derived earlier in Refs. 3 and 4.

We seek the solution of the set (2.4) in the form

$$p_x=p_x{}^{(0)}(t)+\delta p_x,\quad x=(\omega_0/k_0)t+\xi(t), \tag{2.11}$$

where $\delta p_x \ll p_x^{(0)}$ and $\dot{\xi}\ll\omega_0/k_0$ are small corrections to the nonstationary ground state (2.8)–(2.10). We obtain from (2.7) the connection between the momentum perturbation δp_x and the perturbations of the velocity $\dot{\xi}$ and of the coordinate ξ:

$$\delta p_x/mc = (\dot{\xi}/c)\gamma^3_{\rm ph}\,(1+\omega_H{}^2\beta_{\rm ph}{}^2t^2)^{1/2}$$

$$+\,\omega_H{}^2\beta_{\rm ph}{}^2\gamma_{\rm ph}t\xi/c(1+\omega_H{}^2\beta_{\rm ph}{}^2t^2)^{1/2}. \tag{2.12}$$

The second equation of the set (2.12) yields a relation between the rate of change $\delta\dot{p}_x$ of the momentum perturbation and the displacement ξ of the particle in the wave:

$$\delta\dot{p}_x/mc=-(eE_0/mc)\,\sin k_0\xi-\beta_{\rm ph}\gamma_{\rm ph}\omega_H^2t(1+\omega_H^2\beta_{\rm ph}{}^2t^2)^{-1/2}. \tag{2.13}$$

Eliminating δp_x from (2.13) with the aid of (2.12), we obtain the desired equation for the particle displacement (t):

$$\frac{d}{dt}\left[\gamma(t)\frac{d\xi}{dt}+\frac{\beta^2_{\rm ph}\omega_H{}^2t}{\gamma(t)}\xi\right]+\frac{eE_0}{m\gamma^2_{\rm ph}}\sin k_0\xi=-\frac{\beta_{\rm ph}\omega_H{}^2ct}{\gamma(t)}, \tag{2.14}$$

where the relativistic factor $\gamma(t)$ is determined by Eq. (2.10).

It follows from (2.14) that the particle acceleration along the wave front is accompanied by a slow drift of the particle along the x axis in a direction opposite to the wave motion, and by oscillations about the leading center. To analyze this motion we write the solution of (2.14) as a sum of two terms, $\xi(t) = \bar{\xi}(t) + \delta\xi(t)$. The first, $\bar{\xi}(t)$, corresponds to slow drift of the leading center, while the second corresponds to small oscillations of amplitude $\delta\xi \ll \bar{\xi}$. The location of the leading center is then determined from the equation

$$\sin k_0\bar{\xi}(t)=-\omega_H{}^2\beta_{\rm ph}\gamma^2_{\rm ph}ck_0t/\Omega_b{}^2\gamma(t), \tag{2.15}$$

where $\Omega_b = (eE_0k_0/m)^{1/2}$ is the oscillation frequency of the particles trapped in the potential well of the wave (the bounce frequency). At $\beta_{\rm ph}\omega_H t \gg 1$ the solution of (2.15) can tend asymptotically to a stationary value $\bar{\xi} = \xi_0$, where

$$\sin k_0\xi_0=-\gamma_{\rm ph}B_0/E_0. \tag{2.16}$$

Clearly, the stationary phase shift corresponds to a finite displacement of the particle relative to the bottom of the potential well of the particle-trapping wave. In addition, the condition

$$E_0>\gamma_{\rm ph}B_0 \tag{2.17}$$

is necessary for realization of the asymptotic solution (2.16).

This condition was obtained and discussed in References 3 and 4. Without dwelling in greater detail on the conditions for realizing this asymptotic acceleration regime under various initial conditions (see Ref. 4), we consider small oscillations of the trapped particle about the position of its guiding center. Putting $\delta\xi \ll \bar{\xi}$ in (2.14), we obtain for small oscillations the equation

$$\frac{d}{dt}\left\{\gamma(t)\frac{d\delta\xi}{dt}+\frac{\beta_{\rm ph}{}^2\omega_H^2t}{\gamma(t)}\delta\xi\right\}+\frac{\Omega_b^2\cos k_0\bar{\xi}(t)}{\gamma^2_{\rm ph}}\delta\xi=0. \tag{2.18}$$

