Disorder fields in two-dimensional conformal quantum-field theory and $N=2$ extended supersymmetry

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The spectrum of anomalous dimensions of disorder fields (corresponding to "charge symmetry") in models of two-dimensional conformal field theory, which describe critical points in Z_p-Ising models, is determined. It is shown that these models are related to the conformal theory with $N=2$ extended supersymmetry. Exact "minimal" solutions are constructed for this theory.

1. INTRODUCTION

A series of exact solutions in two-dimensional conformal quantum-field theory with cyclic global-symmetry groups Z_p $(p=2, 3, \ldots)$ was constructed in Ref. 1. The corresponding central charges in the Virasoro algebra[2] are given by

$$c_p=2(p-1)/(p+2), \quad p=2, 3, \ldots . \tag{1.1}$$

We shall use $[Z_p]$ to denote all these models. The $[Z_p]$ models contain the conformal "spin" fields σ_k, $k=1, 2, \ldots, p-1$ that transform in accordance with the representations

$$\Omega\sigma_k(x)=\omega^k\sigma_k(x) \tag{1.2}$$

of the group Z_p, where Ω is the generator of Z_p, $\Omega^p=1$, and

$$\omega=\exp(2\pi i/p). \tag{1.3}$$

Evidently these models describe "self-dual" critical points in the Z_p-invariant generalizations of the Ising model.[1]

The space of states constructed in Ref. 1 is generated by spin fields $\sigma_k(x)$ and "disorder fields" $\mu_k(x)$, $k=1, 2, \ldots, p-1$, corresponding to the elements Ω^k of the group Z_p. In reality, the $[Z_p]$ models are also invariant under "charge conjugation" C:

$$\sigma_k(x)\to\sigma_k^+(x), \tag{1.4}$$

where $\sigma_k^+(x)=\sigma_{p-k}(x)$. The global symmetry of a $[Z_p]$ model is thus seen to contain the dihedral group D_{2p} of order $2p$ (Ref. 3), generated by the elements (1.2) and (1.4). Hence, as explained in Section 2, the theory must contain disorder fields corresponding to odd elements of D_{2p}. We shall call them C-disorder fields. In Section 3, we construct the C-disorder fields for the $[Z_p]$ models, and determine the corresponding spectrum of anomalous dimensions.

It is possible to construct a supersymmetric generalization of the two-dimensional conformal quantum-field theory.[4–6] A conformal theory with $N=2$ extended supersymmetry is constructed in Ref. 7. We shall show in Section 4 that this theory is simply related to the $[Z_p]$ models. This enables us to construct an infinite series of exact solutions for $N=2$ superconfirmal field theory, described in Section 4. Moreover, it is possible to produce arguments showing that this series exhausts all the "minimal" solutions of the $N=2$ superconformal theory with reasonable properties.

2. DISORDER FIELDS IN TWO-DIMENSIONAL STATISTICAL PHYSICS AND FIELD THEORY

Special fields, called "disorder parameters," have been successfully used in the analysis of different two-dimensional statistical models (see Refs. 8 and 9). Although the concept of the disorder parameter (we shall use the phrase *disorder field*) is now standard, it will be useful to present here the definition and general properties of such fields. To be specific, we shall speak of a lattice system with nearest-neighbor interaction; an analogous definition can be introduced for other systems, too.

Consider a two-dimensional statistical system with a discrete global symmetry group G. To each site $\mathbf{x}=(x_1, x_2)$ of a square lattice L, we assign a spin variable $\sigma(\mathbf{x})$, which runs over the orbit of some (generally speaking, reducible) representation R of the group G. The pair Hamiltonian $H(\sigma,\sigma')=H(\sigma',\sigma)$ is G-invariant: $H(\sigma,\sigma')=H[R(g)\sigma, R(g)\sigma']$ for all $g\in G$. Of course, the complete distribution function

$$W(\{\sigma\})=Z^{-1}\exp\Big\{-\beta\sum_{\mathbf{x}\in L}\sum_{a=1,2}H(\sigma(\mathbf{x}),\sigma(\mathbf{x}+\mathbf{e}_a))\Big\}, \tag{2.1}$$

has the same property, where e_a are the basis vectors of L, $\beta=(kT)^{-1}$, and Z is the partition function. The correlation functions are defined in a standard fashion:

$$\langle\sigma(\mathbf{x}_1)\ldots\sigma(\mathbf{x}_N)\rangle=\sum_{\{\sigma\}}\sigma(\mathbf{x}_1)\ldots\sigma(\mathbf{x}_N)W(\{\sigma\}). \tag{2.2}$$

We now introduce the dual lattice $\tilde{L}$, whose sites $\tilde{\mathbf{x}}=(\tilde{x}_1), \tilde{x}_2)$ are the centers of the faces of the lattice L. We take an arbitrary, simple, directed contour γ on L, and make the following change of the summation variables in (2.2):

$$\begin{aligned}&\sigma(\mathbf{x})\to R(g)\sigma(\mathbf{x}), \quad \text{if} \quad \mathbf{x}\in\Lambda_\gamma,\\ &\sigma(\mathbf{x})\to\sigma(\mathbf{x}), \quad \text{if} \quad \mathbf{x}\in\overline{\Lambda}_\gamma,\end{aligned} \tag{2.3}$$

with arbitrary $g\in G$. In these expressions, Λ_γ $(\overline{\Lambda}_\gamma)$ represents the part of the lattice L that lies outside (inside) the contour γ. We then obtain

$$\begin{aligned}&\langle\sigma(\mathbf{x}_1)\ldots\sigma(\mathbf{x}_M)\sigma(\mathbf{x}_{M+1})\ldots\sigma(\mathbf{x}_N)\rangle\\ &=R_1(g)\ldots R_M(g)\langle\sigma(\mathbf{x}_1)\ldots\sigma(\mathbf{x}_M)\sigma(\mathbf{x}_{M+1})\ldots\sigma(\mathbf{x}_N)\varphi_g(\gamma)\rangle,\end{aligned} \tag{2.4}$$

where $R_i(g)$ acts on the field $\sigma(x_i)$. We have assumed that $\mathbf{x}_1, \ldots, \mathbf{x}_M\in\Lambda_\gamma$, $\mathbf{x}_{M+1}, \ldots, \mathbf{x}_N\in\overline{\Lambda}_\gamma$, and that the contour γ is clockwise, as shown in Fig. 1. The factor $\varphi_g(\gamma)$ in (2.4)

