Two-dimensional solitons: magnetic vortices in a uniaxial antiferromagnet

V. P. Voronov and A. M. Kosevich

M. S. Ol'minskiĭ State Pedagogical Institute, Belgorod

(Submitted 22 November 1985)
Zh. Eksp. Teor. Fiz. **90**, 2145–2151 (June 1986)

The dynamical properties of two-dimensional topological solitons (magnetic vortices) in a uniaxial antiferromagnet are analyzed. It is shown that, when the anisotropy energy has a certain form, the antiferromagnetism vector precession frequencies and the magnetic-vortex energy can have limiting values, the limiting energy values being dependent on the topological parameter of the vortex. The limiting values of the antiferromagnetic-vortex energy are found to be equal to those of the magnetic-vortex energy in a uniaxial ferromagnet.

1. Recently there has been an upsurge in interest in the analysis of two-dimensional nonlinear perturbations in magnetic materials, in light of the possibility of their experimental detection.[1] A special place among localized nonlinear perturbations is occupied by two-dimensional topological solitons (magnetic vortices), whose properties in ferromagnets have been described in fairly great detail.[2–6] In the present paper we show that magnetic vortices in antiferromagnets possess a number of properties similar to those possessed by vortices in ferromagnets.

The basic equations governing the dynamics of a two-sublattice uniaxial antiferromagnet have been obtained by Bar'yakhtar and Ivanov.[7] Under the natural assumption that the magnetization of the antiferromagnet is small compared to that of each of the sublattices, the dynamics of the magnetic material is described by the equations for the unit antiferromagnetism vector

$$\mathbf{l}(\sin\theta\cos\varphi;\ \sin\theta\sin\varphi;\ \cos\theta),$$

which, in the angular variables θ and φ, have the form[2,7]

$$\alpha\left[\Delta\theta-\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}\right]$$

$$-\alpha\sin\theta\cos\theta\left[(\nabla\varphi)^2-\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}-gH\right)^2\right]-\frac{\partial W_a}{\partial\theta}=0,$$

$$\alpha\nabla(\sin^2\theta\,\nabla\varphi)-\frac{\alpha}{c^2}\frac{\partial}{\partial t}\left[\sin^2\theta\left(\frac{\partial\varphi}{\partial t}-gH\right)\right]=0. \quad (1)$$

Here $\mathbf{H}$ is an external magnetic field parallel to the selected axis (the z axis); c is a characteristic velocity equal, when $\mathbf{H}=0$, to the smallest spin-wave phase velocity in the linear theory: $c=gM_0(\alpha A)^{1/2}/2$; α and A are respectively the inhomogeneous and homogeneous exchange constants; g is the gyromagnetic ratio; W_a is the anisotropy energy and which depends only on the angle θ of the uniaxial antiferromagnet.

If we introduce the new variable $\tilde{\varphi}=\varphi-gHt$, the equations (1) for the two scalar functions become invariant under the Lorentz transformation with c as the limiting velocity. Therefore, we actually have, when we find some steady-state solutions to the equations (1), an entire family of solutions obtainable from the first Lorentz transformation containing a velocity $V<c$. Let us, bearing this in mind, limit ourselves to the analysis of the static soliton solutions for $\mathbf{H}=0$.

Let us consider the following localized axisymmetric solutions to the system (1) with $\mathbf{H}=0$:

$$\theta=\theta(r),\quad \varphi=\nu\chi-\omega t, \quad (2)$$

where r and χ are the polar coordinates in the plane perpendicular to the preferred axis; $\nu=0, 1, 2,...$ is the parameter playing the role of topological charge of the soliton,[2] and ω is the precession frequency. To the soliton corresponds a solution $\theta(r)$ satisfying the boundary conditions

$$\theta(0)=\pi m;\quad m=0,\ 1;\quad \theta(\infty)=0. \quad (3)$$

The equations (1) possess two integrals of motion; the magnetization field energy

$$E=2\pi aM_0^2\int\left\{\frac{\alpha}{2}\left[\left(\frac{d\theta}{dr}\right)^2+\left(\frac{\nu^2}{r^2}+\frac{\omega^2}{c^2}\right)\sin^2\theta\right]+W_a(\theta)\right\}r\,dr \quad (4)$$

and the number of spin deviations in the soliton:

$$N=-\frac{2\pi a\alpha M_0^2}{\hbar c^2}\int\frac{\partial\varphi}{\partial t}\sin^2\theta r\,dr. \quad (5)$$

The quantity N is proportional to the z component of the total magnetization of the magnetic material. In the formulas (4) and (5) a is the lattice constant.