The solution of Eq. (2.18) with initial conditions

$$\delta\xi(0)=0,\quad \delta\dot{\xi}(0)=\tilde{v}(0)=v_0 \tag{2.19}$$

can be obtained by the WKB method:

$$\delta\xi(t)=\frac{v_0\gamma^{3/2}_{\rm ph}}{\Omega_b\cos^{1/4}k_0\bar{\xi}(t)}(\gamma_{\rm ph}/\gamma)^{1/4+1/2\gamma_{\phi}^2}$$

$$\times\sin\int_0^t dt'\,\frac{\Omega_b\cos^{1/2}k_0\bar{\xi}(t')}{\gamma_{\rm ph}\gamma^{1/2}(t')}. \tag{2.20}$$

At $E_0 \gg \gamma_{\rm ph}B_0$ the particles oscillate near the bottom $k_0\bar{\xi}(t) \ll 1$ of the potential well; it follows then from (2.20) that

$$\delta\xi(t)=\frac{\tilde{v}}{\Omega}\sin\int_0^t dt'\,\Omega(t'), \tag{2.21}$$

where the frequency Ω and the amplitude v of the oscillation velocity decrease with time as

$$\Omega=\Omega_b/\gamma_{\rm ph}\gamma^{1/2},\quad \tilde{v}=v_0\,(\gamma_{\rm ph}/\gamma)^{3/4+1/2\gamma_{\rm ph}{}^2} \tag{2.22}$$

It follows from (2.22) that the laboratory frequency Ω of the oscillations of the trapped particles decreases, $\Omega\sim\gamma^{-1/2}$, in view of the relativistic mass increase, as they are accelerated along the wave front. The respective oscillation amplitudes $\delta\xi(t)$ and $\delta\dot{\xi}(t)$ of the particle positions and velocities decrease for the same reason. If the phase velocity of the potential wave is close to that of light, $\beta_{\rm ph}\approx 1$, which corresponds to $\gamma_{\rm ph}\gg 1$, the oscillation amplitude decreases as $\delta\xi\sim\gamma^{-1/4}$; for the oscillation velocity wave from (2.21) and (2.22) respectively $\delta\dot{\xi}\sim\gamma^{-3/4}$. In the opposite limit

$\beta_{ph} \ll 1$ (or $\gamma_{ph} \approx 1$) the amplitude and velocity of the oscillations is more strongly dependent on the relativistic factor γ, viz., $\delta\xi \sim \gamma^{-3/4}$, $\delta\dot{\xi} \sim \gamma^{-5/4}$.

We conclude this section by presenting in explicit form the conditions under which the solution (2.20) is valid;

$$E_0 > \gamma_{ph} B_0, \quad k_0\delta\xi \ll 1, \quad \Omega_b \gg \gamma_{ph}{}^2 \beta_{ph} \omega_H / \gamma^{1/2}. \tag{2.23}$$

The last inequality corresponds to the adiabaticity condition.

3. SPONTANEOUS EMISSION

An electron moving along a wave front with velocity $u \approx c$ and oscillating in a direction perpendicular to the wave front must radiate. If the electron moves towards the observer with ultrarelativistic velocity, the radiation frequency can be substantially higher than its bounce frequency. For simple estimates we shall use the equation[6] for the radiation power I of a relativistic particle moving along a curvilinear path

$$I = 2e^2c\gamma^4/3R^2, \tag{3.1}$$

where R is the path radius of curvature. The instantaneous value of the radius of curvature is

$$R = c^2/a, \tag{3.2}$$

where $a = \delta\ddot{\xi}$ is the transverse acceleration which can be determined from (2.21):

$$a = -\Omega\tilde{v} \sin \Omega t. \tag{3.3}$$

Averaging (3.1) over the period of the oscillations we obtain

$$I = e^2\Omega^2\tilde{v}^2\gamma^4/3c^3. \tag{3.4}$$

Substituting relations (2.22) in Eq. (3.4) we find that the intensity I radiated by one electron depends on its energy as

$$I = \frac{1}{3}\left(\frac{v_0}{c}\right)^2 \frac{e^2\Omega_b{}^2}{c} \gamma^{3/2 - 1/\gamma_{ph}{}^2} \gamma_{ph}{}^{1/\gamma_{ph}{}^2 - 1/2}. \tag{3.5}$$

Equation (3.5) is valid, generally speaking, for the time interval Δt during which the frequency Ω and the energy of the particle change little, i.e., $\dot{\gamma}\Delta t \ll 1, \Omega\Delta t \gg 1$.

