The theory of acceleration effects accompanying the breakoff in plasma pinch constrictions

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We consider a two-stage "capacitor" model for the evolution of the electromagnetic field accompanying the breakoff in the constrictions of a plasma pinch with a longitudinal current. For a peripheral plasma with a density of the form $n(r) \propto r^s$ we find exact solutions for the waves in the initial and final stages of the pinch evolution.

§1. INTRODUCTION

The development of constrictions on a plasma filament with a current, first predicted in Ref. 1, leads regularly to the current channel breakoff.[2] It is just the breakoff in the constrictions which is the key point which determines the whole set of observed acceleration effects accompanying a powerful pulse discharge in a gas.[3] The dynamics of the electromagnetic field near a constriction which is breaking open is qualitatively similar in nature to the "charging" and "discharging" of a condenser. In the "charging" stage—a relatively slow buildup of the axial electric field—the motion of the pinch boundary leads to the occurrence of a magnetosonic wave (MSW) at the periphery. In that case both the ions and the electrons of the peripheral plasma drift along the radius to the center and the current transferred by the wave is caused by the polarization ion current. It is important that the generation of the wave is accompanied by the "outflow" of the current from the pinch to the periphery (the total current in the system remains constant!). It is just the drift character of the electron motion which makes it possible to "charge" the plasma capacitor appreciably. At the moment of the constriction breakoff the field has reached a magnitude

$$E_z \sim (v/c) B \sim (c_A/c) B.$$

From the moment when the boundaries of the constrictions reach the axis the nature of the particle motion changes suddenly: the presence near the constriction axis of a field $E_z \neq 0$ leads to the acceleration of some of the electrons and to formation of a near-axial electron beam. This is the "trigger" mechanism causing a fast "discharge"[4]—a rarefraction wave of E_z is formed and destroys the field built up in the slow stage.

For the ions in the peripheral plasma the fast phase is equivalent to an instantaneous "shock" which is totally sufficient, as was shown in Ref. 4, to explain not only the experimentally observed energies of accelerated deuterons but also their acceleration in a number sufficient for an explanation of the complete neutron effect of a pulsed plasma pinch.

In the present paper we present in detail the picture of the evolution of the electromagnetic field near a constriction of a pinch with a constant total current I_0, using a simple model the main feature of which is the presence around the pinch of a cold peripheral plasma with a given initial density profile $n = n^0(r)$. We obtain for the slow "charging" stage the simplest solutions for power-law profiles of the form $n^0 \propto r^s$. In the particular case $s = -2$ it turns out that the fast "discharging" phase reduces to a linear equation and also is amenable to an analytical study. The model with a peripheral plasma with a decreasing density $n^0 \propto r^{-2}$ enables us thus to trace completely the evolution of the electromagnetic field in the vicinity of a breaking constriction.

§2. FAST MAGNETOSONIC WAVE AND OUTFLOW OF CURRENT WHEN A CONSTRICTION BREAKS

In the slow "charging" stage the peripheral plasma with a sufficient density has a large permittivity $\varepsilon_{\rm eff} \sim (c/c_A)^2 \gg 1$. A wave on the periphery, which is produced by the radial shift in the pinch boundary $a = a(t)$ in agreement with the condition

$$E_z/B|_{r=a(t)} = -\dot{a}(t)/c, \quad a(0) = a_0, \tag{1}$$

therefore propagates with the Alfvén speed $c_A = B/(4\pi M n)^{1/2}$ and not with the speed of light. We have studied these waves before using a linear equation[5] (here and henceforth we assume cylindrical symmetry):

$$\frac{\partial}{r\partial r} r \frac{\partial}{\partial r} E_z \equiv \Delta_r E_z = \frac{4\pi M n^0(r)}{B_0^2(r)} \frac{\partial^2}{\partial t^2} E_z,$$
$$B_0(r) = \frac{2I_0}{cr}, \quad n^0 \sim r^s. \tag{2}$$

Under pinch conditions, however, the wave can distort the main magnetic field of the current appreciably and, strictly speaking, the applicability of the linear approximation is violated. The simplest nonlinear generalization of (2) is, clearly, the following:

$$\Delta_r A = \frac{4\pi M n^0(r)}{B} \frac{\partial}{\partial t}\left(\frac{1}{B}\frac{\partial A}{\partial t}\right), \quad B = -\frac{\partial A}{\partial r},$$
$$E_z = -\frac{1}{c}\frac{\partial A}{\partial t}. \tag{3}$$

Here $A(r,t)$ is the vector potential of the field, $n^0(r)$ the plasma density which, as before, is assumed to be given. In the case of a constant density $n^0(r) = n_0 = \text{const}$ a simple wave is possible with a parabolic profile

$$A = -\frac{I_0}{c}\left[2 \ln \frac{R}{a_0} - 1 + \frac{r^2}{R^2}\right], \quad B = \frac{2I_0 r}{cR^2},$$
$$E_z = \frac{2I_0 R_A}{cR^2}\left(1 - \frac{r^2}{R^2}\right) \tag{4}$$

and with a front

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$$R=R(t)=(a_0^2+2R_Act)^{1/2}, \quad R_A=\frac{a_0c_{A0}}{c}=I_0/c^2(\pi Mn_0)^{1/2}. \tag{5}$$