It is clear that the function $\theta(r)$ is a solution to the ordinary differential equation

$$\alpha\Delta\theta-\alpha\sin\theta\cos\theta(\nu^2/r^2-\omega^2/c^2)-dW_a/d\theta=0. \quad (6)$$

2. In the simplest case, when the anisotropy energy density in the uniaxial antiferromagnet has the form

$$W_a(\theta)=(\beta/2)\sin^2\theta,\quad \beta>0, \quad (7)$$

Eq. (6) can be rewritten in terms of dimensionless coordinates:

$$\frac{d^2\theta}{dx^2}+\frac{1}{x}\frac{d\theta}{dx}-\left(1+\frac{\nu^2}{x^2}\right)\sin\theta\cos\theta+\varkappa^2\sin\theta\cos\theta=0, \quad (8)$$

where $x=r/l_0$; $l_0^2=\alpha/\beta$; $\varkappa^2=(\omega l_0/c)^2$. As can easily be verified, for $x\to\infty$ we have

$$\theta=(\mathrm{const}\,x^{-1/2})\exp[-x(1-\varkappa^2)^{1/2}]. \quad (9)$$

It follows from (9) that Eq. (8) possesses the localized solu-

 0038-5646/86/061257-04$04.00

tions in question only when $0 < \varkappa^2 < 1$.

It is impossible to solve Eq. (8) analytically and obtain explicit solutions satisfying the boundary conditions (3); therefore, it was solved by numerical methods with the aid of a computer. The numerical integration of Eq. (8) enabled us to find the solutions satisfying the boundary conditions (3) with $m = 1$ and $\nu = 1, 2$, and 3. The dependence, constructed with their aid, of the parameter $\varkappa^2$ on the number N of spin deviations is depicted in Fig. 1, where the abscissas are the values of the ratio N/N_1, in which $N_1 = 2\pi a\alpha M_0^2 \omega l_0^2/(\hbar c^2)$.

Analysis of the obtained soliton solutions and of the soliton parameters leads to the following conclusions. First, the solutions are functions of the dimensionless coordinate x, which are nonzero in a narrow interval around $x = 0$; therefore, they are, from the standpoint of the macroscopic description, not different from functions with a δ-function singularity. Second, the value of the derivative $dN/d\omega > 0$, a fact which, according to the results obtained in Ref. 7, may be evidence of the instability in the region $0 < \varkappa^2 < 1$ of the soliton solutions for an antiferromagnet with the anisotropy (7).

For $\varkappa^2 = 1$, Eq. (8) assumes the simpler form

$$\frac{d^2\theta}{dx^2}+\frac{1}{x}\frac{d\theta}{dx}-\frac{\nu^2}{x^2}\sin\theta\cos\theta=0. \tag{10}$$

The analytic solution to Eq. (10) with the boundary conditions (3) is known[8,9]:

$$\theta=2\,\mathrm{arctg}\,(R/x)^{|\nu|}, \tag{11}$$

where R is an arbitrary parameter playing the role of soliton dimension. Let us, using (11), compute the integrals of motion (4) and (5):

$$E=E_0(4|\nu|+2N/N_1), \qquad E_0=\pi a\alpha M_0^2, \tag{12}$$

$$N=N_1\frac{2R}{\nu^2}\frac{\pi}{\sin(\pi/\nu)}. \tag{13}$$

The formula (12) gives the dependence of the soliton energy on the number of spin deviations. It follows from (12) and (13) that the number N of spin deviations is infinite in the case when $\omega = c/l_0$ and $\nu = 1$. This dependence of $\varkappa^2$ on N, or, more exactly, of ω on N, is equivalent to the corresponding dependence for the magnetic soliton in a one-dimensional antiferromagnet.[7] For $\nu > 1$ the quantities N and E tend to some finite limits as $\varkappa^2 \to 1$.

3. Let us consider an antiferromagnet with an anisotropy energy of the more general form:

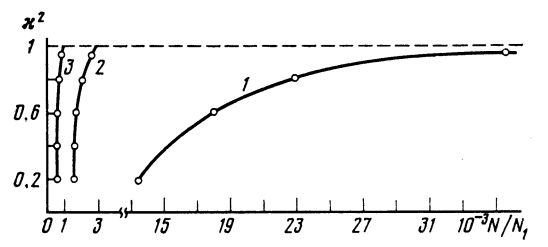

FIG. 1. Results of the analysis of the dependence $\varkappa^2(N/N_1)$. The number on the curve is the value of the parameter ν.