The intensity of the radiation into a solid angle do in a direction $\mathbf{n}$ and in a frequency interval $(\omega, \omega + d\omega)$ is given by (see, e.g., Ref. 6):

$$\frac{dW}{d\omega\, do} = \frac{e^2}{4\pi^2c^3}\left(\frac{\omega}{\omega'}\right)^4 |[\mathbf{n}[\mathbf{n} - \frac{\mathbf{u}}{c}, \mathbf{a}(\omega')]]|^2, \tag{3.6}$$

where $\omega' = \omega(1 - \mathbf{un}/c)$ and $\mathbf{a}(\omega')$ is the Fourier transform of the acceleration

$$\mathbf{a}(\omega') = \int_{-\infty}^{+\infty} dt\, \mathbf{a}(t) e^{i\omega' t}. \tag{3.7}$$

In the derivation (3.6), the velocity $\mathbf{u} = (\omega_0/k_0)\mathbf{e}_x + u_y \mathbf{e}_y$ was regarded as constant and the only variable was the acceleration $\mathbf{a}(t)$. It follows from (3.3) and (3.6) that the radiation frequency ω is

$$\omega = \Omega(1 - \mathbf{un}/c)^{-1}. \tag{3.8}$$

Clearly, the frequency ω of the emitted waves depends on the angle θ between the velocity and the wave vector of the wave. For emission strictly along the beam ($\theta = 0$), the wave frequency is a maximum, $\omega = \omega_m$, where

$$\omega_m = 2\gamma^2\Omega, \tag{3.9}$$

and the frequency Ω in (3.9) depends, according to (2.21), on the particle energy. Substituting expression (2.22) for the oscillation frequency of the trapped particle in (3.9), we obtain the dependence of the radiation frequency on the relativistic factor of the particle:

$$\omega_m = 2\gamma^{3/2}\Omega_b/\gamma_{ph} \tag{3.10}$$

It is easily seen from (3.6) that the radiation intensity differs substantially from zero only in a narrow cone with angle $\Delta\theta \sim \gamma^{-1}$. In this sense, the angular distribution of the radiation is perfectly analogous to the corresponding distribution of the radiation emitted by ultrarelativistic particles in arbitrary external electromagnetic fields.[6]

We estimate now the power radiated by one electron accelerated to an energy $\mathscr{E} \sim 1$ GeV. Putting in (3.5) $v_0/c \sim 0.1$, $\gamma_{ph} = 2$, $\Omega_b \approx 10^{13}$ sec^{-1} we obtain $I \sim 3 \cdot 10^8$ W. The characteristic radiated wavelength is in this case of the order of several tens of angstroms. If a total number $N \sim 10^{11}$–10^{12} electrons are accelerated along the wave front, the total power of the incoherent spontaneous emission can reach ~ 100 W, a value comparable with the radiation power of other sources emitting in this wavelength band.[7]

The equations obtained in this section for the intensity and for the frequency are similar to the corresponding equations of the theory of undulator radiation except, however, that in our case the bounce frequency and the oscillation amplitude of the particles trapped by the potential wave depend on the relativistic factor γ, i.e., on the particle energy. Another difference is due to the frequency range. The maximum energy of the accelerated particles depends only on the size of the region in which the electrons interact with the external accelerating fields, and can be very high (a variant in which the particles are accelerated to energies $\mathscr{E} \sim 1$ TeV is discussed, e.g., in Ref. 3). The frequency multiplication is due in our case, just as in the case of undulator radiation, to the relativistic Doppler effect, but the bounce frequency Ω_b can be several orders higher than the frequency of the transverse electrons in an undulator. In our situation the wavelength of the generated radiation can therefore be decrease both by accelerating the particles to higher energies and by increasing the bounce frequency, i.e., by going from the macroscopic spatial period used in undulators to the microscopic spatial period of the external electromagnetic field.

4. STIMULATED EMISSION

The foregoing analysis is valid only when the action of a photon emitted by one particle on the emission by another particle can be neglected. Obviously, this is not the case at high current densities. It is then necessary to consider the stimulated emission. If the electron energy spread in a coordinate frame moving at the directional beam velocity is large enough, and the beam density is so low that the wavelength of the scattered radiation is less than the Debye radius of the electrons, one can speak of induced incoherent scattering by individual electrons. In the opposite limit of a denser and colder beam, the emission is stimulated and coherent, and is caused by excitation of natural collective oscillations of the beam. We note in this connection that the solution (2.20) indicates that the beam is "cooled" by deceleration, i.e., that the velocity and beam-density spreads decrease, and thus the conditions under which the radiation becomes coherent become less stringent.