It follows from (1) that the pinch-radius evolution that generates the wave (4) is determined by the condition $A(r=a(t),t)=0$. This gives for the boundary motion

$$a(t)=R\left(1-2\ln\frac{R}{a_0}\right)^{1/2} \tag{6}$$

and the constriction must break at the time $t=t_{\rm br}=(e-1)a_0/c_{A0}$, where $e=2.71828$

It is important to note that at the front when $r=R(t)$ the magnetic field of the wave (4) continuously changes into the unperturbed pinch field $B_0(r)=2I_0/cr$. This guarantees the conversion of the total current in the pinch-wave system, whereas the current in the main channel decreases as follows:

$$I_1(t)=I_0\left(1-2\ln\frac{R}{a_0}\right) \tag{7}$$

and at the moment of the constriction breakoff the *whole* of the current shifts to the periphery.

The wave (4) thus describes all the laws that govern the constriction breakoff stage. However, this model as a whole is not satisfactory, since the condition $n^0(r)=\text{const}$ violates near the constriction the continuity equation and requires an additional particle source. It is, however, not difficult to get rid of this restriction. To do this we use the single-fluid hydrodynamics of a cold, perfectly conducting plasma. As we have from the definition of the vector potential that $E_z=-c^{-1}\partial A/\partial t$ and $B=-\partial A/\partial r$, the plasma radial velocity equals

$$v_r=-\frac{cE_z}{B}=-\frac{\partial A}{\partial t}\Big/\frac{\partial A}{\partial r}.$$

We find next the current density from the equation of motion

$$j_z=-\frac{Mnc}{B}\left(\frac{\partial}{\partial t}v_r+v_r\frac{\partial}{\partial r}v_r\right) \tag{8}$$

and, substituting (8) into Ampère's law

$$\frac{\partial}{r\partial r}rB=\frac{4\pi}{c}j_z,$$

we get a nonlinear equation which is more accurate than (3):

$$\Delta_rA=\frac{4\pi Mn}{\partial A/\partial r}\left[\frac{\partial}{\partial t}\frac{\partial A/\partial t}{\partial A/\partial r}-\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial A/\partial t}{\partial A/\partial r}\right)^2\right]. \tag{9}$$

The plasma density $n(r,t)$ which occurs in (9) is determined by using the magnetic-field freezing-in theorem $d(B/nr)/dt=0$, which gives

$$n=n_0\frac{a_0}{r}\frac{B}{B_0}\psi(A). \tag{10}$$

The subscript zero marks here the initial values on the unperturbed boundary of the pinch $r=a_0$ and $\psi(A)$ is an arbitrary function which is uniquely determined by the state of the plasma ahead of the wavefront. For a power-law density profile ahead of the front $n=n_0(r/a_0)^s$ the function $\psi(A)$ equals

$$\psi=\exp[-\nu cA/I_0], \quad \nu=1+s/2. \tag{11}$$

The wave front then clearly propagates according to the law

$$\dot R(t)=c_A|_{r=R}=c_{A0}\left(\frac{a_0}{R}\right)^\nu, \quad R=a_0\left[1+\frac{t}{\tau}\right]^{1/(\nu+1)},$$
$$\tau=a_0^2/cR_A(\nu+1) \tag{12}$$

which for the particular case $s=0$ is the same as (5).

Notwithstanding the complexity of the nonlinear Eqs. (9) to (11), we have discovered that one can indicate from them a particular rather simple exact solution of the form

$$A(r,t)=-\frac{2I_0}{c}\left[\ln\frac{R(t)}{a_0}-\Omega(\xi)\right], \quad \xi=\frac{r}{R(t)}. \tag{13}$$

Substitution of this expression into (9), (10) gives an equation for the self-similar part of the potential $\Omega(\xi)$

$$2\Omega'[(\xi+(\Omega')^{-3}\exp[-2\nu\Omega])\Omega''+\Omega']+(s+2\nu\xi\Omega')$$
$$\times\exp[-2\nu\Omega]=0, \tag{14}$$

whose solution is the function

$$\Omega(\xi)=\nu^{-1}\ln[1+\nu(1-\xi)], \tag{15}$$

which satisfies the conditions $\Omega(1)=0$, $\Omega'(1)=-1$ for the continuity of the vector potential (13) and the magnetic field at the wavefront. This enables us to find for any exponent $s>-4$ all the wave characteristics. For the field components E_z, B, and the plasma density n we find

$$E_z=-\frac{\partial A}{c\partial t}=\frac{2I_0}{cR}\frac{c_{A0}}{c}\left(\frac{a_0}{R}\right)^\nu\frac{1-\xi}{1-\nu(\nu+1)^{-1}\xi},$$
$$\xi=\frac{r}{R(t)},$$
$$B=-\frac{\partial A}{\partial r}=\frac{2I_0}{cR}\frac{1}{\nu+1}\frac{1}{1-\nu(\nu+1)^{-1}\xi}, \quad a(t)<r<R(t),$$
$$n=n_0\left(\frac{R}{a_0}\right)^s\xi^{-1}\left[(\nu+1)\left(1-\frac{\nu}{\nu+1}\xi\right)\right]^{-3}. \tag{16}$$

We show in Figs. 1a,b,c, plots of these quantities. The constriction radius $a(t)$ and the current I_1 in the main pinch vary as follows (Figs. 1d,e):

$$a(t)=R[1-\nu^{-1}((R/a_0)^\nu-1)],$$
$$I_1(t)=I_0(a_0/R)^\nu a(t)/R, \tag{17}$$

so that the contriction breaks off, and the current is completely shifted to the periphery, at the time

$$t=t_{\rm br}=\tau[(1+\nu)^{(1+\nu)/\nu}-1]. \tag{18}$$

The quantity τ is here given by Eq. (12).