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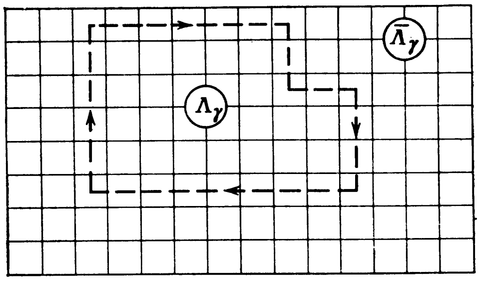

FIG. 1. Directed contour γ on a dual lattice.

arises from the transformation of the distribution function W under the replacements defined by (2.3). It is given by

$$\varphi_g(\gamma)=\prod_{\tilde{l}\in\gamma}\exp\{\beta H(\sigma(\mathbf{x}_l^*)\sigma(\mathbf{x}_l))-\beta H(\sigma(\mathbf{x}_l^*),R(g)\sigma(\mathbf{x}_l))\}. \qquad (2.5)$$

The product in this expression is evaluated over all the sides $\tilde{l}\in\tilde{L}$ comprising the contour γ. Each such side crosses a given side $l\in L$ that joins neighboring sides $\mathbf{x}_l^*$ and $\mathbf{x}_l\in L$, where $\mathbf{x}_l\in\Lambda_\gamma$, $\mathbf{x}_l^*\in\bar{\Lambda}_\gamma$.

We can associate a disorder field $\varphi_g(\gamma\tilde{\mathbf{x}})$ with each element $g\in G$. The quantity φ_g is given by a product such as (2.5), but now evaluated over some nonclosed simple contour $\gamma\tilde{\mathbf{x}}$ joining the fixed site $\mathbf{x}_0\in\tilde{L}$ to the site $\tilde{\mathbf{x}}\in L$. The choice of the point $\tilde{\mathbf{x}}_0$ is immaterial; we shall remove it to infinity: $\tilde{\mathbf{x}}_0=(-\infty,0)$. It is clear from (2.4) that the dependence of expectation values of the form

$$\langle\varphi_{g_1}(\gamma\tilde{\mathbf{x}}_1)\ldots\varphi_{g_M}(\gamma\tilde{\mathbf{x}}_M)\sigma(\mathbf{x}_{M+1})\ldots\sigma(\mathbf{x}_N)\rangle \qquad (2.6)$$

on the shape of the contours $\gamma_i\tilde{\mathbf{x}}_i$ is "weak." The correlation function (2.6) will change only under "nontrivial" deformations, say, of the contour $\gamma_k\tilde{\mathbf{x}}_k:\gamma_k\tilde{\mathbf{x}}_k\to\gamma_k'\tilde{\mathbf{x}}_k$, for which the closed contour $\delta\gamma_k=\gamma_k(\gamma_k')^{-1}$ surrounds any of the points x_i, $i=1,2,\ldots,M$ $(i\neq k)$ or x_j, $j=M+1,\ldots,N$. The transformation of (2.6) under these nontrivial deformations of the contours will be described below.

Henceforth, the scaling limit of the statistical system under consideration[10] will be implied. When long-range correlations are investigated, there is no point in distinguishing between the sites of L and $\tilde{L}$; the coordinates $\tilde{\mathbf{x}}_i$ and $\mathbf{x}_j$ can be regarded as continuous:[1)] $\tilde{\mathbf{x}}_i\to\mathbf{x}_i$, $\mathbf{x}_j\to\mathbf{x}_j$, $\mathbf{x}_j\in\mathbf{R}^2$. Because of the above weak dependence of (2.6) on the shape of the contours, we can discard the arguments γ_i in this expression and write (2.6) in the form

$$\langle\varphi_{g_1}(x_1)\ldots\varphi_{g_M}(x_M)\sigma(x_{M+1})\ldots\sigma(x_N)\rangle. \qquad (2.7)$$

However, we must then recall that (2.7) is a multivalued function of the variables x_i, $x_j\in\mathbf{R}^2$ with a particular monodromy group.

Expressions such as (2.7) must now be given an unambiguous meaning. For the moment, we introduce the common notation $\Phi_a(x)$ for the fields $\varphi_g(x)$, $\sigma(x)$. Consider the case where the N points x_i, $i=1,2,\ldots,N$ lie on the same straight line of "constant time:" $x_i=(\eta_i,\tau)$, where $\eta_1<\eta_2<\ldots<\eta_N$. We shall adopt the convention that, with this choice of x_i, the expression

$$\langle\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\ldots\Phi_{a_N}(x_N)\rangle \qquad (2.8)$$

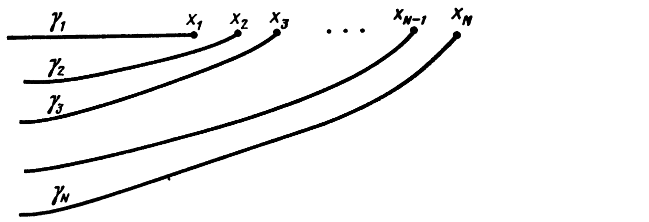

FIG. 2. Mutual disposition of the contours $\gamma_i x_i$ corresponding to the correlation function (2.8).

will correspond to the disposition of the contours $\gamma_i x_i$ shown in Fig. 2. Here, we have associated the fictitious contours $\gamma_k x_k$ with the spin field $\sigma(x_k)$ as well [this may be viewed as the replacement $\sigma(x)\to\sigma(\gamma x)\equiv\varphi_e(\gamma x)\sigma(x)$, where $e\in G$ is the unit element of G, which can also be an identity since $\varphi_e\equiv I$ is the unit operator]. When the disposition of the points $x_i\in\mathbf{R}^2$ is arbitrary, the expression given by (2.8) is taken in the sense of continuous continuation.[2)]

The fields φ_g, σ are not, in general, mutually local, and the corresponding symbols $\varphi_g(x)$, $\sigma(x)$ do not commute under the expectation value symbol. The following "single-time" commutation relations are readily evaluated:

$$\begin{aligned}&\varphi_g(\eta_1,\tau)\sigma(\eta_2,\tau)=\sigma(\eta_2,\tau)\varphi_g(\eta_1,\tau),\ \eta_1<\eta_2,\\&\varphi_g(\eta_1,\tau)\sigma(\eta_2,\tau)=R(g)\sigma(\eta_2,\tau)\varphi_g(\eta_1,\tau),\ \eta_2>\eta_1\end{aligned} \qquad (2.9)$$

and

$$\begin{aligned}&\varphi_{g_1}(\eta_1,\tau)\varphi_{g_2}(\eta_2,\tau)=\varphi_{g_2}(\eta_2,\tau)\varphi_{g_2^{-1}g_1g_2}(\eta_1,\tau),\quad\eta_1<\eta_2,\\&\varphi_{g_1}(\eta_1,\tau)\varphi_{g_2}(\eta_2,\tau)=\varphi_{g_1g_2g_1^{-1}}(\eta_2,\tau)\varphi_{g_1}(\eta_1,\tau),\quad\eta_1>\eta_2.\end{aligned} \qquad (2.10)$$