$$W_a(\theta)={}^1\!/_2\beta\sin^2\theta-{}^1\!/_4 b\sin^4\theta, \qquad \beta>0, \qquad b>0. \tag{14}$$

Equation (6), expressed in terms of the dimensionless variables, for the function θ assumes, when (14) is taken into account, the form

$$\frac{d^2\theta}{d\rho^2}+\frac{1}{\rho}\frac{d\theta}{d\rho}-\frac{\nu^2}{\rho^2}\sin\theta\cos\theta+\Omega\sin\theta\cos\theta-\frac{1}{4}\sin4\theta=0, \tag{15}$$

where we have introduced the following notation:

$$\rho=r/r_0, \quad r_0^2=\frac{c^2}{\omega_1^2-\omega_2^2}=\frac{2\alpha}{b}, \quad \omega_1=gM_0(A\beta)^{1/2}/2,$$

$$\omega_2=gM_0\left[\left(\beta-\frac{b}{2}\right)A\right]^{1/2}\Big/\,2, \quad \Omega=\frac{\omega^2-\omega_2^2}{\omega_1^2-\omega_2^2}.$$

Naturally, Eq. (15) goes over into Eq. (8) in the limit as $b \to 0$. To verify this, we should go back to the independent variable x, and then take the limit as $b \to 0$ ($r_0 \to \infty$), noting that

$$l_0^2\lim_{b\to0}(\Omega/r_0^2)=\varkappa^2-1.$$

It is easy to verify that, as $\rho \to 0$, the solution to Eq. (15) which is of interest to us behaves like

$$\theta=\pi-(\rho/\rho_0)^{|\nu|}\,(\rho_0=\mathrm{const}),$$

while as $\rho \to \infty$,

$$\theta=\mathrm{const}\,\rho^{-1/2}\exp[-\rho(1-\Omega)^{1/2}].$$

To determine those admissible values of the parameter Ω at which Eq. (15) can possess localized solutions, let us multiply (15) term by term by $\rho^2(d\theta/d\rho)$ and integrate the result obtained over ρ in the range from $\rho = 0$ to $\rho = \infty$.

Integrating by parts, we easily find that

$$\Omega\int_0^\infty\sin^2\theta\rho\,d\rho=\frac{1}{4}\int_0^\infty\sin^2 2\theta\rho\,d\rho. \tag{16}$$

The relation (16) can, after simple transformations, be represented in the form

$$(\Omega-1)\int_0^\infty\sin^2\theta\rho\,d\rho=-\int_0^\infty\sin^4\theta\rho\,d\rho. \tag{17}$$

From the positiveness of the integrals in (16) and (17) it follows that soliton solutions of the type under investigation can exist only in the parameter range $0 < \Omega < 1$.

In the case considered by us the integrals of motion (4) and (5) can, when allowance is made for (14), be transformed into

$$E=E_0\int_0^\infty\left\{\left(\frac{d\theta}{d\rho}\right)^2+\left(\frac{\nu^2}{\rho^2}+\frac{\omega^2+\omega_2^2}{\omega_1^2-\omega_2^2}\right)\times\sin^2\theta+\frac{1}{4}\sin^2 2\theta\right\}\rho\,d\rho, \tag{18}$$

$$N=N_2\int_0^\infty\sin^2\theta\rho\,d\rho, \tag{19}$$

where $N_2 = 2E_0\omega/(\hbar(\omega_1^2-\omega_2^2))$.

By computing the variation of the energy (18), integrating by parts, and using (19), we can easily show that the normal differential relation

$$\delta E=\hbar\omega\delta N \tag{20}$$

connecting E and N is satisfied for the antiferromagnetic vortex. The relation (20) coincides with the corresponding differential expression for the self-localized magnetization wave in a ferromagnet.[2]

As has been noted before,[6] the characteristics of the topological solitons manifest themselves in the region of small N ($N \ll N_2$). To small N values corresponds a small localization length R ($R \ll r_0$), a situation which leads to the appearance of a small parameter that allows us to carry out an analytic investigation of the properties of the solutions to Eq. (15). Let us carry out (order of magnitude) estimates of the terms entering into Eq. (15). The first, second, and third terms are of the same order of magnitude:

$$\frac{d^2\theta}{d\rho^2}\sim r_0^2\frac{d^2\theta}{dr^2}\sim\frac{r_0^2}{r}\frac{d\theta}{dr}\sim\frac{r_0^2}{r^2}\nu^2\sim\frac{r_0^2}{R^2}\gg 1.$$

