Electron-phonon interaction in impure metals and superconductors

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Using the Keldysh diagram technique, we obtain collision integrals which describe the energy relaxation of electrons in normal metals and superconductors with high impurity concentrations. The calculations were carried out using both a coordinate system moving with the lattice and the usual laboratory system. The analysis we performed highlights the important role played by the process of inelastic scattering of electrons by impurities in relaxing the electronic energy. We also discuss the relation between the energy relaxation times for electrons and phonons.

1. INTRODUCTION

Recently, there has been a resurgence of interest in the study of energy relaxation of electrons in impure metals. The electron energy relaxation time τ_ε determines the most important characteristics of metals, such as the dephasing time for the electronic wave function,[1] the cooling time for the electron gas, and also a number of parameters which characterize the superconducting state, e.g., the relaxation times for the amplitude and phase of the order parameter.[2,3]

The energy relaxation of electrons is due to electron-electron and electron-phonon interactions. However, while theoretical results pertaining to the electron-electron relaxation mechanism[1] have been confirmed by a multitude of experiments, there are as yet no clear answers for questions relating to relaxation via the electron-phonon channel in impure metals. In the case of extremely impure metals, when $ql \ll 1$ (q is a characteristic phonon wave vector, and l is the mean free path between impurity scatterings), the energy relaxation of electrons has been calculated by a number of workers.[4–7] However, the results of these calculations do not agree among themselves, and the cause of the disagreement remains unclear. The difficulty which arises in comparing Refs. 4 to 7 is connected with the use by these authors of various methods to derive the kinetic equations, as well as their use of various frames of reference to describe the electron-phonon interaction. In Ref. 4, the calculation was performed in a coordinate system moving with the lattice, and a dependence $\tau_\varepsilon^{-1} \sim T^4 l$ was found. In Refs. 5–7, the laboratory system was used, which led to $\tau_\varepsilon^{-1} \sim T^2 l$ for Refs. 5 and 6, while Ref. 7 contains results which agree qualitatively with those derived in Ref. 4. It is no wonder that experimentalists[8,9] encounter difficulty in interpreting their measured data, and occasionally "pick and choose" among theoretical results which do not agree with one another.

To us, it seems timely to analyze the whole problem of electron-phonon interactions in impure metals and semiconductors on the basis of a single approach. The collision integral describing the energy relaxation of electrons in a normal metal can be derived using the Keldysh diagram technique. The calculation is performed by two methods: in Sec. 2 we use the laboratory frame, while in Section 3 we use the comoving frame. The calculation in the lab system allows us to clarify the roles played by the various physical processes which scatter electrons, and to carry out a comparison with the results of Refs. 4–7; the second technique offers a number of advantages in doing calculations. The collision integrals obtained in both systems are identical, while the energy relaxation time agrees with the result of Ref. 4 obtained by the method of Kadanoff and Baym using the two-particle Green's function. In Section 4 energy relaxation in superconductors with high concentrations of impurities is investigated on the basis of the formalism developed in Section 3. The importance of this problem derives from the fact that the macroscopic characteristics of superconductors turn out to be very sensitive to the form of the quasiparticle distribution function, which is determined by energy relaxation processes. In Section 5 we analyze the results obtained and establish a relation between the electron and phonon relaxation times. Finally, we will discuss the errors committed in Refs. 5–7.

2. ENERGY RELAXATION OF ELECTRONS IN THE LABORATORY SYSTEM

The interaction Hamiltonian, including the interaction of electrons among themselves, elastic electron-impurity scattering, the electron-phonon interaction and inelastic electron-impurity scattering, takes the form

$$H_{int} = \sum_{\mathbf{p}\sigma} \sum_{\mathbf{p}'\sigma'\mathbf{q}} \frac{1}{2} V_{ee}(\mathbf{q})\, c^+_{\mathbf{p},\sigma} c^+_{\mathbf{p}',\sigma'} c_{\mathbf{p}'+\mathbf{q},\sigma'} c_{\mathbf{p}-\mathbf{q},\sigma}$$
$$+ \sum_{\mathbf{p}\sigma} \sum_{\mathbf{q}}{}' \tilde{V}_{e\text{-}imp}(\mathbf{q})\, c^+_{\mathbf{p},\sigma} c_{\mathbf{p}-\mathbf{q},\sigma} \sum_{R_\beta^0} \exp(-i\mathbf{q}\mathbf{R}_\beta^0)$$
$$+ \sum_{\mathbf{p}\sigma} \sum_{\mathbf{q}\lambda}{}' \tilde{B}(\mathbf{q})\, c^+_{\mathbf{p}+\mathbf{q},\sigma} c_{\mathbf{p},\sigma} (b_{\mathbf{q},\lambda} + b^+_{-\mathbf{q},\lambda})$$
$$+ \sum_{\mathbf{p}\sigma} \sum_{\mathbf{k}} \sum_{\mathbf{q}\lambda}{}' \tilde{\gamma}(\mathbf{k},\mathbf{q},\lambda)\, c^+_{\mathbf{p},\sigma} c_{\mathbf{p}-\mathbf{k},\sigma} (b_{\mathbf{q},\lambda} + b^+_{-\mathbf{q},\lambda})$$
$$\times \sum_{R_\beta^0} \exp[-i(\mathbf{k}-\mathbf{q})R_\beta^0]. \qquad (1)$$

The prime next to the summation signs indicates terms with $\mathbf{q} = 0$ are omitted; $c^+_{\mathbf{p},\sigma}$ is a creation operator for an electron with momentum $\mathbf{p}$ and spin σ, while $b^+_{\mathbf{q},\lambda}$ is a creation opera-

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tor for a phonon with wave vector $\mathbf{q}$ and polarization index λ. The electron-electron and electron-impurity potentials are respectively equal to

$$V_{ee}(\mathbf{q})=4\pi e^2/q^2, \quad \tilde{V}_{e\text{-}imp}(\mathbf{q})=-4\pi e^2(Z_{imp}-Z_{ion})/q^2, \qquad (2)$$

where Z_{ion} is the valence of an atom of the host lattice, Z_{imp} that of the impurity atom, and R_β^0 is the equilibrium impurity position.

In the electron-phonon interaction vertex $\tilde{B}(\mathbf{q})$, as opposed to the usual expressions,[10] we do *not* include screening

$$\tilde{B}(\mathbf{q})=-i\tilde{V}_{e\text{-}ion}(\mathbf{q})\mathbf{q}\mathbf{e}_\lambda N/(2MN\omega_{q,\lambda})^{1/2},$$
$$\tilde{V}_{e\text{-}ion}(\mathbf{q})=-4\pi e^2 Z_{ion}/q^2, \qquad (3)$$

N is the number of unit cells, $\omega_{q,\lambda}$ is the phonon frequency, $\mathbf{e}_\lambda$ is the phonon polarization vector and M the ion mass.

Scattering by impurities with emission or absorption of a phonon is described by the vertex

$$\tilde{\gamma}(\mathbf{k},\ \mathbf{q},\ \lambda)=-i\tilde{V}_{e\text{-}imp}(\mathbf{k})\mathbf{k}\mathbf{e}_\lambda/(2MN\omega_{q,\lambda})^{1/2}. \qquad (4)$$

From the condition of electrical neutrality, it follows that the electron density n_e satisfies the equation

$$n_e=Z_{ion}N_{ion}+Z_{imp}N_{imp}=ZN, \qquad (5)$$

where Z is the average charge in a unit cell, and N_{imp} is the number of impurities per unit volume.