The "bypassing relations," determining the monodromy properties of the correlators (2.8), are particularly important. We shall write out these relations, using the symbol $\Phi_a^*\Phi_b$ to represent the continuous continuation of the correlation function $\langle X\Phi_a(x_1)\Phi_b(x_2)Y\rangle$ (X, Y are any products of the fields Φ_a) in the variable x_1 along a closed contour, taken clockwise once around the point x_2 (Fig. 3):

$$\begin{aligned}&\varphi_g*\sigma=\varphi_gR(g)\sigma,\quad\sigma*\sigma=\sigma\sigma,\\&\varphi_{g_1}*\varphi_{g_2}=\varphi_{(g_1g_2)g_1(g_1g_2)^{-1}}\varphi_{g_1g_2g_1^{-1}}.\end{aligned} \qquad (2.11)$$

We note further that the correlation functions (2.7) are nonzero only when $g_1g_2\ldots g_M=e$. Under the global symmetry transformations of G, the disorder fields φ_g transform in accordance with the rule

$$\sigma(x)\to R(h)\sigma(x),\quad\varphi_g(x)\to\varphi_{h^{-1}gh}(x). \qquad (2.12)$$

The scaling limit of the statistical system implies the existence of a critical point.[1] We shall refer to it as the G-critical point if the correlation length diverges at this point for all fields φ_g, $g\in G$. The theory then contains the conformal fields $\varphi_g(x)$. The total (closed with respect to the operator algebra[2]) space of fields $\{F\}$, describing long-range cor-

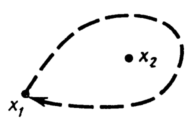

FIG. 3. Contour participating in the definition of the symbol $\Phi_a^*\Phi_b$ in (2.11) and below.

relations at the G-critical point, can be naturally expanded into the sum of subspaces

$$\{F\}=\bigoplus_{g\in G}\{\varphi_g\}, \tag{2.13}$$

where $\{\varphi_g\}$ contains all the fields arising under all the possible "compositions" of the field φ_g and the spin field σ. It is clear that the operator algebra[2] satisfies the following "selection rules":

$$\{\varphi_{g_1}\}\{\varphi_{g_2}\}\in\{\varphi_{g_1g_2}\}, \tag{2.14}$$

where the left-hand side represents the product (at different points) of any fields from the corresponding subspaces. It is, in principle, possible to give a detailed classification (in accordance with the representations of the group G) of the mutual localization properties of the fields in each of the subspaces $\{\varphi_g\}$ (this is done for a special case in Section 3. Here, we only emphasize that all the fields in $\{\varphi_e\}$ are mutually local.

We note one further property of the space $\{F\}$ of the conformal field theory describing the G-critical point. If the group G contains a cyclic subgroup Z_p (generated, say, by the element $g_1\in G$, $g_1^p=e$), the subspaces $\{\varphi g_1^k\}$, $k=1, 2,\dots,p-1$ will include (nonlocal) fields with spins that are multiples of $1/p$. This property is readily derived from the above relationships, and is useful in the analysis of global symmetries in models of conformal field theory.

3. C-DISORDER FIELDS IN MODELS $[Z_p]$

A series of models of two-dimensional conformal field theory was constructed in Ref. 1. These models will be denoted by $Z, p=2, 3, \dots$. As already noted in the Introduction, the consequence of "charge invariance" (1.4) is that a model $[Z_p]$ actually has a more extensive global symmetry group D_{2p}, i.e., the dihedral group of order $2p$ (Ref. 3). (From now on it is assumed that $p\geqslant 3$.) Let $\Omega_k=\Omega^k$, $k=0, 1,\dots,p-1$; $\Omega^p=E$[3)] denote the elements of the cyclic subgroup $Z_p\in D_{2p}$, and let R_k, $k=0, 1,\dots,p-1$ denote the "odd" elements of D_{2p}, where $R_k=C\Omega^k$. In terms of this notation, the multiplication table for D_{2p} is

$$\begin{aligned}&\Omega_k\Omega_l=\Omega_{k+l},\quad R_k\Omega_l=R_{k+l},\\ &R_kR_l=\Omega_{l-k},\quad \Omega_kR_l=R_{l-k}.\end{aligned} \tag{3.1}$$

A model $[Z_p]$ contains conformal spin field $\sigma_k(x)$, $k=1, 2,\dots, p-1$ (Ref. 1), where each pair $(\sigma_k,\sigma_k^+)(\sigma_k^+\equiv\sigma_{p-k})$ forms the basis of the two-dimensional representation of D_{2p}:

$$\begin{aligned}&\Omega_k\sigma_l=\omega^{kl}\sigma_l,\quad \Omega_k\sigma_l^+=\omega^{-kl}\sigma_l^+,\\ &R_k\sigma_l=\omega^{kl}\sigma_l^+,\quad R_k\sigma_l^+=\omega^{-kl}\sigma_l,\end{aligned} \tag{3.2}$$

where ω is given by (1.3).

There are also $p-1$ disorder fields μ_k, $k=1, 2,\dots, p-1$, corresponding to the elements Ω_k of the symmetry group (3.1). According to (2.11), the fields μ_k are mutually local (this is due to the commutative property of the group z_p), but they are nonlocal with respect to σ_k. The bypassing relations

$$\mu_k*\sigma_l=\omega^{kl}\mu_k\sigma_l \tag{3.3}$$

are a consequence of (3.2). The transformations Ω_k act on the μ_l in a trivial manner and, in the case of charge conjugation $\mathbf{C}$

$$\mu_k\to\mu_k^+, \tag{3.4}$$

where $\mu_k^+\equiv\mu_{p-k}$. The anomalous dimensions of the fields[4)] and μ_k are found in Ref. 1, where the space of the fields $\{F_\mu\}$, which are closed with respect to the operator algebra, is also constructed.