Is is useful to note that for any $s>-4$ the constriction at the moment of breakoff moves with the Alfvén velocity $v_r|_{t=t_{\rm br}}=-c_{A0}$ and the field and density distributions (16) are qualitatively similar. On the whole they are simplest for the choice $s=-2$ for which we have in the wave region $a(t)<r<R(t)$ the relations

$$B=B(t)=\frac{2I_0}{cR(t)}, \quad E_z=B(t)\frac{c_{A0}}{c}\left(1-\frac{r}{R(t)}\right),$$
$$n=n_0\frac{a_0^2}{rR(t)}. \tag{19}$$

In conclusion we emphasize that the applicability of the

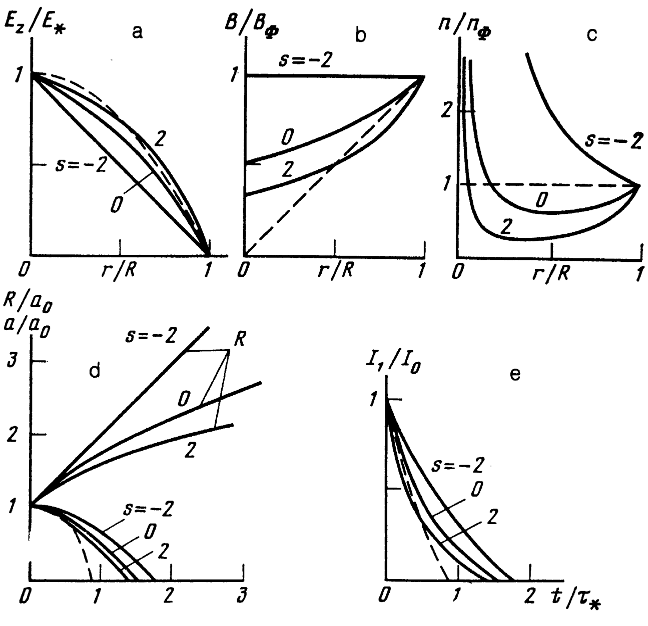

FIG. 1. Evolution of the fields E_z (a) and B (b), of the density n (c), of the radius $R(t)$ of the MSW and the boundary $a(t)$ of the constriction (d), and also of the pinch current $I_1(t)$ (e) during the stage of the constriction breakoff. The parameter of the curves is s. The dashed line is the regime (4) with a constant plasma density. We use: $E_* = E_z(r=0,t)$, $B_\Phi = 2I_0/cR$, $n_\Phi = n^0(r=R)$, $\tau_* = a_0^2/cR_A$.

single-fluid description requires that the ions be magnetized. Since the characteristic time for the development of the wave is of the order of $t_{\text{char}} \sim a_0/c_{A0}$, the ions are magnetized under the condition

$$1 \ll t_{\text{char}}\omega_{Bi} \sim \frac{a_0\omega_{Bi0}}{c_{A0}} = \frac{a_0\omega_{0i}}{c} = \frac{c_{A0}/c}{\varepsilon}, \quad \omega_{0i} = (4\pi e^2 n_0/M)^{1/2}, \tag{20}$$

where we use the notation

$$\varepsilon = c_{A0}/a_0\omega_{0i} = B_0/4\pi|e|n_0a_0 \equiv (R_e/a_0)^2, \quad R_e = (I_0/2\pi|e|cn_0)^{1/2} \tag{21}$$

R_e is a parameter which is important in what follows in the fast "discharge" stage. The magnetization of the ions thus imposes a stringent condition on the quantity ε: $\varepsilon \ll c_{A0}/c \ll 1$. We consider in Appendix I a two-fluid model and show that the MSW equations (9), (10) are valid also for the weaker inequality $\varepsilon \ll 1$.

§3. FORMATION OF A NEAR-AXIAL ELECTRON BEAM

In what follows we restrict ourselves to the case $s = -2$, when $n_0(r) \propto r^{-2}$. During the evolution of the wave (16) the running number of particles included in the motion is, up to the moment of breakoff, equal to

$$N_1(t) = 2\pi \int_{a(t)}^{R(t)} dr\, rn(r,t) \tag{22}$$

and is, by virtue of the equation of continuity, exactly equal to the running number of particles which are up to the same time "covered" by the front.

$$N_2(t) = 2\pi \int_{a_0}^{R(t)} dr\, rn^0(r). \tag{23}$$

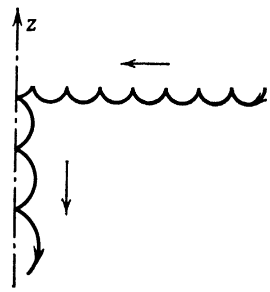

FIG. 2. Qualitative picture of the electron drift in the near-axial region of the contriction. The arrows indicate the direction of the drift.