In the Keldysh diagram technique,[11] the electron and phonon Green's functions, along with the electron and phonon self-energies, are represented by matrices

$$\hat{G}=\begin{pmatrix}0 & G^A\\ G^R & G^C\end{pmatrix}, \quad \hat{D}=\begin{pmatrix}0 & D^A\\ D^R & D^C\end{pmatrix},$$
$$\hat{\Sigma}=\begin{pmatrix}\Sigma^C & \Sigma^R\\ \Sigma^A & 0\end{pmatrix}, \quad \hat{P}=\begin{pmatrix}P^C & P^R\\ P^A & 0\end{pmatrix}. \qquad (6)$$

The electron Green's function averaged over impurity positions equals

$$G^R(\mathbf{p},\ \varepsilon)=[G^A(\mathbf{p},\ \varepsilon)]^*=(\varepsilon-\xi_p+i/2\tau)^{-1},$$
$$\xi_p=(p^2-p_F^2)/2m, \qquad (7)$$

where τ is the electron momentum relaxation time due to impurity scattering, p_F is the Fermi momentum and m is the electron mass. We will neglect renormalization of the electron spectrum due to the electron-phonon interaction.

In order to solve the problem of energy relaxation in a spatially-homogeneous electron system, we focus our attention on times much longer than the electronic momentum relaxation time τ; then G^C can be case in the form

$$G^C(\mathbf{p},\ \varepsilon)=S(\varepsilon)[G^A(\mathbf{p},\ \varepsilon)-G^R(\mathbf{p},\ \varepsilon)], \qquad (8)$$

where $n_\varepsilon=[S(\varepsilon)+1]/2$ can be interpreted as the electron energy distribution function. We will obtain a kinetic equation for the function n_ε.

The Green's function for phonons equals

$$D^R(\mathbf{q},\ \omega)=[D^A(\mathbf{q},\ \omega)]^*=(\omega-\omega_q+i0)^{-1}-(\omega+\omega_q+i0)^{-1}. \qquad (9)$$

Here we assume that any quasilocal modes which appear when impurities are introduced do not affect the initial segment of the phonon spectrum, and also that the velocity of sound is already renormalized.[12] For equilibrium phonons, we have

$$D^C(\mathbf{q},\ \omega)=(2N_\omega+1)[D^R(\mathbf{q},\ \omega)-D^A(\mathbf{q},\ \omega)],$$
$$N_\omega=[\exp(\omega/T)-1]^{-1}, \qquad (10)$$

where T is the temperature of the heat bath.

The vertices which mediate energy transfer in the Keldysh techniques are represented by objects of the form Q^l_{ij}, where the upper index is for bosons, the lower electrons. The vertices corresponding to the electron-phonon interactions and inelastic scattering of electrons by impurities take the form $\tilde{B}^l_{ij}=\tilde{B}K^l_{ij}$, $\tilde{\gamma}^l_{ij}=\tilde{\gamma}K^l_{ij}$, where

$$K_{ij}^1=2^{-1/2}\delta_{ij}, \quad K_{ij}^2=2^{-1/2}(\sigma_x)_{ij}, \qquad (11)$$

σ_x is the well-known Pauli Matrix. The matrix $\tilde{V}_{e\text{-}imp}(\sigma_x)_{ij}$ corresponds to a vertex which describes elastic scattering off an impurity.

The renormalization of any scalar vertex $\hat{w}^l$ (i.e., independent of the direction of the vector $\mathbf{p}$) due to impurities is shown graphically in Fig. 1 for the ladder approximation. As a result, we are led to a renormalized vertex $\hat{W}^l$ which satisfies the following matrix equation

$$\hat{W}^l=\hat{w}^l+\frac{1}{\pi\nu\tau}\int\frac{d\mathbf{p}}{(2\pi)^3}\hat{\sigma}_x\hat{G}(\mathbf{p},\varepsilon)\hat{W}^l\hat{G}(\mathbf{p}+\mathbf{q},\varepsilon+\omega)\hat{\sigma}_x, \qquad (12)$$

where ν is the density of states and τ is the scattering of an electron off an impurity, which will be determined later by (20) including screening. In particular, as regards the Coulomb vertex $w^l=K^l_{ij}$, to lowest order in $(p_Fl)^{-1}$,[13] we have

$$W_{22}^1=\frac{1}{2^{1/2}(1-\zeta_0)}, \quad W_{12}^1=\frac{S(\varepsilon)\zeta_0}{2^{1/2}(1-\zeta_0)},$$
$$W_{21}^1=-\frac{S(\varepsilon+\omega)\zeta_0}{2^{1/2}(1-\zeta_0)},$$
$$W_{11}^1=\frac{1}{2^{1/2}}\left[\frac{1}{1-\zeta_0^*}-2S(\varepsilon)S(\varepsilon+\omega)\,\mathrm{Re}\frac{\zeta_0}{1-\zeta_0}\right], \quad W_{22}^2=0, \qquad (13)$$
$$W_{12}^2=W_{21}^2=\frac{1}{2^{1/2}}, \quad W_{11}^2=\frac{[S(\varepsilon+\omega)-S(\varepsilon)]\zeta_0^*}{2^{1/2}(1-\zeta_0^*)},$$

where

$$\zeta_n=\frac{1}{\pi\nu\tau}\int\frac{d\mathbf{p}}{(2\pi)^3}G^A(\mathbf{p},\varepsilon)G^R(\mathbf{p}+\mathbf{q},\varepsilon+\omega)x^n, \quad x=\frac{\mathbf{pq}}{pq}. \qquad (14)$$

We write out the value of ζ_n (which we will need later) in the long-wavelength limit ($ql\ll1$, $\omega\tau\ll1$):

$$\zeta_0=1+i\omega\tau-{}^1/_3(ql)^2, \quad \zeta_1=-i[1+\zeta_0(i\omega\tau-1)]/ql,$$
$$\zeta_2=-i(i\omega\tau-1)\zeta_1/ql. \qquad (15)$$

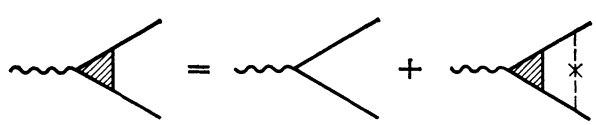

FIG. 1. Renormalization of vertex function due to impurities within the ladder approximation.

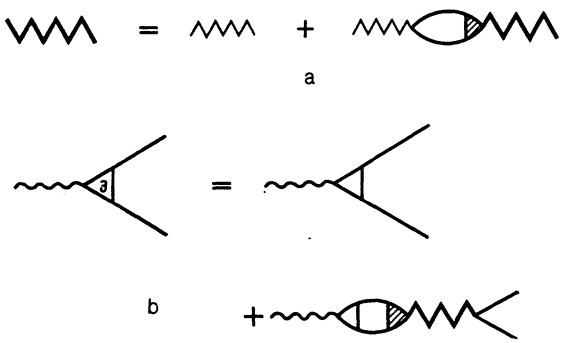

FIG. 2. (a) Equation for the screened Coulomb potential, (b) Equation which describes screening of the vertex function.

