Instability of periodic waves described by the nonlinear Schrödinger equation

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The stability against two-dimensional perturbations of one-dimensional periodic envelope waves described by the nonlinear Schrödinger equation is investigated. Media whose transverse dispersion and nonlinearity have opposite signs are considered. Dispersion equations are obtained for long-wave perturbations of waves with an arbitrary Jacobi modulus and also for perturbations with arbitrary wave numbers in the weakly nonlinear wave approximation. A qualitative analysis of these dispersion equations is carried out with a view to elucidating the general characteristics of the stability problem for specific types of periodic waves, classifying the results obtained in previous investigations, and interpreting the numerical calculations. The conclusions, arrived at semi-heuristically by Zakharov (Zh. Eksp. Teor. Fiz. **53**, 1735 (1967) [Sov. Phys. JETP **26**, 994 (1968)]), that in the class of two-dimensional perturbations all types of periodic waves are unstable in media with an arbitrary nonlinearity sign is confirmed. The error in the results obtained by certain authors with respect to the one-dimensional stability problem for waves in media with positive nonlinearity is discussed.

1. INTRODUCTION

As is well known,[1] among the simplest steady-state solutions to the one-dimensional nonlinear Schrödinger equation (NSE)—nonlinear parabolic equation—are, besides solitons, the periodic (cnoidal) waves. The present paper is devoted to the problem of the stability of such waves. This problem was first investigated by Zakharov.[2] Subsequently, these investigations were continued by a number of other authors, including Rowlands and Infeld,[3–6] Martin *et al.*,[7] and Pavlenko and Petviashvili.[8–10]

Zakharov[2] considered the class of two-dimensional perturbations in media with positive transverse dispersion and arbitrary nonlinearity sign. He arrived at the conclusion that, irrespective of the sign of the nonlinearity, all the periodic solutions to the NSE with a constant phase are unstable. His method (based on the variational principle) has been criticized by Vakhitov and Kolokolov.[11] And Pavlenko and Petviashvili[8–11] have made assertions about the properties of periodic waves, that are at variance with Zakharov's result.[2] Since these assertions have not been refuted by Zakharov himself or anybody else, the impression has been created that the this result[2] is erroneous. This may explain why no mention is made of Ref. 2 in recent publications, including Ref. 12.

In the present paper we use methods different from the method used by Zakharov,[2] and show that his result, though obtained in a rather heuristic manner, is in fact correct, whereas all the alternative assertions made by Pavlenko and Petviashvili[8–10] are erroneous.

A summary of the information that we have about the one-dimensional stationay waves under investigation here is given in Sec. 2. Also given in Sec. 2 are the basic equations for the perturbations. In Sec. 3 the perturbation dispersion equation is derived by the method of power series expansion in the small wave numbers, and this equation is analyzed in Sec. 4. In Sec. 5 the instabilities are investigated by the method of power series expansion in the small Jacobi modulus s^2. The results of the investigation are summarized and discussed in Sec. 6.

2. BASIC EQUATIONS

2.1. Stationary waves. We shall investigate the stability of the one-dimensional periodic waves described by the equation

$$L_0\varphi=0, \tag{2.1}$$

where $\varphi=\varphi(\xi)$ is the amplitude of the envelope wave, ξ is the coordinate in the rest frame of the wave, and the operator L_0 is defined by the relation

$$L_0=\partial^2/\partial\xi^2+\alpha+\beta\varphi^2. \tag{2.2}$$

The parameter $\alpha=\pm 1$ and $\beta=\pm 1$ are determined by the signs of the dispersion and the nonlinearity.

For $\alpha=\pm 1$ and $\beta=1$, as $\varphi(\xi)$, we can take the function[1,10]

$$\varphi(\xi)=b_1\,\mathrm{cn}\,(g_1\xi,\ s)\equiv\varphi_c,\quad b_1=2^{1/2}sg_1,\quad g_1=|2s^2-1|^{-1/2}, \tag{2.3}$$

where cn is the Jacobi elliptic function and s^2 is the modulus of this function. For $\alpha=-1$ the modulus s^2 varies within the limits $\frac{1}{2}<s^2\leqslant 1$, whereas for $\alpha=1$ we have $0\leqslant s^2<\frac{1}{2}$. For $\alpha=-1$ and $\beta=1$ Eq. (2.1) possesses, besides (2.3), a solution of the form[1,10]

$$\varphi(\xi)=b_2\,\mathrm{dn}\,(g_2\xi,\ s)\equiv\varphi_d,\quad b_2=2^{1/2}g_2,\quad g_2=(2-s^2)^{-1/2}, \tag{2.4}$$

corresponding to the interval $0\leqslant s^2\leqslant 1$.

In the case $\alpha=1$ and $\beta=-1$ Eq. (2.1) admits the solution[1]

$$\varphi(\xi)=b_3\mathrm{sn}(g_3\xi,\ s)\equiv\varphi_s,\quad b_3=2^{1/2}sg_3,\quad g_3=(1+s^2)^{-1/2}, \tag{2.5}$$

with the modulus s^2 lying in the range $0\leqslant s^2\leqslant 1$.

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We call the waves described by functions φ of the type (2.3)–(2.5) cn, dn, and sn waves respectively.

2.2. Perturbations. To describe the perturbations of waves of the type (2.3)–(2.5), we shall proceed from a NSE of the following form for the complex envelope Ψ of the wave (cf. Refs. 1 and 13):

$$i\partial\Psi/\partial t+\partial^2\Psi/\partial x^2+\gamma\partial^2\Psi/\partial y^2+\beta|\Psi|^2\Psi=0. \tag{2.6}$$