At the moment of the breakoff of the constriction, N_1 has reached the value $N_1 = N_{1\max} = 2\pi n_0 a_0^2$. One checks easily that for $t > t_{\text{br}}$, if the MSW evolves further, this number is conserved whereas N_2 continues to grow! This could be called the "sticking" of an excess number $\delta N(t) = N_2(t) - N_{1\max}$ of particles to the axis of the system. In the near-axial region, however, the nature of the particle motion changes drastically (Fig. 2) and the radial current caused by the growth of $\delta N(t)$ is changed into an axial current.

As the basic factors giving rise to the axial motion of the particles we select the drift caused by the effective "shift in direction" of the magnetic field when a particle intersects the axis, and the acceleration of the particles oscillating near the axis by the axial electric field. For a particle initially incident on the axis along the radius, the non-uniform drift velocity is equal to $|v_z| = 2|v_{r0}|/\pi$ and clearly depends on the sign of the charge. When a field $E_z \neq 0$ is present in the vicinity of the axis, in first instance the plasma electrons are accelerated, and furthermore almost freely, so that

$$\frac{d}{dt}v_{ze} \approx -\frac{|e|}{m}E_z. \tag{24}$$

We estimate the current of the axial electron beam I_n. The radial current (per unit length) $I_r^{(1)}$ of the electrons which reach the vicinity of the axis is equal to

$$I_r^{(1)}(t) = -2\pi rnv_r = 2\pi c\left(\frac{nr}{B}\right)E_z \tag{25}$$

and is proportional to the field E_z, since $nr/B = \text{const}$ because the magnetic field is frozen in the plasma. At a certain time t', $dN = I_r^{(1)}(t')dt'$ electrons with zero velocity v_z reach the axis during a time dt'.

We neglect the contribution of the non-uniform drift. Then by the instant t the electrons reach according to (24) a speed

$$v_z = -\frac{|e|}{m}\int_{t'}^{t} dt''\, E_z.$$

Summing over all particles arriving at the axis by the instant t, we find the axial current

$$I_n(t) = \frac{e^2}{m}\int_0^t dt'\, I_r^{(1)}(t')\int_{t'}^t dt''\, E_z = \frac{|e|}{4\varepsilon mc}A^2(t), \tag{26}$$

where $A(t)$ is the vector potential of the field ($E_z = -c^{-1}\partial A/\partial t$) and ε is the parameter (21). One can generalize Eq. (26) also for relativistic electrons if instead of (24) we use for the determination of v_z the "relativistic" formula

$$\frac{d}{dt}[v_z/(1-v_z^2/c^2)^{1/2}]=-\frac{|e|}{m}E_z. \quad (27)$$

For the beam current we now find instead of (26)

$$I_n(t)=\frac{|e|A^2}{2\varepsilon mc}[1+(1+(eA/mc^2)^2)^{1/2}]^{-1}, \quad (28)$$

which, as one should expect, goes over into the non-relativistic estimate (26) if $|eA|\ll mc^2$.

§4. INITIAL PHASE OF THE "DISCHARGE". ESTABLISHMENT OF THE RELATIVISTIC ELECTRON BEAM (REB) CURRENT

As shown in §2, we have on the axis $E_z>0$ directly at the moment of the breakup of the constriction, so that the beam current (26) increases rapidly. The vector potential of the field, during the stage when the beam current grows, can be written in the form

$$A=A^0(r,t)+\tilde{A}(r,t),$$

where $\tilde{A}$ is the field of the axial current and A^0 the potential of the MSW (13), which is approximately equal to (we assume $s=-2$).

$$A^0(r,t)=-\frac{2I_0}{c}\left[\ln\frac{R(t)}{a_0}-1+\frac{r}{R(t)}\right]$$
$$\approx-\frac{2I_0}{c}\left[\frac{r}{R_*}+\frac{c_{A0}t}{R_*}\left(1-\frac{r}{R_*}\right)\right] \quad (29)$$

at the time immediately after the breakup. We put $t=0$ at the time of the breakup of the constriction; $R_*=R(t_{\rm br})$ is the radius of the MSW front at the time of the breakup. The structure of the field in the beam region will not be given in detail; we note that the beam occupies a region with a small radius a_B. We describe the reaction of the plasma surrounding the beam by Eqs. (A8) (see Appendix I).

Since it is primarily the plasma electrons which respond to the change in the field we can as a first step neglect the effect of the "fast" field $\tilde{A}$ on the ions, putting $G\approx A^0(r,t)$ in (A8). The ions continue "by inertia" the motion dictated by the MSW. For the vector potential $\tilde{A}$ outside the beam we then get from (A.8) the linear equation

$$\Box\tilde{A}=(\varepsilon r)^{-2}\tilde{A}, \quad r>a_B \quad (30)$$

(the linearity is caused by the convenient choice $n^0(r)\propto r^{-2}$). When $\varepsilon\ll1$ it is clear that the plasma strongly screens the electromagnetic field so that the phenomena described here evolve in the immediate vicinity of the pinch.