Including screening of the Coulomb potential and electron-phonon vertex in the random-phase approximation leads to the equations shown schematically in Fig. 2. We remark that the screened Coulomb potential has the same matrix structure as the G-function. The expressions for all the matrix elements $\hat{V}_{ee}$ were obtained in Ref. 13. Below, we write down expressions we will need later for V_{ee}^{R} and V_{ee}^{A}:

$$V_{ee}^{R}(\mathbf{q},\omega)=[V_{ee}^{A}(\mathbf{q},\omega)]^{*}=V_{ee}(\mathbf{q})/\varepsilon^{R}(\mathbf{q},\omega),$$
$$\varepsilon^{R}(\mathbf{q},\omega)=1-V_{ee}(\mathbf{q})P_{0}^{R}(\mathbf{q},\omega), \tag{16}$$

where P_0 is the loop shown in Fig. 2:

$$P_{0}^{R}(\mathbf{q},\omega)=-\nu\frac{1-(1-i\omega\tau)\zeta_{0}}{1-\zeta_{0}}. \tag{17}$$

For $ql\ll 1$, $\omega\tau\ll 1$, we have

$$\varepsilon^{R}(\mathbf{q},\omega)=1+4\pi e^{2}\nu D/(-i\omega\tau+Dq^{2}),\quad D=v_{F}^{2}\tau/3. \tag{18}$$

In the case of elastic and inelastic scattering of electrons by impurities, the relevant momentum transfers are large enough so that $ql\gg 1$; then $\varepsilon^{R}(\mathbf{q},\omega)$ can be taken in the static limit:

$$\varepsilon^{R}(\mathbf{q},\omega)=1+\varkappa^{2}/q^{2},\quad \varkappa^{2}=4\pi e^{2}\nu,\quad \nu=mp_{F}/\pi^{2}. \tag{19}$$

For isotropic scattering,

$$V_{e\text{-}imp}=-4\pi e^{2}(Z_{imp}-Z_{i0n})/\varkappa^{2},\quad 1/\tau=\pi\nu N_{imp}V_{e\text{-}imp}^{2}. \tag{20}$$

Then inelastic scattering vertex for electrons by impurities coincides with (4) if we use Eq. (20) for $\hat{V}_{e\text{-imp}}$.

In the long-wavelength limit the screened electron-phonon vertex B_{ij}^{l} depends significantly on the frequency ω due to the dispersion of the dielectric permeability (18):

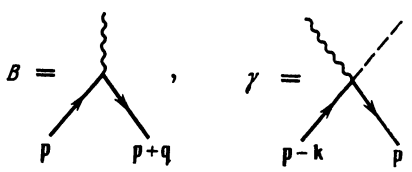

FIG. 3. Vertices which describe electron-phonon scattering along with inelastic electron-impurity scattering.

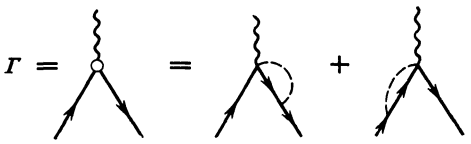

FIG. 4. The effective vertex Γ.

$$B_{ij}^{2}=\frac{1}{\varepsilon^{A}(\mathbf{q},\omega)}\bar{B}_{ij}^{2}(\mathbf{q}),\quad B_{11}^{1}=B_{22}^{1}=(-B_{12}^{2})^{*},$$
$$B_{12}^{1}=B_{21}^{1}=-2(2N_{\omega}+1)\operatorname{Re}B_{12}^{2}. \tag{21}$$

If we use the vertices B and γ (Fig. 3), it is not hard to write down the electron self-energy diagram; however, the number of diagrams which results is extremely large. Therefore it is convenient to introduce effective vertices in which the diagrams have already been partially summed. A similar procedure was employed in Ref. 5 for the temperature Green's function.

Let us introduce the effective vertex Γ, whose defining equation is shown graphically in Fig. 4. After some calculations we obtain

$$\Gamma_{22}^{1}=\mathbf{pe}_{\lambda}/2^{1/2}\tau(2MN\omega_{q,\lambda})^{1/2},\quad \Gamma_{11}^{1}=-\Gamma_{22}^{1},\quad \Gamma_{12}^{1}=S(\varepsilon)\Gamma_{22}^{1},$$
$$\Gamma_{21}^{1}=-S(\varepsilon+\omega)\Gamma_{22}^{1},\quad \Gamma_{11}^{2}=[S(\varepsilon)-S(\varepsilon+\omega)]\Gamma_{22}^{1},$$
$$\Gamma_{12}^{2}=\Gamma_{21}^{2}=\Gamma_{22}^{2}=0. \tag{22}$$

Using Γ, we introduce the vertex L shown in Fig. 5

$$L_{ij}^{2}=-P_{1}^{A}(\mathbf{q},\omega)V_{ee}^{A}(\mathbf{q},\omega)K_{ij}^{2}\approx-P_{1}^{A}(\mathbf{q},\omega)K_{ij}^{2}/P_{0}^{A}(\mathbf{q},\omega), \tag{23}$$

where P_1 is the loop containing the vector vertex Γ proportional to the first power of x. The last equation in (23) is correct in the limit of strong screening $V_{ee}(\mathbf{q})P_{0}(\mathbf{q},\omega)\gg 1$. Taking into account (15) and (17), we obtain

$$L_{ij}^{2}=\frac{-i\omega p_{F}\zeta_{1}^{*}}{1-(1+i\omega\tau)\zeta_{0}^{*}}\frac{K_{ij}^{2}}{(2MN\omega_{q,\lambda})^{1/2}}=\frac{\omega_{p_{F}}}{ql}\frac{K_{ij}^{2}}{(2MN\omega_{q,\lambda})^{1/2}}. \tag{24}$$

The vertices B_{ij}^{l} and L_{ij}^{l} can be conveniently combined by introducing the vertex $g_{ij}^{l}=B_{ij}^{l}+L_{ij}^{l}$:

$$g_{ij}^{2}=ig\left[1+i\omega\tau\frac{{}^{4}/_{15}(ql)^{2}+i\omega\tau}{{}^{1}/_{3}(ql)^{2}}\right]K_{ij}^{2},\quad g=\frac{2}{3}\varepsilon_{F}\frac{\mathbf{qe}_{\lambda}}{(2MN\omega_{q,\lambda})^{1/2}} \tag{25}$$

We can express the vertices L_{ij}^{1} and g_{ij}^{1} as functions of the vertices L_{ij}^{2} and g_{ij}^{2} respectively in precisely the same way as we did for the vertices B_{ij}^{l} in (21). We note that the quantity g coincides with the vertex which is commonly used for describing the electron-phonon vertex in pure metals.[10]

FIG. 5. The vertex L which takes into account screening in the inelastic electron-impurity process, and the effective vertex g.

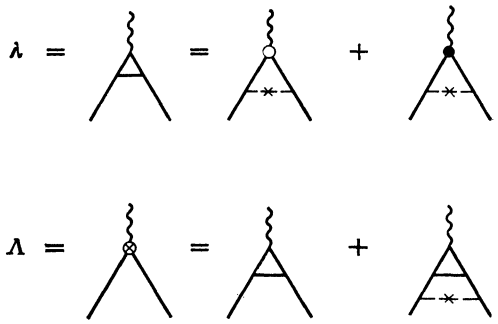

FIG. 6. Renormalization of the vertex $\Gamma + g$ due to impurities.

The vertices g_{ij}^{l} and Γ_{ij}^{l} for low momentum transfers and frequencies are significantly renormalized due to the electron-impurity interaction. Since the vertex Γ_{ij}^{l} is a vector, i.e., depends on $\mathbf{p}\mathbf{e}_{\lambda}$, we cannot directly write down an equation like (12) for it. However, such an equation can be written for the vertex Λ, as shown in Fig. 6, where the bare scalar vertex λ is obtained from the vertices Γ and g by inserting a single impurity line. As a result, we obtain

$$\Lambda_{11}^{2}=ig\frac{{}^{4}/_{15}(ql)^{2}-i\omega\tau}{{}^{1}/_{3}(ql)^{2}}[S(\varepsilon+\omega)-S(\varepsilon)],$$

$$\Lambda_{12}^{2}=\Lambda_{21}^{2}=\Lambda_{22}^{2}=0. \tag{26}$$

So as to avoid cumbersome statements here and later on, let us write out only the expressions for vertices with a phonon index of 2. We remark that when we use only these vertices, we obtain that part of the collision integral proportional to $(2N_{\omega}+1)[S(\varepsilon+\omega)-S(\varepsilon)]$.