Here x is the coordinate on which the steady state of the wave depends (the longitudinal coordinate), y is the transverse coordinate, $\gamma = \pm 1$ is a parameter characterizing the sign of the transverse dispersion of the waves, and the meaning of the quantity β is given in Subsec. 2.1. The coordinate x is connected with the coordinate ξ, introduced in Subsec. 2.1, by the relation $\xi = A(x - 2pt)$, where A and p are real constants. Making in (2.6) the standard substitution

$$\Psi=Af(\xi, y, t)\exp[-i(\alpha A^2+p^2)t+ipx], \tag{2.7}$$

we obtain

$$\partial^2 f/\partial\xi^2+\alpha f+\beta|f|^2 f=-2i\partial f/\partial\tau-\gamma\partial^2 f/\partial\eta^2, \tag{2.8}$$

where $\tau = 2A^2t$ and $\eta = Ay$. As in Ref. 13, we obtain

$$f=\varphi(\xi)+u(\xi,\eta,\tau)+iv(\xi,\eta,\tau), \tag{2.9}$$

where $\varphi(\xi)$ is determined by Eqs. (2.1) and (2.2) and u and v are real functions characterizing the perturbation of the steady state of the wave. From (2.8) it follows that the functions u and v satisfy the equations

$$\begin{aligned} L_1u&=2\partial v/\partial\tau-\gamma\partial^2u/\partial\eta^2,\\ L_0v&=-2\partial u/\partial\tau-\gamma\partial^2 v/\partial\eta^2, \end{aligned} \tag{2.10}$$

where the operator L_1 is defined by the relation

$$L_1=\partial^2/\partial\xi^2+\alpha+3\beta\varphi^2. \tag{2.11}$$

Following Refs. 3 and 4, we seek the solution to the equations (2.10) in the form

$$(u,v)=[U(\xi),V(\xi)]\exp(-i\omega\tau+ik\xi+ik_\perp\eta)+\text{c.c.} \tag{2.12}$$

Here $U(\xi)$ and $V(\xi)$ are periodic functions of ξ, c.c. stands for complex conjugate, ω and $k_\perp$ are the frequency and transverse wave number of the perturbations, and the quantity k has the meaning of the "mean" ξ th wave number of the perturbations (in a different terminology, k is the Bloch quasimomentum). According to (2.10) and (2.12), the functions U and V satisfy the equations

$$\begin{aligned} L_1U&=-2i(\omega V+kU')+\varkappa^2U,\\ L_0V&=2i(\omega U-kV')+\varkappa^2V, \end{aligned} \tag{2.13}$$

where the prime denotes differentiation with respect to ξ and $\varkappa^2\equiv k^2+\gamma k_\perp^2$.

3. DISPERSION EQUATION FOR LONG-WAVE PERTURBATIONS OF PERIODIC WAVES WITH AN ARBITRARY JACOBI MODULUS

Assuming ω, k, and $k_\perp$ to be small parameters, we solve Eq. (2.13) through a power series expansion in these quantities. Proceeding in much the same way as Zakharov and Rubenchik do in Ref. 13 (see also Refs. 3 and 4), we arrive at a biquadratic equation for ω_2, which we represent in the form

$$\omega^4+A\omega^2+B=0, \tag{3.1}$$

where A and B connected with the wave numbers by the relations

$$\begin{aligned} A&=-2(a_{12}k^2+\gamma a_{13}k_\perp^2)/a_{11},\\ B&=(a_{22}k^4+2\gamma a_{23}k^2k_\perp^2+a_{33}k_\perp^4)/a_{11}. \end{aligned} \tag{3.2}$$

The expressions for the coefficients a_{ik} $(i, k = 1, 2, 3)$ are given in Appendix A. There we explain that the coefficients a_{ik} corresponding to the various types of waves are connected by relations determined by the Jacobi transformations.

Thus, the dispersion equation in the long-wave approximation is characterized by the formulas (3.1) and (3.2) and the expressions for the a_{ik} given in Appendix A. It is essentially the same dispersion equation obtained in Refs. 3–6, but represented in a different form. We thus confirm the correctness of the dispersion equation obtained in Refs. 3–6.

The formulas (3.1) and (3.2) and the expessions for the a_{ik} given in Appendix A indicate that, contrary to the assertions made in Ref. 9, the general problem of periodic-wave perturbation cannot be investigated with the use of the energy method or any similar variant of the variational approach.

4. ANALYSIS OF THE DISPERSION EQUATION IN THE LONG-WAVE APPROXIMATION

The dispersion equation (3.1) with relations of the form (3.2) for A and B allows us to investigate long-wave perturbations with arbitrary relation between k and $k_\perp$ and for arbitrary s^2. Below we, however, limit ourselves to the analysis of certain limiting cases only. In Sec. 4.1 we shall consider longitudinal perturbations: $k\neq 0$, $k_\perp = 0$; in Subsec. 4.2, transverse perturbations: $k = 0$, $k_\perp\neq 0$. In Subsec. 4.3 we investigate oblique perturbations $((k,k_\perp)\neq 0)$ with $s^2\ll 1$; in Subsec. 4.4, oblique perturbations with $1 - s^2\ll 1$.

4.1. Longitudinal perturbations. Let $k_\perp = 0$. In this case we find from (3.1), (3.2) that

$$\omega^2=k^2[a_{12}\pm(a_{12}^2-a_{11}a_{22})^{1/2}]/a_{11}. \tag{4.1}$$

Equation (4.1) was obtained by Rowlands.[3] Assuming s^2 to be arbitrary, Rowlands[3] analyzed this equation numerically, and thus showed that for all s^2 the cn and dn waves are unstable, while the sn waves are stable. In contrast, we shall show this analytically.

Using (A.2)–(A.4), we find

$$\omega_c^2=k^2g_1^2s_1^2[s_1^2+s^2\mu\pm is_1s(1-\mu)]^2/(s_1^2+s^2\mu^2)^2, \tag{4.2}$$

$$\omega_d^2=-k^2g_2^2s_1^2[\lambda-s_1^2\pm s_1(1-\lambda)]^2/(\lambda^2-s_1^2)^2, \tag{4.3}$$

$$\omega_s^2=k^2g_3^2[1-s^2\rho\pm s(1-\rho)]^2/(1-s^2\rho^2)^2. \tag{4.4}$$

The subscripts c, d and s attached to ω^2 indicate the types of waves we are investigating. The remaining designations are explained in Appendix A.