In the notation of (2.13), this space is given by the sum

$$\{F_\mu\}=\bigoplus_{k=0}^{p-1}\{\mu_k\}$$

(we are assuming that $\mu_0=I$). Moreover, in accordance with the arguments put forward in Section 2, $\mathbf{C}$-disorder fields Φ_k, $k=0, 1,\dots,p-1$ should be present in $[Z_p]$, where these fields correspond to the odd elements R_k of the group D_{2p}. These fields will be constructed below, and we shall thereby describe the complete space of fields

$$\{F\}=\{F_\mu\}\oplus\{\Phi\}=\bigoplus_{k=0}^{p-1}\{\mu_k\}\bigoplus_{k=0}^{p-1}\{\Phi_k\} \tag{3.5}$$

for the $[Z_p]$ model. We shall use φ_k to denote any fields in the subspaces $\{\Phi_k\}$, where φ_k, $k=0, 1,\dots,p-1$ form p-component multiplets with identical conformal dimensions $(\Delta,\bar\Delta)$. By virtue of (3.1) and (3.2), the fields φ_k are nonlocal with respect to σ_k,μ_k; the corresponding bypassing relations are

$$\varphi_k*\sigma_l=\omega^{kl}\varphi_k\sigma_l^+,\quad \varphi_k*\mu_l=\varphi_{k+2l}\mu_l^+. \tag{3.6}$$

A $[Z_p]$ model has infinite-dimensional symmetry described by the algebra of "para-Fermion" currents $\psi_k(z)$, $\bar\psi_k(\bar z)$, $k=1, 2,\dots,p-1$ (Ref. 1), where a and $\bar z$ are complex coordinates in $\mathbf{R}^2$:

$$z=x_1+ix_2,\quad \bar z=x_1-ix_2. \tag{3.7}$$

The fields $\psi_k(z)$ and $\bar\psi_k(\bar z)$ are looked upon as the products (suitably regularized—see Ref. 1)

$$\psi_k\infty\mu_k\sigma_k,\quad \bar\psi_k\infty\mu_k\sigma_k^+ \tag{3.8}$$

and have conformal dimensions $(\Delta_k,0)$ and $(0,\Delta_k)$, where

$$\Delta_k=k(p-k)/p \tag{3.9}$$

are equal to the spins of these fields. The current algebra $\psi_k(z)$ is determined by the operator expansions[1]

$$\psi_{k_1}(z_1)\psi_{k_2}(z_2)=c_{k_1,k_2}(z_{12})^{-2k_1k_2/p}[\psi_{k_1+k_2}(z_2)+O(z_{12})], \tag{3.10a}$$

$$\begin{aligned}&\psi_k(z_1)\psi_k^+(z_2)\\ &=(z_{12})^{-2k(p-k)/p}\{I+[k(p-k)(p+2)/p(p-1)]\\ &\qquad\times z_{12}^2T(z_2)+O(z_{12}^3)\},\end{aligned} \tag{3.10b}$$

where $z_{12}=z_1-z_2$, $\psi_k^+=\psi_{p-k}$, $\psi_0\equiv I$, $T(z)$ is the corresponding component of the energy-momentum tensor, the "structure constants" are

$$c_{k_1,k_2}^2=\frac{(k_1+k_2)!(p-k_1)!(p-k_2)!}{k_1!k_2!(p-k.-k_2)!p!} \tag{3.11}$$

and the sum k_1+k_2 in (3.10) and (3.11) is understood to be taken over the modulus of p. Here and below, we concentrate our attention on the field $\psi_k(z)$, bearing in mind the fact that the analogous relationships are valid for the current $\bar\psi_k(\bar z)$. The bypassing relations follow from (3.8):

$$\psi_k * \varphi_l = \omega^{kl} \psi_k^+ \varphi_{l-2k}, \quad \psi_k^+ * \varphi_l = \omega^{-kl} \psi_k \varphi_{l+2k}. \tag{3.12}$$

The double bypassing relation

$$\psi_k \mathop{*}\limits^{*} \varphi_l = \omega^{-2k(p-k)} \psi_k \varphi_l, \tag{3.13}$$

which follows from (3.12), shows that the product $z^{\Delta_k} \psi_k(z) \varphi_l(0,0)$ is a two-valued analytic function of z with a root branch point $z = 0$. This enables us to write the corresponding operator expansion in the form

$$\psi_k(z)\varphi_l(0,0) = z^{-\Delta_k} \sum_{n=-\infty}^{\infty} z^{n/2} A_{-n/2}^{(k)} \varphi_l(0,0), \tag{3.14}$$

where $A^{(k}_{-n)/2\varphi_l} \in \{\Phi_{l-k}\}$ are certain new C-disorder fields. The formula given by (3.14) is the definition of the operators $A^{(k)}_{-n/2}$, $n = 0, \pm 1, \pm 2, \ldots$, acting in the space

$$\{\Phi\} = \mathop{\oplus}_{k=0}^{p-1} \{\Phi_k\}.$$

By virtue of (3.12), we can also write the expansions

$$\psi_k^+(z)\varphi_l(0,0) = z^{-\Delta_k} \sum_{n=-\infty}^{\infty} z^{n/2} e^{i\pi n} U_{ll'}^{(-k)} A_{-n/2}^{(k)} \varphi_{l'}(0,0), \tag{3.15}$$

which show that the operators $A^{(k)}_{n/2}$ satisfy the relations

$$A_{n/2}^{(p-k)} \varphi_l = e^{i\pi n} U_{ll'}^{(-k)} A_{n/2}^{(k)} \varphi_{l'}. \tag{3.16}$$

where $U^{(k)}$ are the unitary matrices $U^{(k)}_{ll'} = \omega^{-k^2+kl} \delta_{l-2k,l'}$ which can be written as the powers

$$U^{(k)} = U^k, \quad U_{ll'} \equiv U_{ll'}^{(1)} = \omega^{l-1} \delta_{l-2,l'}, \quad U^p = E. \tag{3.17}$$

By analogy with (3.14), we can define the operators $A^{(k)}_{n/2}$ in terms of contour integrals:

$$A_{n/2}^{(k)} \varphi_l(0,0) = \oint_\gamma \frac{dz}{2\pi i} z^{\Delta_k - 1 + n/2} \psi_k(z) \varphi_l(0,0), \tag{3.18}$$

where the symbol $\oint_\gamma$ represents integration over the closed contour γ, taken twice around the point $z = 0$.