Therefore, in order to describe the electron-phonon interaction in impure metals we must take into account the vertices g, γ, Γ and Λ. The kinetic equation for the function n_{ε} which determines the electron energy relaxation was obtained in Refs. 13, 14:

$$\frac{\partial n_{\varepsilon}}{\partial t}=\frac{1}{2\pi\nu}\int\frac{d\mathbf{p}}{(2\pi)^{3}}[G^{A}(\mathbf{p},\varepsilon)-G^{R}(\mathbf{p},\varepsilon)]$$
$$\times[(2n_{\varepsilon}-1)(\Sigma^{A}(\mathbf{p},\varepsilon)-\Sigma^{R}(\mathbf{p},\varepsilon))-\Sigma^{C}(\mathbf{p},\varepsilon)]. \tag{27}$$

The expression standing on the right side of (27) is the collision integral, and will be denoted I_{ε}.

Let us first investigate longitudinal phonons. The electron self-energy diagrams $\Sigma(\mathbf{p},\varepsilon)$ to first order in the electron-phonon interaction but including the electron-impurity interaction to all orders in the approximation $p_F l\gg 1$ is shown in Fig. 7. We will now turn to a direct calculation of these diagrams for $q_T l\ll 1$ where $q_T = T/u_l$, and u_l is the longitudinal sound velocity.

The first diagram gives the following expression for the collision integral

$$I_1(\varepsilon)$$
$$=\frac{8}{\pi\nu}\int\frac{d\mathbf{p}\,d\mathbf{q}\,d\omega}{(2\pi)^{7}}g^{2}R(\varepsilon,\omega)\operatorname{Im}G^{A}(\mathbf{p},\varepsilon)\operatorname{Im}G^{A}(\mathbf{p}+q,\varepsilon+\omega)$$
$$\times\operatorname{Im}D^{R}(\mathbf{q},\omega), \tag{28}$$

$$R(\varepsilon,\omega)=N_{\omega}n_{\varepsilon}(1-n_{\varepsilon+\omega})-(1+N_{\omega})(1-n_{\varepsilon})n_{\varepsilon+\omega}. \tag{29}$$

The reader can convince himself that the second, third and fourth diagram make no contribution to the energy relaxation. The fifth, sixth, ninth and tenth diagrams are related to the corresponding collision integrals

$$I_5={}^{4}/_{5}I_1,\quad I_6=-I_1,\quad I_9=-I_5,\quad I_{10}=-I_1.$$

The seventh diagram does not depend on the parameter $q_T l$, and its contribution to the collision integral is in order of magnitude $I_7\sim I_1(q_T l)^{-2}$. However, I_7 cancels with the contribution from the eight diagram I_8 calculated to lowest order in $q_T l$ (for which the internal G-function must be expanded in $q_T l$); we thus obtain $I_7 + I_8 = 9/5 I_1$. In this way, the full contribution to the collision integral for all the diagrams is found to be $4/5\ I_1$. We thus find for the energy relaxation time when $\varepsilon\ll T$

$$\frac{1}{\tau_{\varepsilon}}=-\frac{\delta I(\varepsilon)}{\delta n_{\varepsilon}},\quad \frac{1}{\tau_{\varepsilon l}}=\frac{\pi^{4}\beta}{5}\frac{p_F l T^{4}}{(p_F u_l)^{5}},$$
$$\beta=\left(\frac{2}{3}\varepsilon_F\right)^{2}\frac{\nu}{2MNu_l^{2}}. \tag{30}$$

For transverse phonons, we need include only the diagrams Σ_7 and Σ_8. If we neglect the difference in the sound velocities, so that $u_l = u_t = u$, then we obtain $\tau_{\varepsilon t}=4/3\tau_{\varepsilon l}$. Finally, for one longitudinal and two transverse phonon branches we have $\tau_{\varepsilon}=2/5\tau_{\varepsilon l}$, which agrees with the results obtained by Schmidt.[4] In the general case $u_l\neq u_t$, so that τ_{ε} has the form

$$\frac{1}{\tau_{\varepsilon}}=\frac{\pi^{4}\beta}{5}\frac{p_F l T^{4}}{(p_F u_l)^{3}}\left[1+\frac{3}{2}\left(\frac{u_l}{u_t}\right)^{5}\right]. \tag{31}$$

3. ENERGY RELAXATION OF ELECTRONS IN THE MOVING SYSTEM

As was shown in Section 2, in order to describe the interaction of electrons with long-wavelength phonons in the

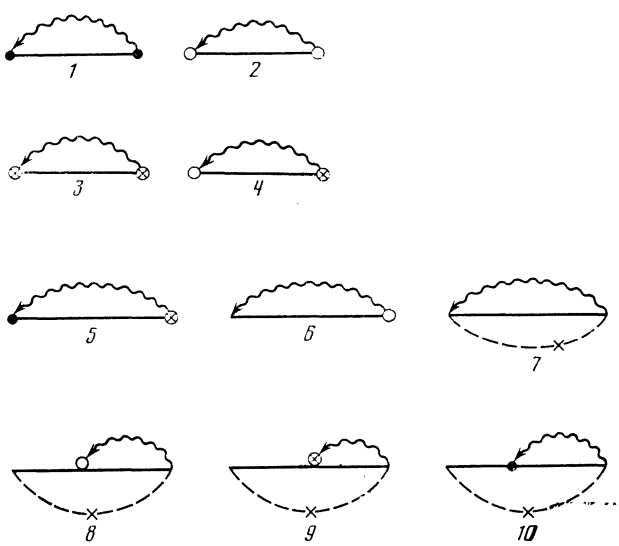

FIG. 7. Self-energy diagrams for electrons in the lab system. By diagrams 4–6 and 8–10 we mean to include two diagrams apiece: one is shown in the figure, while the other differs from it by an interchange of vertices.

laboratory system it is necessary to calculate directly the scattering of electrons by vibrating impurities. It is therefore convenient to transform to a coordinate system moving with the lattice in which the impurity is stationary. The effective electron-phonon interaction in this co-moving system appears as an additional term linear in the lattice displacement when we transform the kinetic energy of an electron from the lab system to the co-moving system. The co-moving system is used by a number of workers[15-17,4] (see also the recent review in Ref. 18). According to Ref. 17, the electron distribution function does not change when we go from the lab to the co-moving system; consequently, the collision integrals in both systems should be one and the same. The following calculations confirm these statements. In Ref. 4, the finite-temperature two-particle Green's function equation was used to derive the collision integral. To use the Keldysh technique, we require vertices with tensor structure. The electron-phonon interaction vertex has the form

$$(\Gamma_0)_{ij}{}^{l} = -\frac{(\mathbf{pq})\,(\mathbf{pe}_\lambda)}{m(2MN\omega_{q,\lambda})^{1/2}} K_{ij}{}^{l}. \tag{32}$$

For longitudinal phonons,

$$(\Gamma_0)_{ij}{}^{l} = -3gx^2 K_{ij}{}^{l}, \quad x = \mathbf{pq}/pq.$$

By a process analogous to the one described in the previous section, we can include the effects of the interelectron Coulomb interaction, and are led to a screened vertex (Fig. 2b):

$$(\Gamma_s)_{ij}{}^{2} = -3g[x^2 + P_2{}^{A}(\mathbf{q},\omega)V^{A}(\mathbf{q},\omega)]K_{ij}{}^{2}$$

$$\approx -3g[x^2 - P_2{}^{A}(\mathbf{q},\omega)/P_0{}^{A}(\mathbf{q},\omega)]K_{ij}{}^{2}. \tag{33}$$

The last equality in (33) is correct under strong-screening conditions, i.e., $V_{ee}(q)P_0(\mathbf{q},\omega) \gg 1$; $P_2(\mathbf{q},\omega)$ is the loop containing the vertex $x^2 K_{ij}^{l}$.