It can be seen from (4.2) that, for all $0 < s^2 < 1$, the one-dimensional perturbations of the cn waves are characterized by two pairs of frequencies with complex ω^2, so that $(\mathrm{Re}\,\omega, \mathrm{Im}\,\omega) \neq 0$. The corresponding instabilities may be called oscillatory or "traveling" instabilities. Thus, our analytic formula (4.2) confirms Rowlands' numerical analysis.[3] It also confirms Pavlenko and Petviashvili's numerical calculations[10] pertaining to the case $s^2 > 0.5$, but refutes their conclusion, repeated subsequently in Petviashvili and Yan'kov's review article,[14] that the cn waves are stable when $s^2 < 0.2$. Therefore, it is clear that Petviashvili and Yan'kov's hypothesis[14] (see also Ref. 8) that the reason for the stability may be connected with the total integrability of the NSE is dubious.

Further, it follows from (4.3) that the dn waves, being also unstable for all s^2, are characterized by two pairs of frequencies with real $\omega^2 < 0$, so that the corresponding instabilities are aperiodic. Finally, as can be seen from (4.4), the sn waves, unlike the cn and dn waves, are stable for all s^2.

For $s^2 \ll 1$ it follows from (4.2)–(4.4) that

$$\omega_c^2 = (1 \pm is) k^2, \tag{4.5}$$

$$\omega_d^2 = (-32k^2/s^4,\ -k^2/2), \tag{4.6}$$

$$\omega_s^2 = (1 \pm s) k^2. \tag{4.7}$$

Notice that the second pair of purely imaginary dn-wave perturbation frequencies (4.6) are none other than the negative square of the growth rate in the long-wave limit of the modulation instability of a monochromatic wave (see, for example, Ref. 15). As to the first pair of frequencies for such perturbations, their meaning will be elucidated in Sec. 5 in terms of the interaction of the Fourier harmonics of the dn wave.

The cn- and sn-wave perturbation frequencies ω_c and ω_s are equal to each other when the corrections of the order of s are neglected (see (4.5) and (4.7)). This is not surprising, since in the corresponding approximation the two waves are characterized by functions of one and the same form, differing only in a ξ-coordinate shift. In the case of the sn wave a correction of the order of s is insignificant, which cannot be said of the cn wave: the instability of this wave is revealed precisely when allowance is made for such a correction. This instability can also be interpreted in terms of the interaction of the Fourier harmonics (see Sec. 5).

For $(1 - s^2) \ll 1$ we find from (4.2)–(4.4) that (see Appendix B)

$$\omega_c^2 = k^2 (1 \pm 2i\varepsilon^{1/2}) \varepsilon, \tag{4.8}$$

$$\omega_d^2 = -k^2 (1 \pm 2\varepsilon^{1/2}) \varepsilon, \tag{4.9}$$

$$\omega_s^2 = k^2 (1/2,\ \lambda^2\varepsilon^2/8), \tag{4.10}$$

$$\varepsilon = (1 - s^2)/\lambda^2. \tag{4.11}$$

Taking account of (B.7) and (B.8), we conclude that in the case of the cn and dn waves

$$\varepsilon = 4h^2 e^{-h}, \tag{4.12}$$

where h is the soliton spacing. Using (4.8), (4.9), and (4.12), we observe that, as the soliton spacing increases, the growth rate of the cn- and dn-wave instabilities decreases exponentially:

$$\mathrm{Im}\,\omega_c = 4kh_1^2 e^{-h_1}, \quad \mathrm{Im}\,\omega_d = 2kh_2 e^{-h_2/2}. \tag{4.13}$$

According to Petviashvili and Yan'kov,[14] numerical calculations[10] also indicate that the growth rate of the instability of a periodic lattice of very sparsely distributed solitons is exponentially small. Noting this result, Petviashvili and Yan'kov[14] claim that "this...was to be expected," without explaining why. The formulas (4.13) confirm the numerical results obtained in Ref. 10, and furnish an analytic explanation for them.

Let us note that Rowlands[3] also considered the asymptotic behavior of the growth rate of the cn-wave perturbations for $s^2 \to 1$. He arrived at the conclusion that in this case the growth rate decreases like $(s_1 \ln s_1)^{3/2}$. This is an error: According to (4.8), it is not the growth rate, but $\mathrm{Im}(\omega_c^2)$ that decreases according to this law.

4.2. Transverse perturbations For $k = 0$ we find from (3.1) that

$$\omega_{c\pm}^2 = -(1 \pm 2) \gamma k_\perp^2 g_1^2 s^2 \nu\mu/(s_1^2 + s^2\mu^2), \tag{4.14}$$

$$\omega_{d\pm}^2 = -(1 \pm 2) \gamma k_\perp^2 g_2^2 \chi\lambda/(\lambda^2 - s_1^2), \tag{4.15}$$

$$\omega_{s\pm}^2 = (1 \pm 2) \gamma k_\perp^2 g_3^2 s^2 \sigma\rho/(1 - s^2\rho^2). \tag{4.16}$$

Here ω_+ corresponds to even perturbations, ω_- to odd perturbations.

The formulas (4.14)–(4.16) indiciate that transverse perturbations of all the three types of wave are unstable no matter what signs the nonlinearity and the transverse dispersion have (cf. Refs. 2 and 13). The even or odd perturbations are unstable, depending on the signs of these quantities.

For $s^2 \ll 1$ we obtain from (4.14)–(4.16) the relations

$$\omega_{c\pm}^2 = -(1 \pm 2) \gamma k_\perp^2 s^2/4, \tag{4.17}$$

$$\omega_{d\pm}^2 = -(1 \pm 2) \gamma k_\perp^2/2, \tag{4.18}$$

$$\omega_{s\pm}^2 = (1 \pm 2) \gamma k_\perp^2 s^2/4. \tag{4.19}$$

In the opposite limiting case, when $s_1^2 \ll 1$,

$$\omega_{c\pm}^2 = \omega_{d\pm}^2 = -(1 \pm 2) \gamma k_\perp^2/3, \tag{4.20}$$

$$\omega_{s\pm}^2 = (1 \pm 2) \gamma k_\perp^2/6. \tag{4.21}$$

Notice that the formulas (4.20) coincide with the results obtained in Ref. 13 for a single soliton, and that the formulas (4.15), (4.18), and (4.20) for dn waves with $\gamma = -1$ are in qualitative agreement with the numerical results obtained in Ref. 7. Let us also point out a certain paradoxical aspect of the expressions (4.18) for the squares of the frequencies of the transverse perturbations of dn waves with small s^2: neither of these two expressions coincides with the well-known formula for the square of the frequency of the transverse perturbations of a monochromatic wave. The causes of this paradox are elucidated in Subsecs. 4.3 and 5.2.