The general solution of Eq. (30) in the form of a divergent electromagnetic wave can by found by a "fictitious-current" method similar to the one suggested in Ref. 5. Here we shall not give it, since the purely wave stage is very short-lived, $t_{\rm char}\sim a_B/c$, and after that time the field near the beam becomes quasi-stationary. We have thus approximately

$$\tilde{A}(r,t)\approx\alpha(t)(a_B/r)^{1/\varepsilon}, \quad (31)$$

where $\alpha(t)$ is an arbitrary function. We match the "external

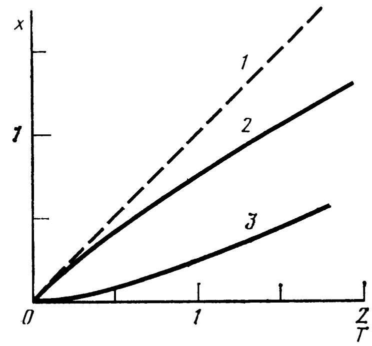

FIG. 3. Evolution of the reduced vector potential and of the axial electron current with time in the stage of the formation of the REB. We have put: $I_*=mc^3/2|e|\varepsilon$; 1: $x^0(T)$, 2: $x(T)$, 3: I_n/I_*.

current" (31) with the beam current $I_n(t)$ through the condition

$$r\tilde{B}|_{r=a_B}=2I_n(t)/c. \quad (32)$$

We further take it into account that at the boundary of the beam the total vector potential is equal to

$$A(t)\approx-ctE_0+\alpha(t), \quad E_0=\frac{2I_0}{cR_*}\frac{c_{A0}}{c}. \quad (33)$$

Finally, combining Eqs. (28) and (31)–(33) we get an algebraic equation for the beam current

$$I_n=\frac{mc^3}{2\varepsilon|e|}(T-x), \quad T=x+x^2[1+(1+x^2)^{1/2}]^{-1}, \quad (34)$$

where we write $T=|e|E_0t/mc$; $x=-|e|A/mc^2$ is the reduced vector potential. The function $x(T)$ is shown in Fig. 3 and illustrates the inductive delay of the growth of the current.

At the start of the acceleration process when $T\approx0$ we have $x(T)\approx T$ and the beam current increases rapidly:

$$I_n\approx\frac{mc^3}{4\varepsilon|e|}T^2. \quad (35)$$

By the time $T\sim1$ the value $x\sim1$ is reached and the electrons become relativistic, $|v_z/c|\approx x(T)\sim1$. The front of the REB should thus be formed after a time of order

$$t\sim t_{\rm acc}=mc/|e|E_0. \quad (36)$$

Subsequently, at $T\gg1$ we would have $x(T)\approx T/2$ so that the current $I_n\approx mc^3T/4\varepsilon|e|$ would continue to grow, while the field E_z would be established at a level $E_z\approx E_0/2$. The residual field is obviously due to the ion inertial motion which we postulated earlier to be given by the MSW. In fact, the REB evolution is accompanied by a growth in the r-component of the electrical field, which approximately equals (see (A.6))

$$E_r\approx\frac{\tilde{A}}{\varepsilon r}\approx\frac{2I_n(t)}{cr}(a_B/r)^{1/\varepsilon}. \quad (37)$$

This field stops the ions and as a consequence switches off the main source—the radial drift of the plasma to the axis, which maintains the non-zero electron-accelerating field E_z.

Using the REB growth rate found above, we can estimate the time for the stopping of the ions, which determine the saturation of the REB current as follows

$$t\sim t_{\rm stop}=3\omega_{0i}^{-1}(a_B/a_0)^{1/2}. \quad (38)$$

If the beam radius is not too small, or to be more precise, if $(a_B/a_0) > (m/M^2)(c/c_{A0})^3/4$ we shall have $t_{\rm stop} > t_{\rm acc}$ and it is possible to form the REB. We shall not consider the concluding stages of the REB in more detail, the more so since in the infinite cylindrical model considered by us there are no factors which determines its duration. In a real case the constriction has a finite length L so that the beam electrons must "drain off" at least after a time $\Delta t \sim L/c$ after their flow to the axis stops.

§5. ELECTROMAGNETIC RAREFACTION WAVE (EMW)

The concluding "discharging" stage—the annihilation of the electrical field accumulated by the constriction which is breaking up—is described neglecting the fact that the size of the REB, the time it takes to generate it, and the radial stopping of the ions are all finite. The appearance of the REB then instantaneously "places" the field E_z on the axis and the ions behind the discharge wavefront are immobile.