To calculate the loops via the Keldysh techniques, including the impurity ladders, we take advantage of the following method, which noticeably simplifies the calculations and is especially effective in the superconducting case. Let us investigate, e.g.,

$$P_0{}^{A}(\mathbf{q},\omega) = -2i\int \frac{d\mathbf{p}\,d\varepsilon}{(2\pi)^4} W_{ij}{}^{2} G_{ni}(\mathbf{p},\varepsilon) K_{mn}{}^{1} G_{jm}(\mathbf{p}+\mathbf{q},\varepsilon+\omega). \tag{34}$$

Since summation is implied by the pairs of repeated indices in the right side of (34), we can convince ourselves that it is proportional to the vertex W_{11}^2 determined by equation (12). This allows us to perform the $\mathbf{p}$ integration in (34):

$$P_0{}^{A}(\mathbf{q},\omega) = -2^{1/2}i\int \frac{d\varepsilon}{2\pi} (N_{imp}V_{e\text{-}imp}^2)^{-1} W_{11}{}^{2}(\mathbf{q},\omega,\varepsilon). \tag{35}$$

Expression (35) is exact only in the sense that the ladder approximation is exact. If we want to calculate $P_0(\mathbf{q},\omega)$ via formula (35), it is necessary for W_{11}^2 to be known to first-order accuracy in $(p_F l)^{-1}$:

$$W_{11}{}^{2}(\mathbf{q},\omega,\varepsilon) = [S(\varepsilon+\omega) - S(\varepsilon)]\zeta_0{}^{*}/2^{1/2}(1-\zeta_0{}^{*}) + S(\varepsilon+\omega)\zeta_0{}^{A} - S(\varepsilon)\zeta_0{}^{R}, \tag{36}$$

where

$$\zeta_n{}^{A} = (\zeta_n{}^{R})^{*} = \frac{1}{\pi\nu\tau}\int \frac{d\mathbf{p}}{(2\pi)^3} G^{A}(\mathbf{p},\varepsilon) G^{A}(\mathbf{p}+\mathbf{q},\varepsilon+\omega) x^n,$$

$$\zeta_0{}^{A} = -\frac{i}{2^{1/2}\tau(\varepsilon_F+\varepsilon)}. \tag{37}$$

We remark that $\zeta_0^{A} \ll \zeta_0$, but because the first term in (36) when substituted into (35) integrates to a finite limit, while the second and third terms are infinite, their contribution is found to be first order.[19,13] After performing the integration in (35), we obtain expression (17).

In order to calculate the loop $P_2(\mathbf{q},\omega)$ using the method described above, it is necessary to include the renormalization of the vector vertex $x^2 K_{ij}^{l}$ due to electron-impurity interactions. Proceeding in the same way as we did to renormalize the vector vertex Γ (Fig. 6), we obtain as a result the vertex Γ_c:

$$(\Gamma_c)_{12}{}^{2} = (\Gamma_c)_{21}{}^{2} = (\Gamma_c)_{22}{}^{2} = 0,$$

$$(\Gamma_c)_{11}{}^{2} \tag{38}$$

$$= [S(\varepsilon+\omega) - S(\varepsilon)]\zeta_2{}^{*}/2^{1/2}(1-\zeta_0{}^{*}) + S(\varepsilon+\omega)\zeta_2{}^{A} - S(\varepsilon)\zeta_2{}^{R}.$$

The loop $P_2(\mathbf{q},\omega)$ is connected with the vertex $(\Gamma_c)_{11}^2$ by a relation analogous to (35):

$$P_2{}^{A}(\mathbf{q},\omega) = -2^{1/2}i\int \frac{d\varepsilon}{2\pi} (N_{imp}V_{e\text{-}imp}^2)^{-1} (\Gamma_c)_{11}{}^{2}$$

$$= -\nu\left[\frac{1}{3} - \frac{i\omega\tau\zeta_2{}^{*}}{1-\zeta_0{}^{*}}\right]. \tag{39}$$

Substituting the resulting value of $P_2(\mathbf{q},\omega)$ into (33), we obtain

$$(\Gamma_s)_{ij}{}^{2} = g\left[1 - 3x^2 + \frac{i\omega\tau(\zeta_0{}^{*} - 3\zeta_2{}^{*})}{1-\zeta_0{}^{*} - i\omega\tau\zeta_0{}^{*}}\right] K_{ij}{}^{2}. \tag{40}$$

After renormalizaiton of the vertex Γ_s by the impurity ladder, which contains both vector and scalar parts, we obtain the vertex Γ_f:

$$(\Gamma_f)_{12}{}^{2} = (\Gamma_f)_{21}{}^{2} = (\Gamma_f)_{22}{}^{2} = 0, \tag{41}$$

$$(\Gamma_f)_{11}{}^{2} = \frac{[S(\varepsilon+\omega) - S(\varepsilon)]\,(\zeta_0{}^{*} - 3\zeta_2{}^{*})}{2^{1/2}(1-\zeta_0{}^{*} - i\omega\tau\zeta_0{}^{*})} g$$

$$= \frac{4}{5\cdot 2^{1/2}}[S(\varepsilon+\omega) - S(\varepsilon)]g.$$

We note that the last equality in (41) is correct for any relation between ω and Dq^2.

The electron self-energy diagrams, including the vertices Γ_s and Γ_f, are shown in Fig. 8. It is easily seen that Σ_3 reduces to zero upon integration with respect to angle, while $\Sigma_2 = 0$ due to the index structure of the vertices. Calculating the collision integral including Σ_1, we obtain a result which coincides precisely with formula (30) which we evaluated in

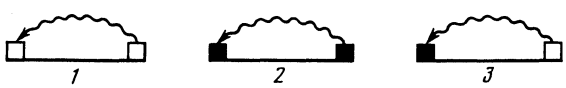

FIG. 8. Diagrams for the electron self-energy in the comoving system: □ is the vertex Γ_s, ■ is the vertex Γ_f.

the laboratory system. For transverse phonons we need take into account only the vertex Γ_0, and again we obtain agreement with the results of Section 2.

4. ENERGY RELAXATION IN IMPURE SUPERCONDUCTORS

In superconductors, energy relaxation of electrons which are excited near the transition temperature T_c takes place with an energy transfer of $\delta\varepsilon \sim T$. In the low-temperature region $T \ll \Delta$, where Δ is the gap in the electronic excitation spectrum; as before, the scattering processes are accompanied by energy transfers $\delta\varepsilon \sim T$. For recombination processes, however, $\delta\varepsilon \sim \Delta$. In order to simplify things, we limit ourselves to the case in which both of these processes are modified by impurity scattering, which obtains when the inequality

$$\max\{\Delta\tau,\ T\tau\} \ll u/v_F, \tag{42}$$

is satisfied; this is meaningful when $ql \ll 1$, where q is a characteristic momentum transfer. In the opposite limit, the electron-phonon collision integral was evaluated in Refs. 20–22.