4.3. Oblique perturbations of waves with $s^2 \ll 1$. Using (3.1), (3.2), (A.2), and (A.4), we find that when $s^2 \ll 1$ the squares of the frequencies

$$\omega_c^2=k^2-s^2\gamma k_\perp^2/4\pm is(k^4+\gamma k^2k_\perp^2-s^2k_\perp^4/4)^{1/2}, \tag{4.22}$$

$$\omega_s^2=k^2+s^2\gamma k_\perp^2/4\pm s(k^4+\gamma k^2k_\perp^2+s^2k_\perp^4/4)^{1/2}, \tag{4.23}$$

correspond to oblique perturbations of cn and sn waves. These relations illustrate the transition from the formulas (4.5) and (4.7), which characterize the squares of the frequencies of the longitudinal perturbations of cn and sn waves, to the formulas (4.17) and (4.19) for the squares of the transverse perturbations of such waves. Using (4.22), we find that ω_c^2 ceases to be complex and becomes real at $k_\perp^2/k^2>4/s^2$ in media with $\gamma=1$ and at $k_\perp^2/k^2>1$ in media with $\gamma=-1$. From (4.23) it follows that the frequencies ω_s^2 remain real for any relation between k and $k_\perp$ if $\gamma=1$, and are complex if $\gamma=-1$ and

$$1<k_\perp^2/k^2<4/s^2. \tag{4.24}$$

Using (3.1), (3.2), and (A.3), we find that the oblique perturbations of dn waves with $s^2\ll1$ described by the dispersion equation

$$16(2\omega^2+k^2+\gamma k_\perp^2)k^2+s^4(\omega^2+3\gamma k_\perp^2/2)(\omega^2-\gamma k_\perp^2/2)=0. \tag{4.25}$$

Setting here $k_\perp=0$ or $k=0$, we arrive at the formula (4.6) or (4.18).

Let us, using (4.25), follow the transition from (4.6) to (4.18) as the ratio $k_\perp^2/k^2$ increases. The second pair of roots of Eq. (4.6) are most sensitive to the presence of $k_\perp\neq0$. These roots become modified at $k_\perp^2\gtrsim k^2$. In this case instead of the second equality in (4.6) we have

$$\omega_d^2=-(k^2+\gamma k_\perp^2)/2. \tag{4.26}$$

This dispersion equation is well known: it describes the modulation instability of a monochromatic wave with $(k,k_\perp)\neq0$ in the case when $\gamma=\pm1$ and the stabilization of this instability at $k_\perp\gg k$ in media with $\gamma=-1$ (cf. Refs. 15 and 16). For $k_\perp^2/k^2$ lying in the interval (cf. (4.24))

$$1\ll k_\perp^2/k^2\ll1/s^4, \tag{4.27}$$

instead of (4.26) we have

$$\omega_d^2=-\gamma k_\perp^2/2, \tag{4.28}$$

while the first pair of roots of (4.6) still remain the same. At the right boundary of the indicated interval, i.e., for

$$k_\perp^2/k^2\approx1/s^4, \tag{4.29}$$

the two pairs of roots are found to be of the same order of magnitude, and are, as a result, substantially mixed up with each other. Finally, at still higher $k_\perp^2/k^2$ values, when $k_\perp^2/k^2\gg1/s^4$, the dispersion equation (4.25) reduces to the set of equations (4.18) for the squares of the frequencies, $\omega_{d\pm}^2$.

From the foregoing it follows, in particular, that in the case of strictly transverse perturbations ($k=0$) the standard formula (4.26) does not follow from Eq. (4.25). This circumstance should, however, be regarded not as implying the inapplicability of the formula (4.26) when $k=0$, but as resulting from the insufficient accuracy of the formula (4.2, which we derived without making allowance for the terms, like ω^4/k^2 and $k_\perp^4/k^2$, that are, within the framework of the long-wave approximation used by us, formally small. This will become clear from Subsec. 5.2, where we obtain the corresponding generalization of the formula (4.25) by the method of power series expansion in s^2.

4.4. Oblique perturbations of waves with $1-s^2\ll1$. Using (3.1), (3.2), and (A.2)–(A.4), we find that for $s_1^2\ll1$

$$\omega_c^2=\varepsilon k^2-\gamma k_\perp^2/3\pm2(-\varepsilon^3k^4+\varepsilon^2\gamma k_\perp^2k^2/3+k_\perp^4/9)^{1/2}, \tag{4.30}$$

$$\omega_d^2=-\varepsilon k^2-\gamma k_\perp^2/3\pm2(\varepsilon^3k^4+\varepsilon^2\gamma k_\perp^2k^2/3+k_\perp^4/9)^{1/2}, \tag{4.31}$$

$$\omega_s^2=(1+\lambda)k^2/4+\gamma(1-\lambda/2)k_\perp^2/6$$
$$\pm[k^4(1+\lambda)^2(1-\lambda^2\varepsilon^2)/16+\gamma(1+3\lambda/4)k_\perp^2k^2/6$$
$$+(1-\lambda)k_\perp^4/9]^{1/2}. \tag{4.32}$$

Let us consider with the aid of (4.30)–(4.32) the transition from the formulas (4.8)–(4.10), which characterize the longitudinal perturbations, to the formulas (4.20) and (4.21) for the transverse perturbations as $k_\perp/k$ increases. In the case of the cn wave the role of $k_\perp$ is important when $k_\perp^2/k^2\gtrsim\varepsilon^{3/2}$. If

$$\varepsilon^{3/2}\ll k_\perp^2/k^2\lesssim\varepsilon, \tag{4.33}$$

then instead of (4.8) and (4.20) we obtain from (4.30) the equation

$$\omega_c^2=\varepsilon k^2-(1\pm2)\gamma k_\perp^2/3. \tag{4.34}$$

The formula (4.34) indicates that the instabilities of cn waves with complex frequencies disappear when $k_\perp^2/k^2\gtrsim\varepsilon^{3/2}$. In this case the perturbations remain stable right up to $k_\perp^2/k^2\approx\varepsilon$. For $k_\perp^2/k^2>\varepsilon$, (4.34) goes over to (4.20), and describes the a periodic instabilities discussed in Subsect. 4.3. In contrast to the case of cn waves, the growth rate of dn-wave perturbations becomes sensitive to the presence of $k_\perp$ only when $k_\perp^2/k^2\gtrsim\varepsilon$. In this case from (4.31) we have

$$\omega_d^2=-\varepsilon k^2-(1\pm2)\gamma k_\perp^2/3. \tag{4.35}$$

In contrast to the $k_\perp=0$ case, when the two branches of the dn-wave perturbations are unstable, only one of the perturbation branches remains unstable in the case when $k_\perp^2/k^2>\varepsilon$ (see (4.20)).