Estimation of the fourth and fifth terms yields values of the order of unity; therefore, Eq. (15) can, in the leading approximation in the small parameter $(R/r_0)\ll 1$, be replaced by Eq. (10), which possesses the self-similar solution (11). Substituting (11) into the expression (17), we find, after simple transformations, that

$$(1-\Omega)\int_0^\infty\frac{y^{1/|\nu|}}{(1+y)^2}dy=4\int_0^\infty\frac{y^{1/|\nu|+1}}{(1+y)^4}dy, \tag{21}$$

where $y=(\rho/R)^{2|\nu|}$. Each of the integrals in (21) can be expressed in terms of a gamma function, and the properties of the latter lead to the following relation between the parameters of the localized solution:

$$\Omega={}^1/_3(1+2/\nu^2). \tag{22}$$

The existence of limiting (as $N\to 0$) antiferromagnetism-vector precession frequencies follows from the formula (22).

It is worth noting that, for $\nu=1$ and $\nu=2$, the limiting values of Ω in the antiferromagnetic soliton are equal to those of the magnetization-vector precession frequencies in the ferromagnetic soliton, or vortex.[6] The limiting precession frequencies behave quite differently in the case of large values of the parameter ν. Thus, for $\nu\to\infty$, in the case of an antiferromagnet $\Omega\to 1/3$, while in the case of a ferromagnet the frequency tends to zero.

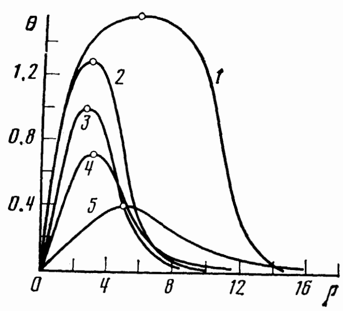

FIG. 3. Dependence of θ on ρ for the magnetic soliton: 1) $\Omega=0.1$; 2) $\Omega=0.3$; 3) $\Omega=0.5$; 4) $\Omega=0.7$; and 5) $\Omega=0.9$ (the points indicate the maximum values of the function).

Substituting the solution (11) into (18), and taking account of (22), we obtain the following expression for the limiting values of the energy:

$$E=E_0\left[4|\nu|+\left(\Omega+\frac{\omega^2+\omega_2^2}{\omega_1^2-\omega_2^2}\right)\frac{N}{N_2}\right]. \tag{23}$$

The formula (23) gives the soliton energy as a function of the number N of spin deviations up to terms of the order of $(N/N_2)^2$ in the case when $N\to 0$. As follows from (23), the soliton energy in an antiferromagnet tends, as $N\to 0$, to its limiting value $E=4\pi a\alpha M_0^2|\nu|$, which is equal to the limiting value of the soliton energy in the case of a ferromagnet.[6]

4. The complete solution to Eq. (15) with the boundary conditions (3) was found numerically by the "shooting" method with the use of a computer and a qualitative analysis of the ($\rho=\infty$) limiting integral curves in the phase plane (see Ref. 2). From the magnetization distribution $\theta(\rho)$ found we constructed for the cases $m=1$ and $\nu=1, 2$, and 3 the dependence of the number of spin deviations in the soliton on the dimensionless frequency Ω. This dependence is depicted in Fig. 2, where the numbers on the curves are the values of the parameter ν. Analysis of the solutions obtained and of the soliton parameters allows us to draw the following conclusions. First, the solution with $\nu=1$ goes over, as $N\to 0$, into a δ-function linear singularity. Therefore, the numerical integration of Eq. (15) for the extremely small $N(N\to 0)$ values corresponding to $\Omega\to 1$ meets with considerable difficulties. In view of this, the last point to the left on the curve 1 (Fig. 2) was obtained by us for the value $\Omega=0.83$. Second, as $N\to 0$, the value of the parameter Ω, as

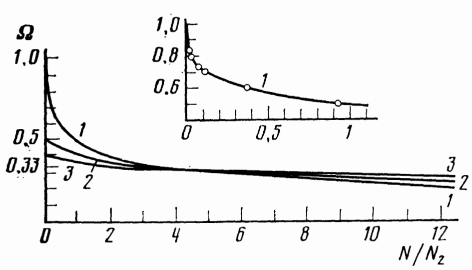

FIG. 2. Dependence of the parameter Ω on N/N_2 (the points indicate the results of the numerical calculation).