We describe the stimulated coherent emission by using the equations of relativistic hydrodynamics with a self-consistent electromagnetic field, neglecting the thermal spread of the beam electrons:

$$\frac{\partial \mathbf{p}}{\partial t}+(\mathbf{v}\nabla)\mathbf{p}=e\left\{\mathbf{E}+\frac{1}{c}[\mathbf{vB}]\right\},\quad \frac{\partial n}{\partial t}+\operatorname{div} n\mathbf{v}=0,$$

$$\operatorname{rot}\mathbf{B}=\frac{4\pi}{c}en\mathbf{v}+\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{4.1}$$

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},\quad \operatorname{div}\mathbf{E}=4\pi en.$$

As shown above, in a coordinate frame moving a the phase velocity of the wave the ground state $v_0(t)$ of the beam-electron motion is a superposition of relativistic motion along the wave front (along the y axis) and fast oscillations of the electrons along the x axis with slowly decreasing amplitude:

$$v_0(t)=u\mathbf{e}_y+\tilde{v}\cos\Omega t\mathbf{e}_x, \tag{4.2}$$

where u, $\tilde{v}$, and Ω are slow functions of the time, i.e., vary little over the period of the oscillations $(d\ln u/dt)$, $(d\ln\tilde{v}/dt)$, $(d\ln\Omega/dt)\ll\Omega$, and are determined by Eqs. (2.22).

We linearize the system (4.1) with respect to the ground state (4.2), assuming that all the perturbed quantities depend only on the coordinate y, i.e., we confine ourselves to one-dimensional interaction of the waves:

$$\begin{gathered}\frac{\partial}{\partial t}\delta n+u\frac{\partial}{\partial y}\delta n=-n_b\frac{\partial}{\partial y}\delta v_y,\\ \frac{\partial}{\partial t}\delta p_x+u\frac{\partial}{\partial y}\delta p_x=e\left\{\delta E_x+\frac{1}{c}u\delta B_z\right\},\\ \frac{\partial}{\partial t}\delta p_y+u\frac{\partial}{\partial y}\delta p_y=e\left\{\delta E_y-\frac{1}{c}\tilde{v}\cos\Omega t\delta B_z\right\},\\ \frac{\partial}{\partial y}\delta B_z=\frac{4\pi en_b}{c}\delta v_x+\frac{1}{c}\frac{\partial}{\partial t}\delta E_x+\frac{4\pi e\delta n}{c}\tilde{v}\cos\Omega t,\\ \frac{\partial}{\partial y}\delta E_x=\frac{1}{c}\frac{\partial}{\partial t}\delta B_z,\quad \frac{\partial}{\partial y}\delta E_y=4\pi e\delta n.\end{gathered} \tag{4.3}$$

We derived the system (4.3) under the assumption that the excited electromagnetic wave is so polarized that the vector of the magnetic component of the wave field is directed along the z axis, and we designated by n_b the unperturbed beam density. We take the Fourier transforms of the set (4.3), representing all the perturbed quantities in the form

$$\delta b(y,t)=\int dk\, e^{iky}\delta b(k,t).$$

After this transformation we can express the set (4.3) in the form of two equations for parametrically coupled oscillators:

$$\begin{gathered}\left(\frac{d}{dt}+iku\right)\gamma^3\left(\frac{d}{dt}+iku\right)\delta n+\omega_L^2\delta n=\frac{ikn_b}{mc}\tilde{v}\cos\Omega t\delta B_z,\\ \left(\frac{d^2}{dt^2}+c^2k^2+\frac{\omega_L^2}{\gamma}\right)\delta B_z=-4\pi ick\tilde{v}\cos\Omega t\delta n,\end{gathered} \tag{4.4}$$

where $\omega_L^2=4\pi n_b e^2/m$ is the square of the electron Langmuir frequency. The derivation of this set of equations took into account that the relation between the momentum perturbation $\delta\mathbf{p}$ and the velocity perturbation δv is

$$\delta p_x=\gamma m\delta v_x,\quad \delta p_y=\gamma^3 m\delta v_y,\quad \gamma=(1-u^2/c^2)^{-1/2}. \tag{4.5}$$