As can be seen from (4.32), finite $k_\perp/k$ values play a role in the perturbations of sn waves with $1-s^2\ll1$ only when $k_\perp/k\gtrsim1$. One of the perturbation branches is then unstable. As in the $s^2\ll1$ case in a medium with $\gamma=-1$, perturbations with complex frequencies also occur when $1-s^2\ll1$.

5. INVESTIGATION OF THE INSTABILITIES OF PERIODIC ENVELOPE WAVES BY THE METHOD OF POWER SERIES EXPANSION IN s^2

5.1. Perturbations of cn *and* sn *waves.* As in Sec. 3, here we proceed from equations of the type (2.13) for U and V, but now we substitute into these equations simplified expressions for L_0 and L_1, which correspond to the approximation $s^2\ll1$. In this approximation we have for the corresponding types of waves the formulas

$$\varphi=(\varphi_c,\varphi_s)=2^{1/2}s(\cos q_c\xi,\ \sin q_s\xi), \tag{5.1}$$

$$L_0=\frac{\partial^2}{\partial\xi^2}+1+s^2\begin{bmatrix}1+\cos 2q_c\xi\\ -1+\cos 2q_s\xi\end{bmatrix},$$
$$L_1=\frac{\partial^2}{\partial\xi^2}+1+3s^2\begin{bmatrix}1+\cos 2q_c\xi\\ -1+\cos 2q_s\xi\end{bmatrix}. \tag{5.2}$$

Here the upper formulas pertain to cn waves and the lower ones to sn waves; and q_c and q_s denote

$$q_c=1+3s^2/4,\quad q_s=1-3s^2/4. \tag{5.3}$$

Let us represent U and V in the form

$$(U,V)=\sum_m(U_m,V_m)\exp(imq\xi), \tag{5.4}$$

where $q=(q_c,q_s)$ and the summation is over all integral m. Taking (5.4) into account, we observe that, owing to the presence in L_0 and L_1 of terms with $\cos 2q\xi$, the mth harmonics are mixed up with the $(m\pm 2)$th harmonics in the equations (2.13). Since the quantities q are close to unity, such intermixing is especially important for the harmonics with $m=\pm 1$. On the other hand, it can be verified that the coupling between these harmonics and all the rest is weak, so that the quantities U_m and V_m with $m\neq\pm 1$ are small (specifically, of the order of s^2 raised to some positive power) compared to $U_{\pm 1}$ and $V_{\pm 1}$. Therefore, we limit ourselves to considering only the harmonics $U_{\pm 1}$ and $V_{\pm 1}$. Then in the case of the cn wave we obtain from (2.13) the system of coupled equations

$$\begin{aligned}&{}^3/_2s^2(U_{\pm1}+U_{\mp1})=-2i(\omega V_{\pm1}\pm ikq_cU_{\pm1})+\varkappa^2U_{\pm1},\\ &{}^1/_2s^2(V_{\mp1}-V_{\pm1})=2i(\omega U_{\pm1}\mp ikq_cV_{\pm1})+\varkappa^2V_{\pm1},\end{aligned} \tag{5.5}$$

while in the case of the sn wave instead of this system we have

$$\begin{aligned}&{}^3/_2s^2(U_{\mp1}-U_{\pm1})=-2i(\omega V_{\pm1}\pm ikq_sU_{\pm1})+\varkappa^2U_{\pm1},\\ &{}^1/_2s^2(V_{\pm1}+V_{\mp1})=2i(\omega U_{\pm1}\mp ikq_sV_{\pm1})+\varkappa^2V_{\pm1}.\end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we obtain the dispersion equation (3.1), with A and B given by

$$\begin{aligned}&A=-[4k^2q^2+\varkappa^2(\varkappa^2\mp s^2)]/2,\\ &B=\{[4k^2q^2-\varkappa^2(\varkappa^2\mp s^2)]^2-4\varkappa^4s^4\}/16.\end{aligned} \tag{5.7}$$

The formulas (5.7) with the upper ($-$) sign and $q=q_c$ pertain to the cn wave, while those with the lower ($+$) sign and $q=q_s$ pertain to the sn wave.

In the case of longitudinal perturbations ($k_\perp=0$, $\varkappa^2=k^2$), from (3.1) and (5.7) we obtain

$$\omega_c^2=k^2\{1+s^2/2+k^2/4\pm[(k^2-s^2)q_c^2+s^4k^2/4]^{1/2}\}. \tag{5.8}$$

Thus, we have another proof of the incorrectness of Pavlenko and Petviashvili's results[8-10] concerning the stability of the cn wave in the region of small s^2. For $k^2\ll s^2$ this formula goes over into Eq. (4.5), which describes the longitudinal instability of the cn wave. It can be seen that this instability stabilizes at

$$k^2>s^2. \tag{5.9}$$

We find with the aid of (3.1) and (5.7) that the longitudinal perturbations of the sn wave are characterized by frequencies the squares of which are given by

$$\omega_s^2=k^2\{1-s^2/2+k^2/4\pm[(k^2+s^2)q_s^2+s^4k^2/4]^{1/2}\}. \tag{5.10}$$

It can be seen that these perturbations are stable not only at small k, as follows from (4.4) and (4.7), but also at any k.