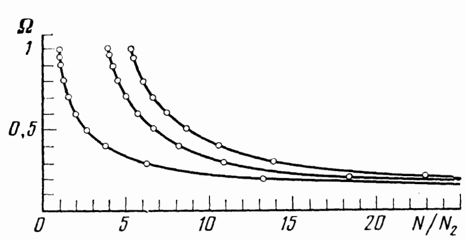

FIG. 4. Results of the analysis of the dependence $\Omega(N/N_2)$ for $m=0$ and $\nu=0$ (bottom curve); $\nu=1$ (middle curve); and $\nu=2$ (top curve). The points indicate the results of the numerical calculation.

given by the formula (22), tends to its limiting value, while the soliton energy tends to $E = 4\pi a \alpha M_0^2|\nu|$, which agrees with the formula (23) for the limiting energy values. Third, the curves describing the N dependence of the parameter Ω intersect each other. Let us note that the curves describing the corresponding dependence in the case of magnetic solitons in a ferromagnet do not intersect (see Fig. 1 in Ref. 6).

5. Finally, let us briefly discuss the solution with $m = 0$. Figure 3 gives some idea of the nature of the solutions to Eq. (15) that satisfy the boundary conditions (3) in the case when $\nu = 1$ and $m = 0$. As follows from Fig. 3, the maximum value of the function $\theta(\rho)$ shifts to the left as Ω increases in the range of values from 0.1 to 0.5, and shifts to the right as Ω increases in the range $0.5 < \Omega < 1$.

As the parameter Ω decreases in the region below $\Omega = 0.1$, the graph of the function $\theta(\rho)$ flattens out, and, as $\Omega \to 0$, the value $\theta_{\max}$ tends to its limiting value $\pi/2$ (curve 1). As $\Omega \to 1$, the solution $\theta(\rho)$ becomes delocalized.

From the magnetization distribution function $\theta(\rho)$ found for the cases $m = 0$ and $\nu = 0$, 1, and 2 we constructed the dependence of the parameter Ω on the number N of spin deviations. This dependence is depicted in Fig. 4. As follows from Fig. 4, as $\Omega \to 1$, the values of N tend to the following limiting values: $N = 0.94N_2$ for $\nu = 0$, $N = 3.87N_2$ for $\nu = 1$, and $N = 5.22N_2$ for $\nu = 2$. Thus, the minimum number N of spin deviations necessary for the generation of a soliton solution increases with increasing topological charge ν. The plots of the N dependence of Ω for different ν values in the $m = 0$ case, unlike the corresponding plots in the $m = 1$ case, do not intersect in the region $N \gg N_2$.

In conclusion, let us note that, in the $\nu = 0$ case, we can, by replacing θ by $\theta/2$, reduce Eq. (15) to the form

$$\frac{d^2\theta}{d\rho^2} + \frac{1}{\rho}\frac{d\theta}{d\rho} - \sin\theta\cos\theta + \Omega\sin\theta = 0. \tag{24}$$

Equation (24) coincides with the equation, (3), obtained in Ref. 10 for the magnetic vortex with $\nu = 0$ in a ferromagnet. The results obtained by us in the $\nu = 0$ case agree with the results obtained in Ref. 10 by integrating this equation with different admissible values of the parameter Ω.

We are grateful to B. A. Ivanov for discussions.

[1] V. S. Gornakov, L. M. Dedukh, and V. I. Nikitenko, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 199 (1984) [JETP Lett. **39**, 236 (1984)].
[2] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Nelineĭnye volny namagnichennosti. Dinamicheski i topologicheskie solitony (Nonlinear Magnetization Waves. Dynamic and Topological Solitons), Nauk. Dumka, Kiev, 1983.
[3] J. Kaczer, Czech. J. Phys. B **12**, 354 (1962).
[4] A. M. Kosevich and V. P. Voronov, Fiz. Nizk. Temp. **7**, 908 (1981) [Sov. J. Low Temp. Phys. **7**, 442 (1981)].
[5] J. Kaczer and V. Krejči, J. Magn. Magn. Mater. 15–18, 1479 (1980).
[6] V. P. Voronov, B. A. Ivanov, and A. M. Kosevich, Zh. Eksp. Teor. Fiz. **84**, 2235 (1983) [Sov. Phys. JETP **57**, 1303 (1983)].
[7] I. V. Bar'yakhtar and B. A. Ivanov, Zh. Eksp. Teor. Fiz. **85**, 328 (1983) [Sov. Phys. JETP **58**, 190 (1983)].
[8] A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
[9] A. M. Perelomov, Usp. Fiz. Nauk **134**, 577 (1981) [Sov. Phys. Usp. **24**, 645 (1981)].
[10] A. S. Kovalev, A. M. Kosevich, and K. V. Maslov, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 321 (1979) [JETP Lett. **30**, 296 (1979)].

Translated by A. K. Agyei