We must now find the commutation relations for the operators $A^{(k)}_{n/2}$. We shall confine our attention to those relations that will be useful below. Consider the double integral

$$\oint_{\gamma_1} \oint_{\gamma} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} (z_1^{1/2} - z_2^{1/2})^{2k/p-1} \times (z_1^{1/2} + z_2^{1/2})^{1-2k/p} z_1^{\Delta_k - 1 + n/2} z_2^{\Delta_k - 1 + m/2} \times \psi_k(z_1) \psi_1(z_2) \varphi_l(0,0). \tag{3.19}$$

The first two factors in (3.19) ensure that the integrand is single-valued on the two-sheet covering of the complex plane of the variable z_1 (and z_2) with the branch point $z_1 = 0$ (correspondingly, $z_2 = 0$), so that the integration contours shown in Fig. 4 are actually closed. The multiple integral (3.19) can be reduced to repeated integrals in two different ways. In the first case, (3.18) is used to evaluate first the integral with respect to z_2 and then, similarly, the integral with respect to z_1. In the second method, we evaluate these integrals in reverse order. To do this, we must first deform the contour γ_1 so that it lies inside γ_2 of Fig. 4, and we must take into account the contribution of the poles $z_1 = z_2$ on the first and second sheets of the integrand in (3.19). The residues at these poles are determined by the expansions (3.10). By equating the results obtained by these two methods of integration, we obtain the required relation

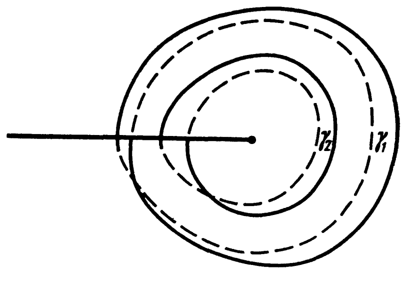

FIG. 4. Contours of integration γ_1 and γ_2 in (3.19).

$$\sum_{q=0}^{\infty} D_{(-\alpha_k,\alpha_k)}^{(q)} [A_{(n-q)/2}^{(k)} A_{(m+q)/2}^{(1)} + A_{(m-q)/2}^{(1)} A_{(n+q)/2}^{(k)}] \varphi_l = \frac{1}{2} (2^{2\alpha_k} c_{k,1} A_{(n+m)/2}^{(k+1)} \delta_{l,l'} + 2^{-2\alpha_k} c_{k-1,1} e^{i\pi m} A_{(n+m)/2}^{(k-1)} U_{ll'}^{(-1)}) \varphi_{l'}, \tag{3.20}$$

where the c_{k_1,k_2} are given by (3.11)

$$\alpha_k = 1 - 2k/p, \tag{3.21}$$

and the constants $D^{(q)}_{(\alpha,\beta)}$ are the coefficients of the power expansion

$$(1-x)^\alpha (1+x)^\beta = \sum_{q=0}^{\infty} D_{(\alpha,\beta)}^{(q)} x^q. \tag{3.22}$$

The other relation that will be useful to us below is obtained considering the integral

$$\oint_{\gamma_1} \oint_{\gamma_2} \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} (z_1^{1/2} - z_2^{1/2})^{1+2/p} \times (z_1^{1/2} + z_2^{1/2})^{-1-2p} z_1^{\Delta_1 - 1 + n/2} z_2^{\Delta_1 - 1 + m/2} \times \psi_1(z_1) \psi_1(z_2) \varphi_l(0,0) \tag{3.23}$$

by the same method, and has the form[5)]

$$\sum_{q=0}^{\infty} D_{(\alpha,-\alpha)}^{(q)} [A_{(n-q)/2}^{(1)} A_{(m+q)/2}^{(1)} + A_{(m-q)/2}^{(1)} A_{(n+q)/2}^{(1)}] \varphi_l = 2^{2\alpha-1} e^{i\pi n} \cos^2\left(\frac{\pi(n+m)}{2}\right) \times \left\{ \frac{p+2}{p} L_{(n+m)/2} + \varkappa_p(n) \delta_{(n+m)/2,0} \right\} U_{ll'}^{(-1)} \varphi_{l'}, \tag{3.24}$$

where $\alpha = 1 + 2/p$,

$$\varkappa_p(n) = n^2/8 - (p-2)/16p, \tag{3.25}$$

and the L_n are the generators of the conformal transformations forming the Virasoro algebra:

$$[L_n, L_m] = (n-m) L_{n+m} + [(p-1)/16(p+2)] (n^3 - n) \delta_{n+m,0}. \tag{3.26}$$

The C-disorder fields φ_l in $\{F\}$ include "invariant fields" Φ_l satisfying the equations

$$A^{(k)}_{n/2}\,\Phi_l=0 \quad \text{for} \quad n>0. \tag{3.27}$$

As usual (see Refs. 1 and 2), the existence of these invariant fields follows from the condition for a lower bound on the spectrum of anomalous dimensions of the theory. The operators $A_0^{(k)}$ act on the field Φ_l as $(p\times p)$ matrices:

$$A_0^{(k)}\,\Phi_l=A^{(k)}_{ll'}\,\Phi_{l'}, \tag{3.28}$$

where $A^0_{ll'}=\delta_{ll'}$, since $\psi_0=1$. In general, the theory may contain a number of such invariant fields (we shall see below that this is actually the case for $p\geqslant 3$). When this is so, we shall denote them by $\Phi_l^{(s)}$ and the corresponding dimensions by $(\Delta_{(s)}, \Delta_{(s)})$.

The anomalous dimensions of the fields $\Phi^{(s)}$ in the models $[Z_p]$ can be determined with the aid of (3.20) and (3.24). Consider, to begin with, (3.20) with $n=m=0$, $\varphi=\Phi$. Using (3.27), this relation reduces to

$$2[A^{(k)}A^{(1)}+A^{(1)}A^{(k)}]\Phi$$
$$=[2^{2\alpha_k}c_{k,1}A^{(k+1)}+2^{-2\alpha_k}c_{k-1,1}U^{(-1)}A^{(k-1)}]\Phi, \tag{3.29}$$

where $U^{(+1)}$ is given by (3.16) and $A^{(k)}$ are the matrices (3.28). Equations (3.29) must be looked upon as a set of equations for the matrix $A^{(k)}$. The solution of this system can be written in the form

$$A^{(k)}=2^{-2k(p-k)/p}\left[\frac{k!\,(p-k)!}{p!}\right]^{1/2}y_kB^k, \tag{3.30}$$

where the numbers y_k must satisfy the equations

$$y_1y_k=(k+1)\,y_{k+1}+(p-k+1)\,y_{k-1}, \tag{3.31}$$

and the matrix B is given by

$$B^2=U^{(-1)} \tag{3.32}$$

whose explicit form will not be necessary for our purposes. Because of (3.16), the above numbers satisfy the condition