In the case of transverse perturbations ($k=0$), from (3.1) and (5.7) we have

$$\omega_c^2=k_\perp^2[k_\perp^2-(1\pm2)\gamma s^2]/4, \tag{5.11}$$
$$\omega_s^2=k_\perp^2[k_\perp^2+(1\pm2)\gamma s^2]/4. \tag{5.12}$$

It can be seen that the transverse long-wave instabilities mentioned in Subsec. 4.2 stabilize when

$$k_\perp^2>3s^2\quad\text{or}\quad k^2>s^2. \tag{5.13}$$

The first inequality pertains to the cn wave in media with positive dispersion, i.e., with $\gamma=1$, and to the sn wave in media with $\gamma=-1$, while the second inequality pertains to the cn wave in the case when $\gamma=-1$ and to the sn wave in the case when $\gamma=1$.

5.2. Perturbations of the dn *wave.* For $s^2\ll 1$ the function φ_d and the operators L_0 and L_1 can be approximated by the expressions

$$\varphi_d=\frac{q}{2}\left(1+\frac{s^2}{4}\cos q\xi+\frac{s^4}{64}\cos 2q\xi)\right), \tag{5.14}$$

$$\begin{aligned}L_0&=\frac{\partial^2}{\partial\xi^2}-\frac{s^4}{16}+\frac{s^2}{2}\cos q\xi+\frac{s^4}{16}\cos 2q\xi,\\ L_1&=\frac{\partial^2}{\partial\xi^2}+q^2+\frac{3s^2}{2}\cos q\xi+\frac{3s^4}{16}\cos 2q\xi,\end{aligned} \tag{5.15}$$

where

$$q=2^{1/2}(1-3s^4/64). \tag{5.16}$$

Let us expand the functions U and V in Fourier series of the form (5.4), and let us limit ourselves to the consideration of the expansion terms with $m=0,\ \pm 1$, and ± 2. It can then be verified that the second harmonics $U_{\pm 2}$ and $V_{\pm 2}$, which are important only at small k and $k_\perp$ values, are connected with the first harmonics $U_{\pm 1}$ and $V_{\pm 1}$ by the approximate relations

$$U_{\pm2}=s^2U_{\pm1}/8,\quad V_{\pm2}=s^2V_{\pm1}/32. \tag{5.17}$$

Taking account of (5.17), we arrive at the following system of equations for U_0, V_0, $U_{\pm 1}$, andf $V_{\pm 1}$:

$$\begin{gathered}(2-\varkappa^2)U_0+2i\omega V_0=-3s^2(U_{+1}+U_{-1})/4,\\ 2i\omega U_0+(\varkappa^2+s^2/16)V_0=s^2(V_{+1}+V_{-1})/4,\\ (\varkappa^2-3s^4/32\mp2kq)U_{\pm1}-2i\omega V_{\pm1}=3s^2(U_0+s^2U_{\mp1}/8)/4,\\ 2i\omega U_{\pm1}+(2+\varkappa^2\pm2kq)V_{\pm1}=s^2V_0/4.\end{gathered} \tag{5.18}$$

It can be seen that, in the $s=0$ approximation, this system of equations splits up into three, one of which corresponds to perturbations with $(U_0,V_0)\neq0$, the second with $(U_1,V_1)\neq0$, and the third with $(U_{-1},V_{-1})\neq0$. In the case of perturbations of the first type we obtain from (5.18) the well known generalization of the dispersion equation (4.26) for the modulation instability of monochromatic waves to the limit of large k and $k_\perp$ values (see, for example, Ref. 15):

$$\omega^2=(k^2+\gamma k_\perp^2)(k^2+\gamma k_\perp^2-2)/4. \tag{5.19}$$

The perturbations with $(U_1,V_1)\neq 0$ and $(U_{-1},V_{-1})\neq 0$ are described by dispersion equations similar to (5.19) with k replaced by $k\pm 2^{1/2}$, i.e., dispersion equations of the form

$$\omega^2=[\varkappa^2(2+\varkappa^2)+8k^2\pm 2^{5/2}k(1+\varkappa^2)]/4. \tag{5.20}$$

From (5.20) it follows that these perturbations, like the perturbations with $(U_0,V_0)\neq 0$, are stable at sufficiently large values of k^2 and $k_\perp^2$. But in the case when $(k^2,k_\perp^2)\ll 1$ we find from (5.20) that

$$\omega^2=\gamma k_\perp^2/2\pm 2^{1/2}k, \tag{5.21}$$

which corresponds to certain variants of the modulation instability of a monochromatic wave.

The Eqs. (4.26) and (5.21) constitute the required set of dispersion equations for the long-wave perturbations of dn waves with $s^2\to 0$. According to (5.18), at finite values of s^2 these dispersion equations become mixed, and are replaced by the following dispersion equation, which is cubic in ω^2:

$$16(2\omega^2+k^2+\gamma k_\perp^2)[k^2-(\omega^2-\gamma k_\perp^2/2)^2/2] +s^4(\omega^2+3\gamma k_\perp^2/2)(\omega^2-\gamma k_\perp^2/2)=0. \tag{5.22}$$

This equation justifies the assertion made at the end of Subsec. 4.3 regarding the limits of applicability of Eq. (4.25). Now let us consider the perturbations with $(\omega^2,k_\perp^2)\gg k^2$, i.e., outside the limits of applicability of Eq. (4.25). Then limiting ourselves to the case of purely transverse perturbations ($k=0$), we find that Eq. (5.22) splits up into two equations, one of which,

$$\omega^4-k_\perp^4/4-s^4(\omega^2+3\gamma k_\perp^2/2)/16=0, \tag{5.23}$$

describes the even perturbations, while the second corresponds to odd perturbations, and coincides with the second equation in (4.18).