If there is no parametric relation ($\tilde{v}=0$), the solution of the set (4.4) describes a linearly polarized transverse electromagnetic wave with nonzero component δB_z propagating independently along the y axis with a longitudinal beam wave due to the oscillations of the beam charge density. Let us examine the solution of the system (4.4) at $\tilde{v}\neq 0$. We neglect in first-order approximation the slow time dependences of the unperturbed quantities $\tilde{v}$, u, and Ω. It is then possible to take in (4.4) the Fourier transforms in frequency:

$$\begin{gathered}\delta n(k,t)=\int d\omega\, e^{-i\omega t}\delta n(k,\omega),\\ \delta B_z(k,t)=\int d\omega\, e^{-i\omega t}\delta B_z(k,\omega)\end{gathered}$$

and obtain in the three-wave approximation, taking into account only the interaction between the density perturbations $\delta m(\omega)$ at the frequency ω and the first-harmonic perturbations $\delta B_z(\omega-\Omega)$ of the magnetic field, the following dispersion equation:

$$[(\omega-ku)^2-\omega_L^2/\gamma^3][(\omega-\Omega)^2-c^2k^2-\omega_L^2/\gamma]=\omega_L^2k^2\tilde{v}^2/4\gamma^3. \tag{4.6}$$

The dispersion equation (4.6), which describes the coherent interaction of the waves, has a counterpart in the theory of free-electron lasers.[8] We seek the solution of the dispersion equation (4.6) at $\tilde{v}/c\ll 1$ near the solutions of the dispersion equations for waves produced as a result of the interaction

$$\omega=\omega_r+\delta\omega,\quad k=k_r,$$

where ω_r and k_r are determined from the conditions

$$(\omega_r-k_ru)^2=\omega_L^2/\gamma^3,\quad (\omega_r-\Omega)^2=c^2k_r^2+\omega_L^2/\gamma. \tag{4.7}$$

If $\omega_r,\Omega\gg\omega_L/\gamma^{1/2}$, Eqs. (4.7) lead to the following equation that relates the frequency of the generated electromagnetic radiation to the frequency Ω:

$$\omega_r=2\gamma^2\Omega. \tag{4.8}$$

This equation is obviously in accord with Eq. (3.9) obtained in the preceding section for the maximum frequency of the spontaneous emission along the beam. The difference is that in the case of the induced process the frequency has a complex increment $\delta\omega$, an equation for which follows from relations (4.6) and (4.7):

$$\delta\omega^2(-2\omega_L/\gamma^{3/2}+\delta\omega)=ck_r\omega_L{}^2\tilde{v}^2/8\gamma^3c^2. \quad (4.9)$$

Equation (4.9) was obtained under conditions such that a slow beam wave is excited, with a dispersion law

$$\omega=ku-\omega_L/\gamma^{1/2}. \quad (4.10)$$

If the wave coupling is weak, $|\delta\omega| \ll 2\omega_L/\gamma^{3/2}$, Eq. (4.10) leads to an expression for the instability growth rate

$$\mathrm{Im}\,\delta\omega\equiv\delta=(\omega_L\omega_r)^{1/2}(\tilde{v}/c)/4\gamma^{3/4}. \quad (4.11)$$

Substituting in (4.11) expression (4.8) for the frequency ω_r, we obtain at $\gamma_{\mathrm{ph}} \approx 1$, taking relations (2.22) into account,

$$\delta=2^{1/2}(\omega_L\Omega_b)^{1/2}(v_0/c)/4\gamma^{5/4}. \quad (4.12)$$

Let us estimate the growth rate (4.12) of radiation generated by a beam of electrons of energy $\mathscr{E}$ = 10 MeV and density $n_b = 10^{12}$ cm^{-3}. Putting $v_0/c \sim 0.1$ and $\Omega_b = 10^{13}$ sec^{-1}, we get $\delta \sim 1.5\cdot 10^9$ sec^{-1}. which corresponds to an e-folding length $L = c/\gamma \sim 10$ cm. The radiation generated is then in the infrared, with frequency $\omega \approx 2\cdot 10^{15}$ sec^{-1}.

Thus, particle acceleration along a wave front to relativistic velocities in a transverse magnetic field is accompanied by radiation whose frequency can exceed substantially the oscillation frequency of the particles trapped by the wave.

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