$$y_k^2=y_{p-k}^2. \tag{3.33}$$

If we now introduce the generating function

$$f(x)=\sum_{n=0}^{p}y_kx^k \tag{3.34}$$

we find that (3.31) reduces to the differential equation which has the elementary integral

$$f(x)=(1+x)^{(p+y_1)/2}(1-x)^{(p-y_1)/2}. \tag{3.35}$$

It is thus clear that (3.31) has solutions satisfying (3.33), provided

$$y_1=p-2s, \quad s=0,\,1,\,2,\ldots\leqslant p/2. \tag{3.36}$$

Consider now (3.24). Substituting $\varphi=\Phi$ and $n=m=0$, we obtain

$$\{4A^{(1)}A^{(1)}-2^{2\alpha}[((p+2)/p)\Delta-\varkappa(0)]U^{-1}\}\Phi=0, \tag{3.37}$$

where Δ is the dimension of the field Φ and $\varkappa$ is given by (3.25). Substituting (3.30) and (3.36), we obtain the spectrum of dimensions of the invariant C-disorder fields $\Phi^{(s)}$ in the $[Z_p]$ model:

$$\Delta_{(s)}=[p-2+(p-2s)^2]/16(p+2), \quad s=0,\,1,\,2,\ldots\leqslant p/2. \tag{3.38}$$

Let $[\Phi^{(s)}]_A$ represent the space of fields spanning all the possible independent monomials of the form

$$A^{(k_1)}_{-n_1/2}\ldots A^{(k_M)}_{-n_M/2}\Phi^{(s)}, \quad n_1,\ldots,n_M>0. \tag{3.39}$$

The space $[\Phi^{(s)}]_A$ corresponds to an irreducible representation of the algebra (3.20), (3.24). The fields comprising this space have dimensions of the form $\Delta_{(s)}+N/2$, $N=0$, $1, 2, \ldots$. The space of C-disorder fields in the $[Z_p]$ model ($p\leqslant 3$) will be written in the form

$$\{\Phi\}=\bigoplus_{s=0}^{[p/2]}[\Phi^{(s)}]_A, \tag{3.40}$$

where $[p/2]$ represents the integral part of $p/2$.

As an example, consider the $[Z_3]$ model. It is well-known[1] that this model describes the critical point of the Potts three-position model,[1] and is equivalent to a definite "minimal model" of the conformal field theory (see Ref. 2). Figure 5 shows the "table of dimensions"[2] for this minimal model. All fields corresponding to the cells in this table are classified in accordance with the representations of the algebra of the para-Fermion currents (3.16). This classification is described in Ref. 1 for the even sector. The odd sector (unshaded cells in Fig. 5) contains the fields $\Phi^{(0)}$, $\Phi^{(1)}$ ($\Delta_{(0)}=1/8$, $\Delta_{(1)}=1/40$), and their immediate "descendants" (3.39) $A_{-3/2}\Phi^{(0)}$ and $A_{-1/2}\Phi^{(1)}$ with dimensions $\Delta_{(0)}+3/2=13/8$ and $\Delta_1+1/2=21/40$ (it can be shown that $A_{-1/2}\Phi^{(0)}=0$).

4. THE Z_p MODELS AND CONFORMAL FIELD THEORY WITH $N=2$ EXTENDED SUPERSYMMETRY

A two-dimensional conformal field theory with $N=2$ extended supersymmetry is proposed was Ref. 8. The finite-dimensional extended superconformal symmetry is generated in this theory by four local currents:[6)] $T(z), S(z), S^+(z)$, $J(z)$. The Fermion currents $S(z)$ and $S^+(z)$ have spin 3/2 (i.e., conformal dimensions 3/2, 0) and the boson currents $J(z)$ and $T(z)$ have spins 1 and 2, respectively. The field $T(z)$ is equal to the corresponding component of the energy-momentum tensor of the theory. The generator algebra is determined by singular terms of the operator expansions

3	13/8	2/3	1/8	0
7/5	21/40	1/15	1/40	2/5
2/5	1/40	1/15	21/40	7/5
0	1/8	2/3	13/8	3

FIG. 5. Table of dimensions in the minimal model describing the critical point of the three-position Potts model[11]. Shaded (unshaded) cells correspond to even (odd) sector.

$$T(z_1)T(z_2)={}^1/_2\tilde{c}(z_{12})^{-4}+2z_{12}^{-2}T(z_2)+z_{12}^{-1}T'(z_2)+O(1), \tag{4.1a}$$

$$T(z_1)J(z_2)=z_{12}^{-2}J(z_2)+z_{12}^{-1}J'(z_2)+O(1), \tag{4.1b}$$

$$T(z_1)S^{(a)}(z_2)={}^3/_2z_{12}^{-2}S^{(a)}(z_2)+z_{12}^{-1}S^{(a)\prime}(z_2)+O(1), \tag{4.1c}$$

$$J(z_1)J(z_2)=(\tilde{c}/12)z_{12}^{-2}+O(1),\quad S^{(a)}(z_1)S^{(a)}(z_2)=O(z_{12}), \tag{4.1d}$$

$$J(z_1)S^{(a)}(z_2)={}^1/_2az_{12}^{-1}S^{(a)}(z_2)+O(1), \tag{4.1e}$$

$$S(z_1)S^+(z_2)={}^2/_3\tilde{c}(z_{12})^{-3}+2(z_{12})^{-2}J(z_2)+z_{12}^{-1}[J'(z_2)+2T(z_2)]+O(1), \tag{4.1f}$$

where $z_{12}=z_1-z_2$, the primes represent differentiation with respect to z_2, $a=\pm1$, $S^{(+1)}=S$, $S^{(-1)}=S^+$, and $O(1)$ and $O(z_{12})$ represent regular parts of the expansions of the indicated order as $z_{12}\to0$. These terms are completely determined by the singular terms written out in (4.1) (see Ref. 13). The numerical parater $\tilde{c}$ in (4.1) is equal to the central charge of the corresponding Virasoro algebra (4.1a). It follows from the structure of the operator expansions (4.1) that the generator algebra of the $N=2$ extended superconformal symmetry has two obvious automorphisms (apart from the trivial one)

$$R_{\rm I}:\quad S\to-S,\quad S^+\to-S^+, \tag{4.2a}$$

$$R_{\rm II}:\quad S\leftrightarrow S^+,\quad J\to-J. \tag{4.2b}$$

Of course, we also have the automorphism $R'_{\rm II}=R_{\rm I}R_{\rm II}$, which is a combination of (4.2a) and (4.2b). Hence, it may be expected that the field theory with the symmetry (4.1) contains fields [and associated representations of the algebra (4.1)] of the following four types:

(a) Neveu-Schwarz fields ϕ (the corresponding sector of the space of fields will be denoted by $\{NS\}$ are local with respect to all three currents S, S^+, J.[7] The following operator expansions are valid:

$$S^{(a)}(z)\Phi(0,0)=\sum_{n=-\infty}^{\infty}z^{n-1}S^{(a)}_{-n-1/2}\phi(0,0), \tag{4.3a}$$

$$J(z)\phi(0,0)=\sum_{n=-\infty}^{\infty}z^{n-1}J_{-n}\Phi(0,0), \tag{4.3b}$$

where $\phi\in\{NS\}$. They define the operators $S^{(a)}_{n+1/2}$ and J_n, $n=0,\pm1,\pm2,\ldots$, acting in the space $\{NS\}$. These operators satisfy the commutation relations given in Ref. 8. The invariant fields $\Phi\in\{NS\}$ satisfy the equations

$$S^{(a)}_{n+1/2}\Phi=J_{n+1}\Phi=0\quad\text{for}\quad n\geqslant0 \tag{4.4}$$

and can be classified in accordance with the charge Q:

$$J_0\Phi^{(Q)}=Q\Phi^{(Q)}. \tag{4.5}$$

According to (4.1e), the fields $S^{(a)}(z)$ have the charge $Q=a/2$. The fields Φ, $S^{(a)}_{-1/2}\Phi$, $S_{-1/2}S^{\pm}_{-1/2}\Phi$ form a supermultiplet and can be combined into a "superfield," as described in Ref. 8.

(b) Ramond fields I. These are denoted by $\chi\in\{R_{\rm I}\}$ and are local with respect to the current $J(z)$, but not local with respect to S and S^+. In this case, we have the bypassing relations (see Section 2)

$$S*\chi=-S\chi,\quad S^+*\chi=-S^+\chi. \tag{4.6}$$

Operator expansions of the form (4.3b) are valid for fields $\chi\in\{R_{\rm I}\}$ but, instead of (4.3a), in this case we have

$$S^{(a)}(z)\chi(0,0)=\sum_{n=-\infty}^{\infty}z^{n-3/2}S^{(a)}_{-n}\chi(0,0), \tag{4.7}$$

where the operators $S^{(a)}_n$, $n=0,\pm1,\pm2,\ldots$ satisfy the commutation relations

$$\{S_n,S_m\}=\{S_n^+,S_m^+\}=0$$
$$\{S_n,S_m^+\}=2(n-m)J_{n+m}+2L_{n+m}+{}^1/_3\tilde{c}(n^2-{}^1/_4)\delta_{n+m,0}, \tag{4.8}$$

where L_n are the generators of the Virasoro algebra and $\{\ldots\}$ represents an anticommutator. The invariant fields $X\in\{R_{\rm I}\}$ satisfy the relations

$$S_n^{(a)}X=I_nX=0\quad\text{for}\quad n>0 \tag{4.9}$$

and, as in case (a), can be classified in accordance with the charge:

$$J_0X^{(Q)}=QX^{(Q)}.$$

(c) Ramond fields II. It is convenient to combine fields corresponding to the automorphisms $R_{\rm II}$ and $R'_{\rm II}$ into the single space $\{R_{\rm II}\}$, labeling them only with the index $\varepsilon=\pm1$. The fields $\eta\in\{R_{\rm II}\}$ are nonlocal with respect to the currents S, S^+, J, but satisfy the bypassing relations

$$S*\eta=\varepsilon S^+\eta,\quad S^+*\eta=\varepsilon S\eta,\quad J*\eta=-J\eta. \tag{4.10}$$

The operator expansions replacing (4.3), in this case have the form

$$J(z)\eta(0,0)=\sum_{n=-\infty}^{\infty}z^{n-1/2}J_{-n-1/2}\eta(0,0), \tag{4.11a}$$

$$S^{(a)}(z)\eta(0,0)=\sum_{n=-\infty}^{\infty}z^{n/2-3/2}S^{(a)}_{-n/2}\eta(0,0), \tag{4.11b}$$

where $S^+_{n/2}=\varepsilon(-)^nS_{n/2}$. The operators $J_{n+1/2}$, $S_{n/2}$, acting in the space $\{R_{\rm II}\}$, satisfy the commutation relations

$$[J_{n+1/2},J_{m-1/2}]=(\tilde{c}/12)(n+{}^1/_2)\delta_{n+m,0}, \tag{4.12}$$

$$\{S_{n/2},S_{m/2}^+\}=(n-m)\sin^2[\pi(n-m)/2]J_{(n+m)/2}+2\cos^2[\pi(n-m)/2][L_{(n+m)/2}+(\tilde{c}/24)(n^2-1)\delta_{n+m,0}]. \tag{4.13}$$

The invariant fields $Y\in\{R_{\rm II}\}$ obey the equations

$$S_{n/2}Y=J_{n-1/2}Y=0\ \text{ for }\ n>0, \tag{4.14}$$

where $\varepsilon S_0^2Y=(2\Delta-\tilde{c}/12)Y_\varepsilon$ and Δ is the dimension of the field Y_ε

We shall show below that there is a series of minimal solutions in $N=2$ superconformal field theory that corresponds to the values

$$\tilde{c}_p=3p/(p+2) \tag{4.15}$$

($p=2,3,4,\ldots$) of the parameter $\tilde{c}$, and we shall calculate the corresponding spectra of anomalous dimensions of the invariant fields Φ, X, Y. Models in this series are simply related to the $[Z_p]$ model described in Section 3.

We now introduce a free massless Bose field

$\tilde{\varphi}(z,\bar{z}) = \varphi(z) + \tilde{\varphi}(\bar{z})$, defined by the two-point functions

$$\langle\varphi(z)\varphi(0)\rangle=-2\lg z,\quad \langle\tilde{\varphi}(\bar{z})\tilde{\varphi}(0)\rangle=-2\lg\bar{z},$$
$$\langle\varphi(z)\tilde{\varphi}(0)\rangle=0 \tag{4.16}$$

(multipoint functions are calculated in accordance with Wick's rule).