Under those conditions the potential G in Eqs. (A.8) does not change with time behind the EMW front, $G = G_0(r)$ and, by virtue of the continuity at the front is equal to

$$G_0(r)=A^0(r,t)\,|_{t=r/c}. \tag{39}$$

The time is reckoned here from the moment the EMW breaks away from the axis, so that its front corresponds to $r = ct$. Differentiating the second equation of (A.8) with respect to the time we get for the field E_z the wave equation (30)

$$\Box E_z=(\varepsilon r)^{-2}E_z,\quad r<ct;\quad E_z|_{r=0}=0. \tag{40}$$

At the front $r = ct$ the wave (40) must be continuously transformed into the MSW field which is approximately constant because $c_A \ll c$

$$E_z\approx E_0\left(1-\frac{r}{R_*}\right),\quad E_0=\frac{2I_0}{cR_*}\frac{c_{A0}}{c},\quad r>ct. \tag{41}$$

We get the continuous solution of this problem by combining the particular solutions $E_z = F_1(r/t)$ and $E_z = tF_2(r/t)$ of Eq. (40). The result will be as follows:

$$E_z=E_0\left[1-\frac{ct}{R_*}\left(1+\frac{\operatorname{th}\zeta}{\varepsilon}\right)\right]e^{-\zeta/\varepsilon},\quad r\leqslant ct\leqslant R_*,$$
$$\zeta=\operatorname{Arcch}(ct/r). \tag{42}$$

The profile of the "discharge" EMW (42) is shown in Fig. 4. It is noteworthy that for some time the field close to the axis is positive, while for

$$t\geqslant\frac{\varepsilon}{1+\varepsilon}\frac{R_*}{c}\approx\varepsilon\frac{R_*}{c}\approx 2{,}72\frac{R_e}{c}\frac{R_e}{a_0} \tag{43}$$

it changes sign. The solution (42) given here is valid only up to the moment $t_* = R_*/c \approx 2.72a_0/c$, when the fast wave reaches the MSW front. The further evolution of the field is qualitatively clear and reduces to the following. At the moment when it reaches the MSW front the fast wave is partially reflected (due to the jump in the derivative of E_z^0) and the reflected wave after that reaches the vicinity of the axis, where it is again reflected and emitted. The non-stationary

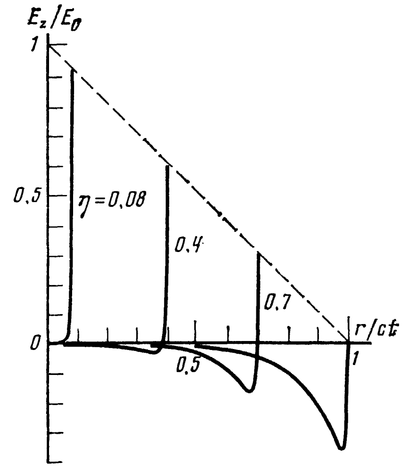

FIG. 4. Profile of the fast electromagnetic rarefaction wave in successive instants of time. The parameter of the curves is $\eta = ct/R_*$; $\varepsilon = 0.2$.

phase lasts a time on the order of several times t_* and is terminated by the total vanishing of the field E_z while the vector potential takes on a stationary value which is approximately equal to

$$A\approx G=G_0(r)=A^0(r,t)\,|_{t=r/c}\approx A^0(r,0)$$
$$=-\frac{2I_0}{c}\begin{cases} r/R_*, & r<R_*\\ \ln(r/R_*), & r>R_*\end{cases}. \tag{44}$$

This corresponds to a re-establishment of the total current of the pinch which is now distributed in the region $r < R_* \approx 2.72a_0$.

§6. CONCLUSION

The sequence of electromagnetic processes accompanying constriction breakup which we earlier called "a capacitor model," reduces to three basic phases that predict an MSW→REB→EMW evolution scheme. This picture in turn is based on a general foundation—the assumption that the pinch is surrounded at the periphery by a cold plasma. This is in our model the only experimentally "unobservable" factor. The experiments so far guarantee only that the plasma density does not exceed $n^0 \sim 10^{15}$ to 10^{16} cm^{-3}.

The picture of the field evolution close to the constriction is consistent if the lengths of the characteristic times of the various stages of the development follow the sequence $\Delta t_{\rm front\ REB} < \Delta t_{\rm EMW} \ll \Delta t_{\rm MSW}$. Using the estimates given in the text one can show that this is satisfied at least in the region $a_0 \lesssim 1$ cm, $I_0 \sim 1$ MA, and $n^0 \sim 10^{15}$ cm^{-3}.

We emphasize, finally, that we have concentrated mainly on a model with a decreasing density of the peripheral plasma $n^0(r) \propto r^{-2}$ solely because it gave us the possibility to solve this example both in the first "charging" stage and in the second "discharging" stage. This case covers the main stages, but in details it may simplify the situation too much. In particular, the linear wave equation (40) of the fast phase does not contain a characteristic length. On the other hand, for other initial profiles, for instance, $n^0(r) = \text{const}$, one can show that the waves may be localized in a region of size $\sim R_e$ (see (21)). This fact may be important for a description of a temporal fine structure such as has been observed in experiments of REB current oscillations.[6]

In conclusion the authors feel that they should express their appreciation to V. D. Shafranov, V. V. Vikhrev, and V. M. Smirnov for discussions of this work and valuable remarks.

APPENDIX I

"Two-potential" equations of the dynamics of a cold two-fluid plasma with inertialess electrons.