For $\gamma=1$ the quantities ω^2 satisfying Eq. (5.23) are real, one of them being negative, and hence corresponding to an instability. For $k_\perp^2\ll s^4$ this ω^2 is given by the first equation in (4.18). Therefore, it remains for us to consider only the $\gamma=-1$ case. In this case we obtain from (5.23) the following expressions for the squares of the frequencies:

$$\omega^2=(s^4\pm D^{1/2})/32, \tag{5.24}$$

$$D=s^8-96s^4k_\perp^2+256k_\perp^4. \tag{5.25}$$

The quantity $D^{1/2}$ is purely imaginary for $k_\perp$ lying in the interval

$$k_{\perp 1}<k<k_{\perp 2}, \tag{5.26}$$

$$k_{\perp 1,2}=s^4(3\mp 2^{3/2})/16. \tag{5.27}$$

In this range of transverse wave numbers there are two types of transverse perturbations characterized by complex frequencies. The perturbations with $k_\perp<k_{\perp 1}$ are stable. Also stable are the perturbations with

$$k_{\perp 2}<k<k_{\perp 3}, \tag{5.28}$$

where $k_{\perp 3}^2=3s^4/8$. But if $k_\perp>k_{\perp 3}$, then, as in the $\gamma=1$ case, the roots ω^2 are real and different in sign, so that one of the perturbation branches is aperiodically unstable. For $k_\perp^2\gg s^4$ the growth rate of this instability corresponding to even perturbations with $(U_0,V_0)=0$, is characterized by the second equation in (4.18), i.e., coincides with the growth rate of odd perturbations.

The foregoing enables us to understand the numerical-analysis results of Martin *et al.*[7] pertaining to even transverse perturbations of dn waves in a medium with $\gamma=-1$. These even perturbations with complex frequencies and wave numbers lying in the interval (5.26) are similar to the traveling instabilities found by Martin *et al.*,[7] while the aperiodic instability of the even perturbations with $k_\perp>k_{\perp 3}$ noted above is similar to one found by these authors.[7]

6. DISCUSSION OF THE RESULTS

We have obtained formulas, (4.2), (4.5), (4.8), (5.8), and (5.9), that indicate the long-wave longitudinal instability of cn waves and the stabilization of this instability as the wave number increases. In contrast to the cn waves, sn waves are stable (within the framework of the assumptions made!) against longitudinal perturbations with any wavelengths. This follows from Eqs. (4.4), (4.7), (4.10), and (5.10). Longitudinal long-wave perturbations of dn waves, like those of cn waves, are also unstable at arbitrary s^2 values. We have obtained expressions for their growth rates: the formulas (4.3), (4.6), and (4.9).

These transverse perturbations of cn and sn waves can, according to the formulas (4.14), (4.16), (4.17), (4.19)–(4.21), and (5.8)–(5.12), grow in the case of arbitrarily small wave numbers, but stabilize at sufficiently large wave numbers. The picture for the transverse instabilities of dn waves is somewhat more complicated. In this case perturbations that grow only when the wave numbers are not too small also develop (see Eqs. (4.15), (4.18), (4.20), and (5.24)–(5.28)).

In considering oblique perturbations, our primary aim is to determine the characteristics of the transition from longitudinal to transverse perturbations. These characteristics are connected with the replacement of unstable perturbation branches by stable ones, and vice versa, and also with the disappearance and appearance of perturbation branches with complex frequencies (see the relation (4.22)–(4.24), (4.27)–(4.29), (4.33)–(4.35), and (5.21).

In accordance with what we said in the Introduction, we investigated the instability of only the simplest steady-state solutions to the NSE. The stability of the steady-state solutions of a more general form has been investigated by Infeld and Ziemkiewich.[6]

In the paper by Pavlenko and Petviashvili[10] it is hypothesized that the long Langmuir-wave trains observed by Antipov *et al.*[17] correspond to cn waves with $s^2<0.2$. This point of view[10] was based on the incorrect conclusion of these authors noted above that such waves possess one-dimensional stability for arbitrary values of the wave numbers of the perturbations. But these waves are stable only against perturbations with $k \gtrsim s$. Since in the case in question s characterizes the wave amplitude (i.e., $s \approx \varphi$), we can expect that only wave trains with relative length $l \lesssim 1/\varphi$ will be one-dimensionally stable.

APPENDIX A

Expressions for the coefficients a_{ik} and the use of the Jacobi transformations

To the three types of functions φ investigated by us (see (2.3)–(2.5)) correspond the three sets of coefficients a_{ik} in the formulas (3.2), so that

$$a_{ik}=(a_{ik}^{c},\ a_{ik}^{d},\ a_{ik}^{s}), \tag{A.1}$$

where the superscripts c, d, and s indicate that we are dealing with cn, dn, and sn waves. The quantities a_{ik}^{c} are given by the relations

$$\begin{gathered} a_{11}^{c}=(\mu^2+s_1^2/s^2)^2/g_1^2s^2, \\ a_{12}^{c}=[(\mu+s_1^2/s^2)^2-(1-\mu)^2s_1^2/s^2]s_1^2/s^2, \\ a_{22}^{c}=g_1^2s_1^4/s^6, \quad a_{13}^{c}=-\mu\nu(\mu^2+s_1^2/s^2)/s^2, \\ a_{23}^{c}=g_1^2s_1^2(3\nu^2-\mu^2)/2s^4, \quad a_{33}^{c}=-3g_1^2\mu^2\nu^2/s^2. \end{gathered} \tag{A.2}$$

Similarly, for the case of dn waves we have

$$\begin{gathered} a_{11}^{d}=(\lambda^2-s_1^2)^2/g_2^2, \quad a_{12}^{d}=-s_1^2[(\lambda-s_1^2)^2+s_1^2(1-\lambda)^2], \\ a_{22}^{d}=g_2^2s_1^4s^4, \quad a_{13}^{d}=-\chi\lambda(\lambda^2-s_1^2), \\ a_{23}^{d}=-g_2^2s_1^2(3\chi^2-s^4\lambda^2)/2, \quad a_{33}^{d}=-3g_2^2\lambda^2\chi^2. \end{gathered} \tag{A.3}$$

Finally, in the case of sn waves we obtain

$$\begin{gathered} a_{11}^{s}=s^4(1-s^2\rho^2)^2/g_3^2, \quad a_{12}^{s}=s^4[(1-s^2\rho)^2+s^2(1-\rho)^2], \\ a_{22}^{s}=g_3^2s_1^4s^4, \quad a_{13}^{s}=\sigma\rho s^6(1-s^2\rho^2), \\ a_{23}^{s}=-g_3^2s^6(3\sigma^2-s_1^4\rho^2)/2, \quad a_{33}^{s}=-3g_3^2\sigma^2\rho^2s^8. \end{gathered} \tag{A.4}$$

In (A.2)–(A.4) we use the following notation:

$$\begin{gathered} \mu=(\lambda-s_1^2)/s^2, \quad \lambda=E(s)/K(s), \quad \rho=(1-\lambda)/s^2, \\ \nu=[(2s^2-1)\lambda+s_1^2]/3s^2, \quad \chi=[(2-s^2)\lambda-2s_1^2]/3, \\ \sigma=[(1+s^2)\lambda-s_1^2]/3s^2, \quad s_1^2=1-s^2, \end{gathered} \tag{A.5}$$

where $K(s)$ and $E(s)$ are the complete elliptic integrals of the first and second kinds.