The component

$$T_0(z)=-{}^1/_4:\partial_z\varphi\partial_z\varphi:(z) \tag{4.17}$$

of the energy-momentum tensor of this free theory forms a Virasoro algebra with central charge $c = 1$. Consider the "composite" theory, containing the $[Z_p]$ fields with certain $p = 2, 3, \ldots$, and the above field $\tilde{\varphi}$ (4.16), which does not interact with $[Z_p]$. We shall use $T_p(z)$ to denote the energy-momentum tensor in a $[Z_p]$ model, and c_p to represent the corresponding value (1.1) of the central charge. The energy-momentum tensor of the composite theory is given by the sum

$$T(z)=T_p(z)+T_0(z) \tag{4.18}$$

and, of course, satisfies the operator expansion (4.1a) with $\tilde{c}$ given by (4.15), since $\tilde{c}_p = c_p + 1$. We also introduce the fields

$$J(z)=(\tilde{c}_p/24)^{1/2}\partial_z\varphi(z), \tag{4.19a}$$

$$S(z)=(2\tilde{c}_p/3)^{1/2}\psi_1(z):\exp\{i\beta_p\varphi(z)\}:, \tag{4.19b}$$

$$S^+(z)=(2\tilde{c}_p/3)^{1/2}\psi_1^+(z):\exp\{-i\beta_p\varphi(z)\}:, \tag{4.19c}$$

where $\psi_1(z)$ and $\psi_1^+(z)$ are the corresponding para-Fermion currents in a $[Z_p]$ model (see Section 3) with spins $\Delta_1 = 1 - 1/p$, and the index

$$\beta_p=[(p+2)/2p]^{1/2} \tag{4.20}$$

in (4.19b) and (4.19c) is chosen so that the fields $S(z)$ and $S^+(z)$ have spins 3/2. It is readily verified [using (3.10)] that the fields (4.18) and (4.19) satisfy the operator expansions (4.1) with $\tilde{c}$ given by (4.15). These composite theories are thus seen to have the symmetry (4.1) for each $p = 2, 3, 4, \ldots$, i.e., they are models of $N = 2$ superconformal field theory.

If we know the structure of the space (3.5) of a $[Z_p]$ model, described in Ref. 1 and in Section 3, we can readily construct invariant fields in the superconformal theory. Without going into details, we now reproduce expressions for the anomalous dimensions [and charges (4.5)] of the invariant fields in all sectors.

(a) The charges (4.5) of the invariant field in the space $\{NS\}$ are given by

$$Q_q=q/2(p+2),\quad q=0,\pm1,\pm2,\ldots,\pm p. \tag{4.21}$$

There is a series of invariant fields $\Phi_q^{(s)}\{NS\}$, $s = 0, 1, 2, \ldots \leqslant 1/2(p - |q|)$ with charge (4.21) and dimensions

$$\Delta_q^{(s)}=[(|q|+2s+1)^2-q^2-1]/4(p+2),$$
$$s=0,1,2,\ldots\leqslant{}^1/_2(p-|q|). \tag{4.22}$$

(b) The space $\{R_{\rm I}\}$ is generated by the invariant fields $X_{q,\alpha}^{(s)}$, $\alpha = \pm 1$ with the following dimensions and charges:

$$Q_{q,\alpha}=(2q-\alpha p)/4(p+2),$$
$$\Delta_{q,\alpha}^{(s)}={}^1/_8+[(|q|+2s+1)^2-(q+\alpha)^2-1]/4(p+2), \tag{4.23}$$

where q and s run over the same values as in (4.21) and (4.22).

(c) The invariant fields $Y^{(s)}$ comprising the space $\{R_{\rm II}\}$ have the dimensions

$$\Delta_{(s)}={}^1/_8+[(p-2s)^2-4]/16(p+2),\quad s=0,1,2,\ldots\leqslant p/2. \tag{4.24}$$

We now turn to the general case of conformal field theory with the symmetry (4.1). If we suppose that p is real (but not necessarily an integer), we can look upon (4.15) as a parametrization of $\tilde{c}$ and (4.19) as the definition of the field φ, c_1, c_1^+. It then follows from (4.1d) that $\varphi(z)$ is a free Bose field (4.16). The fields $c_1(z)$, $c_1^+(z)$, defined in this way, generate an algebra of para-Fermion currents which, for integer values of p, is identical with (3.10), as already explained. When $p > 0$ and not an integer, this algebra does not close, as in (3.10), but contains an infinite number of currents ψ_k, ψ_k^+, $k = \pm 1, \pm 2, \ldots \pm\infty$ and, beginning with a certain k, the dimensions of the field ψ_k become negative. This enables us to consider that the above series of models with $p = 1, 2, 3, \ldots$ and dimensions (4.21)–(4.24) exhausts all field theories with $N=2$ extended superconformal symmetry (4.1) with $\tilde{c}\leqslant 3$.

[1)]It is implied, of course, that the scaling transformation $\mathbf{x}_j\to\Lambda^{-1}\mathbf{x}_j$, $\tilde{\mathbf{x}}_1\to\Lambda^{-1}\tilde{\mathbf{x}}_j$ $(\Lambda\gg1)$ has been carried out and that a suitable renormalization of all fields has been introduced.[10]

[2)]The continuity of the correlation functions (2.8) is, of course, unimportant in this context (in particular, all that we have said is meaningful in lattice theory, too). The condition we have adopted defines the mutual disposition of the contours γx_i corresponding to (2.8) to within "trivial" deformations.

[3)]Here and henceforth, E represents the unit element of the group, and I is the unit operator of field theory.

[4)]The dimensions of the field μ_k are the same as those of σ_k. In general, $[Z_p]$ is self-dual, i.e., all the relations are invariant under the replacement $\sigma_k\to\mu_k$. Apart from the symmetry (1.2), the models are then invariant under the "dual" group $\tilde{Z}_p$: $\tilde{\Omega}\mu_k=\omega^k\mu_k$, $\tilde{\Omega}\sigma_k=\sigma_k$ (Ref. 1).

[5)]Relations (3.20) and (3.24) (and the analogous relations in Ref. 1) provide us with an example of an infinite-dimensional associative algebra with quadratic defining relationships. We note that a finite-generated algebra of this type is considered in Ref. 12.

[6)]In addition to these "right" currents, there are, of course, also the "left" currents $\overline{T}(\bar{z})$, $\overline{S}(\bar{z})$, $\overline{S}^+(\bar{z})$, $\overline{J}(\bar{z})$. Because of the (usually assumed) **P**-invariance $z\to\bar{z}$, the relationships involving left currents will not be written out explicitly.

[7)]It is implied that, in conformal field theory, all fields are local with respect to the energy-momentum tensor.

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