We describe the two-fluid hydrodynamics in cylindrical geometry by the set of equations

$$M\dot{\mathbf{v}}_i=|e|\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_i\mathbf{B}]\right),\quad 0=\mathbf{E}+\frac{1}{c}[\mathbf{v}_e\mathbf{B}],$$

$$\frac{\partial}{\partial t}n_\alpha+\frac{\partial}{r\,\partial r}rn_\alpha v_{r\alpha}=0, \tag{A.1}$$

$$\frac{\partial}{c\partial t}B=\frac{\partial}{\partial r}E_z,\quad \frac{\partial}{r\partial r}rB=\frac{4\pi}{c}\sum_\alpha e_\alpha n_\alpha v_{z\alpha}+\frac{\partial}{c\partial t}E_z,$$

$$\frac{\partial}{r\partial r}rE_r=4\pi\sum_\alpha e_\alpha n_\alpha.$$

Here $\mathbf{B}\equiv B\mathbf{e}_\varphi$ and $E_{r,z}$ are the field components, $v_{r,z\alpha}$ are the velocity components, and the subscripts $\alpha=e,i$ label the particles (electrons, ions).

We introduce the vector potential $A(r,t)$ of the wave such that $E_z=-c^{-1}\,\partial A/\partial t$, $B=-\partial A/\partial r$. The electron fluid is then connected with the field by the relations

$$v_{re}=-\frac{\partial A/\partial t}{\partial A/\partial r},\quad v_{ze}=c\frac{E_r}{B},\quad n_e=n_0\frac{B}{B_0}\frac{a_0}{r}\psi'(A), \tag{A.2}$$

where the last of them is determined by the fact that the magnetic field is frozen into the electrons

$$\frac{d}{dt_e}\left(\frac{B}{rn_e}\right)=0.$$

To describe the ion component of the plasma we introduce one more "potential" $G(r,t)$ that defines the analogs of the "magnetic" field $\mathscr{B}=-\partial G/\partial r$ and "electric" field $\mathscr{E}_z=-c^{-1}\partial G/\partial t$. Clearly, these fields are related through an "induction law"

$$\frac{\partial}{c\,\partial t}\mathscr{B}=\frac{\partial}{\partial r}\mathscr{E}_z. \tag{A.3}$$

We further require that

$$v_{ri}=-c\frac{\mathscr{E}_z}{\mathscr{B}}=-\frac{\partial G/\partial t}{\partial G/\partial r}, \tag{A.4}$$

and it is then easily verified that there is an analog of the "freezing-in theorem"

$$\frac{d}{dt_i}\left(\frac{\mathscr{B}}{rn_i}\right)=0$$

and for the ion component we get

$$v_{zi}=\frac{|e|}{Mc}(G-A),\quad n_i=n_0\frac{\mathscr{B}}{B_0}\frac{a_0}{r}\chi'(G). \tag{A.5}$$

The $\psi(A)$ and $\chi(G)$ in Eqs. (A.2), (A.5) are arbitrary functions which can completely be determined by the state of the plasma ahead of the wavefront.

Using the formulae for the densities $n_{e,i}$ we find from the Poisson equation the r-component of the electric field

$$E_r=\frac{4\pi|e|n_0a_0}{B_0}\frac{\psi(A)-\chi(G)}{r}\equiv\frac{\psi(A)-\chi(G)}{\varepsilon r}, \tag{A.6}$$

where ε is the parameter introduced in the text by Eq. (21). Substituting Eqs. (A.2), (A.4), and (A.5) into the remaining Eqs. (A.1) gives us two equations for the potentials $A(r,t)$ and $G(r,t)$. We give here the result only for the particular form of the profile $n_e^0=n_i^0=n_0(a_0/r)^2$ for the plasma density ahead of the front. We assume furthermore that $B^0=\mathscr{B}^0=2I_0/\mathrm{c}r$. In that case $\psi(A)=A$, $\chi(G)=G$, and we can write the field equations in the form

$$\Box_{r,t}A\equiv\left(\Delta_r-\frac{\partial^2}{c^2\,\partial t^2}\right)A=-\frac{G-A}{\varepsilon^2r^2}\left[1+\left(\frac{c_{A0}}{c}\right)^2\frac{r\mathscr{B}}{a_0B_0}\right],$$

$$\hat{T}G\equiv\frac{\partial}{\partial t}\frac{\partial G/\partial t}{\partial G/\partial r}-\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial G/\partial t}{\partial G/\partial r}\right)^2$$

$$=\frac{c_{A0}^2}{r\varepsilon^2}\frac{G-A}{a_0B_0}\left[1+\left(\frac{c_{A0}}{c}\right)^2\frac{rB}{a_0B_0}\right], \tag{A.7}$$

where the index "zero" indicates the initial values of quantities taken at the pinch boundary. We neglect small corrections $\propto(c_{A0}/c)^2$ on the right-hand side of (A.7) and we then get finally

$$\hat{T}G=-c_{A0}^2r\Box(A/a_0B_0),$$

$$G=A-(\varepsilon r)^2\Box A. \tag{A.8}$$

In the MSW stage we can neglect the fact that the velocity of light is finite. We then find for $\varepsilon\ll1$ from the second equation of (A.8) that $G\approx A$, which yields, after substitution into the first, the MSW equation (for $s=-2$) used in §2. Allowance for the fact that ε is finite determines, as shown below, the fine structure of the wavefront.