Notice that above, when using the elliptic Jacobi functions cn, dn, and sn, we assumed that their modulus s^2 does not fall outside the limits of the interval $0 \leqslant s^2 \leqslant 1$, and that the argument is real. By going beyond these assumptions, and using the real and imaginary Jacobi transformations, we can describe all the three indicated types of waves by some single function, obtain the dispersion equation for the wave characterized by this function, and then construct by means of redesignations effected according to the rules indicated below the dispersion equation for the other two types of waves.

According to Ref. 18, the real and imaginary Jacobi transformations connecting the functions of interest to us have the form

$$\mathrm{cn}(\zeta,\ s)=\mathrm{dn}(s\zeta,\ 1/s), \tag{A.6}$$

$$\mathrm{sn}(\zeta,\ s)=-(1/s)\mathrm{dn}[i(\zeta+C),\ s_1], \tag{A.7}$$

where $C=K(s_1)+iK(s)$. Taking account of these equations, we can express the functions φ_c and φ_s we have introduced [see (2.3) and (2.5)] in terms of φ_d [see (2.4)] as follows:

$$\varphi_c(\xi,\ s)=\varphi_d(\xi,\ 1/s), \tag{A.8}$$

$$\varphi_s(\xi,\ s)=-\varphi_d(i[\xi+C(1+s_1^2)^{1/2}],\ \ s_1). \tag{A.9}$$

It is found that all the changes that occur in the dispersion equation when we go over from the case of a wave of the type (2.4) to the case (2.3) amount to the replacement of the matrix of coefficients a_{ik}^{d} by the matrix a_{ik}^{c}, the replacement being effected according to the following rules:

$$s^2\to 1/s^2, \quad s_1^2\to -s_1^2/s^2, \quad g_2^2\to s^2g_1^2, \quad \lambda\to\mu, \quad \chi\to\nu/s^2. \tag{A.10}$$

It can be verified directly that the system of equations (A.3) goes over into (A.2) when such a replacement is made.

A similar transition from the a_{ik}^{d} to the a_{ik}^{s} can be accomplished according to the following rule. First, we must make the substitutions

$$s^2\to s_1^2, \quad s_1^2\to s^2, \quad g_2^2\to g_3^2, \quad \lambda\to s^2\rho, \quad \chi\to -s^2\sigma. \tag{A.11}$$

Second, we must change the signs of a_{12} and a_{13}. The latter step must be taken because of the fact that the equations (2.13), which correspond to a wave of the type (2.4), reduce to the analogous equations for a wave of the type (2.5) when we go over from a real to an imaginary argument ($\xi \to i\xi$), from real to imaginary wave numbers ($k\to ik$, $k_\perp \to ik_\perp$), and from ω to $-\omega$ ($\omega\to -\omega$).

APPENDIX B

Cnoidal waves for $s^2 \to 1$

In the limiting case $1-s^2 \ll 1$ (i.e., for $s_1^2 \ll 1$) the functions φ_c and φ_d [see (2.3) and (2.4)] describe a set of envelope solitons located sufficiently far apart at distances from each other of the order of

$$1/\lambda=\ln[4/(1-s^2)^{1/2}]. \tag{B.1}$$

Cnoidal waves of this type are also called a periodic lattice of very sparsely distributed solitons. As to the function φ_s (see (2.5)) it corresponds to a set of alternating-sign wave trains, with each train having a length of the order of $1/\lambda$ (a set of kinks). For $s^2 \to 1$ we have [see (A.5)]

$$\mu\to\lambda, \quad \nu\to\lambda/3, \quad \chi\to\lambda/3, \quad \rho\to 1, \quad \sigma\to 2\lambda/3. \tag{B.2}$$

In analyzing the stability of waves with $s^2 \to 1$ we find it also useful to represent the cnoidal waves in the form of a superposition of solitons (in the cases of cn and dn waves) or kinks (in the case of sn waves). This type of representation is characterized by the formulas (cf. Refs. 19 and 20)

$$\varphi_c=2^{1/2}s^{-1}\sigma_1\sum(-1)^n\,\mathrm{sech}\,\sigma_1(\xi-nh_1), \tag{B.3}$$

$$\varphi_d=2^{1/2}\sigma_2\sum\mathrm{sech}\,\sigma_2(\xi-nh_2), \tag{B.4}$$

$$\varphi_s=2^{1/2}s^{-1}\sigma_3\sum(-1)^n\,\mathrm{th}\,\sigma_3(\xi-nh_3), \tag{B.5}$$

where the summation is over all integral n ranging from $-\infty$ to $+\infty$,

$$\sigma_i=\pi g_i/2K(s_1),\quad h_i=2K(s)/g_i,\quad i=1,\ 2,\ 3, \tag{B.6}$$

the quantities g_i being given by the formulas (2.3)–(2.5). According to (B.3)–(B.5), the quantities h_1 and h_2 have the meaning of intersoliton separation; h_3, that of a kink length. In accordance with (B.6), for $s^2 \to 1$, these quantities are given by the relations

$$h_1=h_2=2/\lambda,\qquad h_3=2^{1/2}/\lambda, \tag{B.7}$$

where λ is given by the formula (B.1). Equations (B.1) and (B.7) allow us to express the parameter $1-s^2$ in terms of the intersoliton separation (or in terms of the kink length):

$$1-s^2=16(e^{-h_1},\ e^{-h_2},\ e^{-2^{1/2}h_3}). \tag{B.8}$$

These equalities were used in Sec. 4.

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