The Green's functions in the superconductor are matrices both in the Keldysh indices and in the Nambu indices

$$\hat{G}^R(\mathbf{p},\varepsilon) = (\hat{G}^A(\mathbf{p},\varepsilon))^* = \frac{1}{\xi_p^2 - (E^R)^2}\,[-\xi_p\hat{\sigma}_z + \varepsilon^R\hat{1} + \Delta^R\hat{\sigma}_x], \tag{43}$$

where we introduce the notation

$$\varepsilon^R = \varepsilon(1+i/2\tau\xi_\varepsilon), \quad \Delta^R = \Delta(1+i/2\tau\xi_\varepsilon),$$
$$(E^R)^2 = (\varepsilon^R)^2 - (\Delta^R)^2, \quad \xi_\varepsilon = (\varepsilon^2 - \Delta^2)^{1/2}\operatorname{sign}\varepsilon, \quad |\varepsilon| > \Delta. \tag{44}$$

Here and below we will adhere to the following customary notation: the indices for the Green's function and vertices correspond to the Keldysh space, while the expansion in Pauli matrices is in the Nambu space.

If there is no imbalance in the populating of the branches of the spectrum, then the quasiparticle distribution function is odd in ξ_p, and G^C takes the form

$$\hat{G}^C(\mathbf{p},\varepsilon) = S(\varepsilon)\,[\hat{G}^A(\mathbf{p},\varepsilon) - \hat{G}^R(\mathbf{p},\varepsilon)]. \tag{45}$$

As is clear from the results of Sec. 2 and 3, a description of the electron-phonon interaction in impure metals in the comoving system offers fewer difficulties than the same description in the laboratory system; therefore, in the case of impure superconductors we also will use the comoving system, preserving the same notation for the vertices as in Sec. 3. The bare Coulomb and electron-phonon interaction vertices in the Nambu indices correspond to the matrix σ_z:

$$\hat{w}_{ij}^l = K_{ij}^l \otimes \hat{\sigma}_z, \quad (\hat{\Gamma}_0)_{ij}^l = (\Gamma_0)_{ij}^l \otimes \hat{\sigma}_z, \tag{46}$$

where $\otimes$ denotes the direct product.

Renormalization of the bare vertex $\hat{w}_{ij}^l$ due to impurities in the ladder approximation leads to the matrix equation (12), which in the case of superconductors is represented for each phonon index by a system of sixteen linear equations. The solution of the system so obtained to zero order in $(p_F l)^{-1}$ takes the form

$$\hat{W}_{22}^{\,1} = \frac{1}{2^{1/2}}\left\{\frac{1 - {}^1\!/_2\eta_0(1-A)}{1-\eta_0}\hat{\sigma}_z - \frac{1}{2}\frac{\eta_0 B}{1-\eta_0}i\hat{\sigma}_y\right\},$$
$$\hat{W}_{12}^{\,2} = \frac{1}{2^{1/2}}\left\{\frac{1 - {}^1\!/_2\eta_{0+}^*(1+A)}{1-\eta_{0+}^*}\hat{\sigma}_z - \frac{1}{2}\frac{\eta_{0+}^* B}{1-\eta_{0+}^*}i\hat{\sigma}_y\right\},$$
$$\hat{W}_{22}^{\,2} = 0, \quad \hat{W}_{21}^{\,2} = (\hat{W}_{12}^{\,2})^*, \tag{47}$$
$$\hat{W}_{12}^{\,1} = S(\varepsilon)\,(\hat{W}_{22}^{\,1} - \hat{W}_{12}^{\,2}), \quad \hat{W}_{21}^{\,1} = -S(\varepsilon+\omega)\,(\hat{W}_{22}^{\,1} - \hat{W}_{21}^{\,2}),$$
$$\hat{W}_{11}^{\,2} = S(\varepsilon)\hat{W}_{21}^{\,2} - S(\varepsilon+\omega)\hat{W}_{12}^{\,2} + [S(\varepsilon+\omega) - S(\varepsilon)]\,(\hat{W}_{22}^{\,1})^*,$$

where we introduce the notation

$$\eta_n = \zeta_n(q,\Omega), \quad \eta_{n+} = \zeta_n(q,-\Omega_+), \quad \Omega = \xi_{\varepsilon+\omega} - \xi_\varepsilon,$$
$$\Omega_+ = \xi_{\varepsilon+\omega} + \xi_\varepsilon, \tag{48}$$

where ζ_n is determined by the formula (14) and (15), while the coherent factor similarly is equal to

$$A = [\varepsilon(\varepsilon+\omega) - \Delta^2]/\xi_\varepsilon\xi_{\varepsilon+\omega}, \quad B = \Delta\omega/\xi_\varepsilon\xi_{\varepsilon+\omega}. \tag{49}$$

The renormalization of the vector vertex $x^2 K_{ij}^l \otimes \hat{\sigma}_z$ due to the impurities is carried out by the same methods as were used for a normal metal, and lead to the vertex $\hat{\Gamma}_c$. As before, we write down the solution only for phonon index 2:

$$(\hat{\Gamma}_c)_{22}^{\,2} = 0, \quad (\hat{\Gamma}_c)_{21}^{\,2} = [(\hat{\Gamma}_c)_{12}^{\,2}]^*$$
$$= \frac{1}{2^{1/2}}\frac{\eta_{2+}}{1-\eta_{0+}}\,[(1-A)\hat{\sigma}_z - Bi\hat{\sigma}_y],$$
$$(\hat{\Gamma}_c)_{11}^{\,2} = \frac{1}{2^{1/2}}\hat{\sigma}_z\left\{\frac{(1+A)\eta_2^*}{1-\eta_0^*}\,[S(\varepsilon+\omega) - S(\varepsilon)]\right.$$
$$\left. + (A-1)\left[\frac{S(\varepsilon+\omega)\eta_{2+}^*}{1-\eta_{0+}^*} - \frac{S(\varepsilon)\eta_{2+}}{1-\eta_{0+}}\right]\right\}$$
$$+ \frac{1}{2^{1/2}}i\hat{\sigma}_y\frac{B}{1-\eta_0}\left[\frac{\eta_{2+}^* - \eta_2^*}{1-\eta_{0+}^*}S(\varepsilon+\omega) - \frac{\eta_{2+} - \eta_2^*}{1-\eta_{0+}}S(\varepsilon)\right]. \tag{50}$$

Taking into account the electron-electron interaction leads to a screened vertex

$$(\hat{\Gamma}_s)_{ij}^{\,l} = (\Gamma_s)_{ij}^{\,l} \otimes \hat{\sigma}_z,$$

where for $(\Gamma_s)_{ij}^l$ formula (33) is correct if we understand $P_0(\mathbf{q},\omega)$ and $P_2(\mathbf{q},\omega)$ to be loops calculated for the superconducting state. Relations (35) and (39), which allow us to express the loops through vertices, take the following form in the superconducting state:

$$P_0^A(\mathbf{q},\omega) = -i\,\mathrm{Sp}\frac{\hat{\sigma}_z}{2^{1/2}}\int\frac{d\varepsilon}{2\pi}\,(N_{imp}V_{e\text{-}imp}^2)^{-1}\hat{W}_{11}^{\,2}, \tag{51}$$

$$P_2^A(\mathbf{q},\omega) = -i\,\mathrm{Sp}\frac{\hat{\sigma}_z}{2^{1/2}}\int\frac{d\varepsilon}{2\pi}\,(N_{imp}V_{e\text{-}imp}^2)^{-1}(\hat{\Gamma}_c)_{11}^{\,2}. \tag{52}$$