APPENDIX II. STRUCTURE OF THE FRONT OF A SELF-SIMILAR MAGNETOSONIC WAVE

We neglect in (A.8) the fact that the velocity of light is finite, replacing $\Box A\to\Delta_rA$, and seek for a solution in the form

$$A=-\frac{2I_0}{c}\left[\ln\frac{R}{a_0}+\alpha(\xi)\right],\quad G=-\frac{2I_0}{c}\left[\ln\frac{R}{a_0}+g(\xi)\right],$$

$$\xi=\frac{r}{R(t)},\quad R(t)=a_0+ut, \tag{A.9}$$

where $R(t)$ is the front of the MSW for $\varepsilon=0$ and $u=c_{A0}$. Substitution into the original equation gives

$$g=\alpha-\varepsilon^2\xi(\xi\alpha')',\quad \mu^2(1+g''/g'^2)/g'=(\xi\alpha')', \tag{A.10}$$

where $\mu=u/c_{A0}$ is the Mach number of the wave. If we neglect in (A.10) the fact that ε is finite, the solution can be continued up to the front only when $\mu=1$ and then it equals

$$g=\alpha=\xi-1, \tag{A.11}$$

as should be the case for a single-fluid MSW (§2). For small but finite $\varepsilon\ll1$, Eq. (A.11) holds up to terms $\propto\varepsilon^2$ and $(\mu^2-1)\ll1$ in the whole region $\xi<1$ of the flow. We refine this solution near the "front" $\xi\approx1$. Here we put

$$\xi=1+\varepsilon x,\quad g(\xi)=\varepsilon\tilde{g}(x),\quad \alpha(\xi)=\varepsilon\tilde{\alpha}(x), \tag{A.12}$$

and then we get from Eqs. (A.10), neglecting small terms $\propto\varepsilon\ll 1$

$$\mu^2\tilde{g}''(x)=\tilde{g}'^3(x)(\tilde{\alpha}-\tilde{g}),\qquad \tilde{\alpha}'(x)=1+\frac{\mu^2}{2}[1-1/\tilde{g}'^2(x)], \tag{A.13}$$

where we used the fact that far from the "front," as $x\to\infty$, we must have $\tilde{\alpha}'=\tilde{g}'=1$ in order that there be no field E_r ahead of the wave. It is convenient to write (A.13) as follows

$$\tilde{\alpha}'(x)=p,\quad \tilde{g}'(x)=q,\quad p''(x)=p-q,$$
$$q=\left[1+2\frac{1-p}{\mu^2}\right]^{-1/2}. \tag{A.14}$$

We further restrict ourselves to the soliton approximation, putting $\mu^2\approx 1$, $\eta\equiv(1-p)/\mu^2$, and $|\eta|\ll 1$. We then have $q\approx 1-\eta+\frac{3}{2}\eta^2$ and we get for the function $\eta(x)$ the equation

$$\mu^2\eta''(x)=\eta(\mu^2-1+{}^3/_2\eta), \tag{A.15}$$

whose solution is a soliton of the form

$$\eta(x)=-(\mu^2-1)\,\mathrm{ch}^{-2}\lambda x,\quad \lambda=(\mu^2-1)^{1/2}/2\mu,\quad x=(\xi-1)/\varepsilon. \tag{A.16}$$

It is this solution which determines the structure of the "soliton precursor" of the MSW. In particular, we get for the field components

$$B\approx\frac{2I_0}{cR(t)}(1-\mu^2\eta),\qquad E_z\approx\frac{2I_0}{cR(t)}\mu^3\eta,$$
$$E_r\approx-\frac{2I_0}{cR(t)}\mu(\mu^2-1)^{1/2}\eta\,\mathrm{th}\,\lambda x. \tag{A.17}$$

Since $\eta<0$, the precursor "condenses" the magnetic field.

[1]B. A. Trubnikov, in Fizika plazmy i problema UTR (Plasma Physics and the Problem of Controlled Thermonuclear Reactions) Izd. Akad. Nauk SSSR, Moscow, **1**, 289 (1958).

[2]B. A. Trubnikov and S. K. Zhdanov, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 292 (1985) [JETP Lett. **41**, 358 (1985)]; Ob integriruemosti uravneniĭ gidrodinamiki neustoĭchivykh sred (The integrability of the equations of the hydrodynamics of unstable media) Preprint MIFI, Nr 001-84, 1984.

[3]B. A. Trubnikov, in Fizika plazmy i problema UTR (Plasma Physics and the Problem of Controlled Thermonuclear Reactions) Izd. Akad. Nauk SSSR, Moscow, **4**, 87 (1958); J. Fucai and E. J. Clothiaux, Phys. Rev. Lett. **34**, 863 (1975).

[4]B. A. Trubnikov, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 317 (1986) [JETP Lett. **42**, 389 (1986)]; Fiz. Plazmy **12**, 468 (1986) [Sov. J. Plasma Phys. **12**, No. 4 (1986)].

[5]B. A. Trubnikov and S. K. Zhdanov, Zh. Eksp. Teor. Fiz. **70**, 92 (1976) [Sov. Phys. JETP **43**, 48 (1976)].

[6]G. Herziger, H. Krompholz, and W. Meff, Proc. Internat. Workshop on Plasma Focus Research, Stuttgart, 1983, pp. 31-34, 127-130.

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