As in the normal metal, the vertices W_{11}^2 and $(\Gamma_c)_{11}^2$ must be calculated to first order in $(p_F l)^{-1}$; the vertex W_{11}^2 to zero-order in $(p_F l)^{-1}$ is given by formula (47), while the first order correction takes the form

$$(\hat{W}')_{11}^{\,2} = \hat{\sigma}_z\frac{S(\varepsilon)}{2\tau}\left[\frac{1}{(\varepsilon_F + E^A)^{1/2}} - \frac{1}{(\varepsilon_F - E^A)^{1/2}}\right]$$
$$\times\ \frac{2(A-1) - (1+A)(1 - A\eta_{0+})}{(1-\eta_{0+})^2}\,. \tag{53}$$

For calculating the contribution $(W')_{11}^2$ in (51) the important region is $\varepsilon \sim \varepsilon_F$; therefore, if we neglect terms of order

Δ/ε_F, the result of integrating coincides with the analogous contribution for the normal metal, and so

$$P_0^A(\mathbf{q},\omega)=-\nu\left\{1+\frac{i\tau}{4}\int d\varepsilon\left[\frac{(1+A)\eta_0^*}{1-\eta_0^*}(S(\varepsilon+\omega)-S(\varepsilon))\right.\right.$$
$$\left.\left.+(A-1)\left(\frac{\eta_{0+}^*}{1-\eta_{0+}^*}S(\varepsilon+\omega)-\frac{\eta_{0+}}{1-\eta_{0+}}S(\varepsilon)\right)\right]\right\}. \quad (54)$$

By a transformation analogous to (52), we obtain

$$P_2^A(\mathbf{q},\omega)$$
$$=-\nu\left\{\frac{1}{3}+\frac{i\tau}{4}\int d\varepsilon\left[\frac{(1+A)\eta_2^*}{1-\eta_0^*}(S(\varepsilon+\omega)-S(\varepsilon))\right.\right.$$
$$\left.\left.+(A-1)\left(\frac{\eta_{2+}^*}{1-\eta_{0+}^*}S(\varepsilon+\omega)-\frac{\eta_{2+}}{1-\eta_{0+}}S(\varepsilon)\right)\right]\right\}. \quad (55)$$

Formula (54) is a complete solution to the problem of screening the Coulomb potential in an impure superconductor. We remark that for $\mathbf{q}=0$ formula (54) coincides with the expression $P_0^A(0,\omega)$ for a pure superconductor.[23]

Let us cast the vertex $\hat{\Gamma}_s$ in the form

$$(\Gamma_s)_{ij}^2=gK_{ij}^2\otimes\hat{\sigma}_z[1-3x^2+(3P_2^A-P_0^A)/P_0^A]. \quad (56)$$

The expression $(3P_2^A-P_0^A)/P_0^A$ is small, just as in the normal metal, since $3\eta_2-\eta_0=\frac{4}{15}(ql)^2$. Actually, let us assume, for example, $\omega\ll Dq^2$; then

$$\frac{3P_2^A-P_0^A}{P_0^A}=\frac{-{}^1/_5 i\omega\tau J}{1+{}^3/_4 i\omega\tau J/(ql)^2},$$
$$J=\frac{1}{\omega}\int\frac{d\varepsilon}{2\pi}2A[S(\varepsilon+\omega)-S(\varepsilon)]. \quad (57)$$

The integral J is well known; it is the ratio of the absorption coefficients of sound for the superconducting and normal states. For arbitrary frequencies ω we have $J\sim 1$, and so $3P_2-P_0\ll P_0$ remains correct in other limiting cases also.

After renormalizing the vertex $\hat{\Gamma}_s$ with the impurity ladders, we obtain the vertex $\hat{\Gamma}_f$:

$$(\hat{\Gamma}_f)_{22}^2=0,\quad (\hat{\Gamma}_f)_{21}^2=[(\hat{\Gamma}_f)_{12}^2]^*$$
$$=\frac{\eta_{0+}-3\eta_{2+}}{2^{3/2}(1-\eta_{0+})}[(1-A)\hat{\sigma}_z-Bi\hat{\sigma}_y], \quad (58)$$
$$(\hat{\Gamma}_f)_{11}^2=\frac{\hat{\sigma}_z}{2^{3/2}}\left\{\frac{(1+A)(\eta_0^*-3\eta_2^*)}{1-\eta_0^*}[S(\varepsilon+\omega)-S(\varepsilon)]\right.$$
$$\left.+(A-1)\left[\frac{\eta_{0+}^*-3\eta_{2+}^*}{1-\eta_{0+}^*}S(\varepsilon+\omega)-\frac{\eta_{0+}-3\eta_{2+}}{1-\eta_{0+}}S(\varepsilon)\right]\right\}$$
$$+\frac{1}{2^{3/2}}\frac{i\hat{\sigma}_y B}{1-\eta_0}\left[\frac{(\eta_{0+}^*-\eta_0^*)-3(\eta_{2+}^*-\eta_2^*)}{1-\eta_{0+}^*}S(\varepsilon+\omega)\right.$$
$$\left.-\frac{(\eta_{0+}-\eta_0^*)-3(\eta_{2+}-\eta_2^*)}{1-\eta_{0+}}S(\varepsilon)\right].$$

The kinetic equation for the distribution function $n_\varepsilon=(1/2)[S(\varepsilon)+1]$, which is odd in ξ_p, and which describes the energy relaxation, has the form

$$\frac{\partial n_\varepsilon}{\partial t}=\frac{1}{2\pi\nu}\frac{\xi_\varepsilon}{\varepsilon}\frac{1}{2}\mathrm{Sp}\int\frac{d\mathbf{p}}{(2\pi)^3}[\hat{G}^A(\mathbf{p},\varepsilon)-\hat{G}^R(\mathbf{p},\varepsilon)]$$
$$\times\{(2n_\varepsilon-1)[\hat{\Sigma}^A(\mathbf{p},\varepsilon)-\hat{\Sigma}^R(\mathbf{p},\varepsilon)]-\hat{\Sigma}^\varepsilon(\mathbf{p},\varepsilon)\}, \quad (59)$$

where the trace is taken in order to extract from the matrix collision integral the components proportional to the unit matrix.[2]

The collision integral corresponding to the self-energy (Fig. 8) reduces to zero when integrated over angles. The analysis of the second diagram requires a special investigation, since in the superconducting case the vertex $\hat{\Gamma}_f$ has a complicated structure: $(\Gamma_f)_{12}^2\neq 0, (\Gamma_f)_{21}^2\neq 0$. Writing out the collision integral in the Keldysh indices and including equation (12), we can express I_2 in terms of the vertices dressed by the impurity ladders and in this way we can perform the integration over $\mathbf{p}$ in the collision integral. Let us write down the part of the collision integral proportional to $(2N_\omega+1)[S(\varepsilon+\omega)-S(\varepsilon)]$:

$$I_2\propto\mathrm{Sp}\,[(\hat{\Gamma}_f)_{12}^2(\mathbf{q},\omega)\hat{\sigma}_z\hat{\gamma}_{11}^2(-\sigma,-\omega)\hat{\sigma}_z-(\hat{\Gamma}_f)_{21}^2(-\mathbf{q},-\omega)$$
$$\times\hat{\sigma}_z\hat{\gamma}_{11}^2(\mathbf{q},\omega)\hat{\sigma}_z+(\hat{\Gamma}_f)_{11}^2(\mathbf{q},\omega)\hat{\sigma}_z\hat{\gamma}_{21}^2(-\mathbf{q},-\omega)\hat{\sigma}_z$$
$$-(\hat{\Gamma}_f)_{11}^2(-\mathbf{q},-\omega)\hat{\sigma}_z\hat{\gamma}_{12}^2(\mathbf{q},\omega)\hat{\sigma}_z],$$

where $\hat{\gamma}$ is the "bare" vertex corresponding to $\hat{\Gamma}_f$, as in equation (12), i.e., the vertex obtained from Γ_s by correcting it with one impurity line. Taking into account (58), we obtain $I_2(\varepsilon)=0$, as in the normal metal. In this way, the collision integral for a single longitudinal phonon branch, also for the superconducting case, is determined by the diagram Σ_1:

$$I_1(\varepsilon)=\frac{8}{5}\frac{\tau}{\pi^2}\int dq\frac{d\omega}{2\pi}q^2g^2R(\varepsilon,\omega)$$
$$\times\left[1-\frac{\Delta^2}{\varepsilon(\varepsilon+\omega)}\right]\frac{\varepsilon+\omega}{\xi_{\varepsilon+\omega}}\mathrm{Im}\,D^R(\mathbf{q},\omega). \quad (60)$$

Calculations show that there are relations between the contributions from various phonon branches which enter into the energy relaxation which are similar to those in the normal metal (Section 2). In order to separate out the processes of scattering and recombination in the previous formula it is necessary to go from the electronic representation to the quasiparticle represention.[2] The quasiparticle scattering time τ_s and the recombination time τ_R are computed just as in the case $ql\gg 1$. Near the transition temperature and in the quasiparticle energy region $\Delta\ll\varepsilon\sim T$ of interest for energy relaxation, the times τ_s and τ_R are of the same order as the energy relaxation times for electrons in a normal metal. In the low-temperature region $T\ll\Delta$, the quasiparticle recombination time τ_R for $\varepsilon=\Delta$ is determined by the expression

$$\frac{1}{\tau_R(\Delta,T)}=\pi^{1/2}\left(\frac{2\Delta}{T_c}\right)^{1/2}\left(\frac{T}{T_c}\right)^{1/2}\exp\left(-\frac{\Delta}{T}\right)\frac{4\pi^2}{\tau_\varepsilon(T_c)} \quad (61)$$

while the scattering time for quasparticles with this energy is

$$\frac{1}{\tau_S(\Delta,T)}=\Gamma\left(\frac{9}{2}\right)\zeta\left(\frac{9}{2}\right)\left(\frac{T_c}{2\Delta}\right)^{1/2}\left(\frac{T}{T_c}\right)^{9/2}\frac{4\pi^2}{\tau_\varepsilon(T_c)}, \quad (62)$$

where $\Gamma(x)$ is the gamma function, $\zeta(x)$ is the Riemann zeta function, and τ_ε is the energy relaxation time in the normal metal (31).

5. DISCUSSION OF RESULTS

The calculation of the energy relaxation time τ_ε for electrons in an impure metal under the conditions $T\tau\ll u/v_F$

has led to the result (31), both for the laboratory and comoving coordinate systems. The qualitative dependence of τ_ε on temperature and mean free path of electrons can be obtained from a relation which can be interpreted as an energy-balance equation

$$C_e/\tau_\varepsilon = C_{ph}/\tau_{ph}, \tag{63}$$

where C_e and C_{ph} are the heat capacities for electrons and phonons and τ_{ph} is the electron-phonon relaxation time. For arbitrary values of ql, the time τ_{ph} was calculated by Pippard.[24] The microscopic derivation of the formula for τ_{ph}, presented in the comoving[4] and laboratory[12] frames, confirms the results of Ref. 24. The time τ_{ph} was determined in Refs. 4, 12 as the imaginary part of the phonon self-energy, which is justified since for phonons the drift time coincides with the energy relaxation time. According to Ref. 24, for $ql \ll 1$ we have $\tau_{ph} \sim \omega_q^2 \tau$; assuming that $\omega_q \sim T$, we obtain an expression for τ_ε which coincides in order to magnitude with (30). If, however, following Refs. 5 and 6 we assume that $\tau_\varepsilon^{-1} \propto T^2/p_F^2 ul$, then from (63) it follows that τ_{ph} does not depend on the frequency: $\tau_{ph}^{-1} \sim m/M\tau$.

The errors in Refs. 5 and 6 are first of all connected with the determination of the energy relaxation time from the imaginary part of the electronic self-energy, and in addition are related to the omission of a number of important diagrams. In Ref. 7, the energy relaxation time was found to coincide with (30) in order to magnitude. However, this result was obtained by using a number of groundless and incorrect assumptions. All diagrams which contained the vertices renormalized by the impurity ladder were omitted, as not giving any contribution to the energy relaxation; the diagram Σ_{10} was not investigated at all. After retaining the contributions I_7 and I_8 mentioned in Section 2, the authors of Ref. 7 were led to the conclusion that "the energy exchange between the electron and phonon subsystems is determined purely by the electron-phonon interaction," i.e., by taking into account only the diagram Σ_1 in Fig. 7. It turns out that this is equivalent to throwing away the transverse phonon part. As our calculation shows, inelastic electron scattering off of impurities gives a substantial contribution to the energy relaxation.

In an impure superconductor, when condition (42) is fulfilled, both the scattering and recombination processes for excitations are modified due to impurities. In this case, the collision integral has the canonical form (60) with an effective matrix element dependent on the electron mean free path. This implies that the ratio of the attenuation coefficients for sound in the superconducting and normal states in the low-frequency limit $ql \ll 1$ is given by the BCS formula both for longitudinal and transverse phonons, in agreement with Refs. 16 and 25. The times τ_s and τ_R are larger compared with the same quantities in a pure superconductor by the factor $(ql)^{-1} \gg 1$, where q is a characteristic momentum transmitted by a phonon.

The weakening of the electron-phonon and strengthening of the electron-electron interaction[1] with increasing impurity concentration is reflected in the form of the quasiparticle distribution function in impure or thin superconducting films. This dissipative process whereby one quasiparticle decays into three due to the electron-electron interaction essentially limits stimulated superconductivity in such films. A high impurity concentration can cause the inequality $\tau_{ee} \ll \tau_\varepsilon$ to be satisfied at helium temperatures (τ_{ee} is the electron-electron scattering time[1]). In this case, the nonequilibrium distribution function for quasiparticles is a Fermi function with some effective electron temperature.[26]

Let us discuss briefly the question of what influence a change in the electron-phonon interaction in impure semiconductors will have on the value of the critical temperature T_c. The value of T_c is determined by the interaction of electrons with virtual phonons. A broad range of phonon wave vectors, right up to the Debye wave vector, is important in this interaction. As we have pointed out, in impure superconductors only the interaction of electrons with long-wavelength phonons ($ql \ll 1$) is changed. In view of the smallness of this energy region, we can assume that the corrections to T_c which appear are of order $(p_F l)^{-1} \ll 1$. In this connection, we note that a rise in T_c due to the increase in the effective matrix element for the interaction of electrons with transverse phonons in an impure semiconductor appears from our point of view to be erroneous. The model used in Ref. 27 and in the present work, which is based on the deformation interaction, gives a correct description only for long-wavelength transverse phonons. For short-wavelength phonons, one must take into account the transverse electromagnetic field, as was done in Ref. 12, for the microscopic calculation of the attentuation coefficient of sound. A systematic calculation shows that the matrix element of the electron-phonon interaction, which is proportional to the attenuation coefficient of sound, decreases monotonically both for longitudinal and transverse phonons as the parameter ql decreases. Consequently, the effect predicted in Ref. 27 is impossible. For this same reason, there cannot be any nonmonotonic temperature dependence on τ_ε^{-1}, of the sort obtained in Ref. 28.

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