Quasiparticles and impurities in the quantum Hall effect

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A unified explanation is proposed for the integral as well as the fractional quantization of the Hall effect. It is based on the hypothesis of superfluidity of the two-dimensional electron fluid in a magnetic field and of quasiparticle pinning by the impurities. The number of quasiparticles and their charge are determined only by the requirements of gauge invariance and nondegeneracy of the ground state. In a certain range of field strengths and concentrations, the charge of the superfluid electron fluid is equal to the number of flux quanta through the system, multiplied by a certain rational number with an odd denominator. Arguments similar to those of the Landau superfluidity theory show that a two-dimensional electron fluid in a magnetic field is superfluid under some simple assumptions regarding its spectrum. Some general hypotheses can be substantiated and the wave functions of the ground and weakly excited states can be found explicitly within the framework of the short-range force model for not too high impurity concentrations. The wave functions of an impurity-free system are the same as those conjectured by Laughlin. In the presence of impurities the quasi-holes are localized at the impurities in a certain range of variation of the parameters of the system.

1. INTRODUCTION

The discovery by von Klitzing *et al.*[1] of the integral quantum Hall effect has produced a stream of theoretical work.[2–9] Laughlin[2] showed that if the Fermi level lies in the gap between delocalized states, and the ground state of the electron system is nondegenerate, gauge invariance leads to the values of σ_{xy} being multiples of the quantity $\sigma_0 \equiv e^2/2\pi\hbar$. This idea was developed by Niu and Thouless,[3] and also by Avron and Seiler.[4] They showed that the quantum Hall effect follows from topological considerations under the conditions mentioned above. In a number of works,[5–7] a model of electrons drifting along equipotential lines of a slowly changing random potential has been studied. It follows from intuitive geometrical considerations that there is only a single percolation threshold in such a system, which is identified with the localized levels. Apenko and Lozovik[7] calculated the value of σ_{xy} within the framework of this model and found that it is a multiple of σ_0 if the Fermi level lies above the percolation threshold.

Levine *et al.*[8] considered the problem from the point of view of localization theory. Impurities were modelled as a random potential with the properties of white noise. They found that the renormalization group equations in a strong magnetic field contain a specific term of a topological nature which leads to delocalization close to $\sigma_{xy} = (n + 1/2)\sigma_0$. Khmel'nitskiĭ[9] proposed a simple picture of integral paths of the renormalization group equations in the plane σ_{xx}, σ_{xy}. It follows from it that at absolute zero, the magnetic field dependence of σ_{xy} has the form of a stepwise curve and the transitions between the steps take place discontinuously. This approach gives an explanation of the enormous accuracy in the quantization of σ_{xy}.

In 1982, Tsui *et al.*[10] discovered the fractional quantum Hall effect which has been studied in a number of experiments.[11–14] It was clear from the very beginning that this phenomenon could not be explained without taking account of electron interactions. It is obviously not associated with the formation of a Wigner crystal, since such a crystal would be pinned by impurities. Laughlin[15] proposed a trial wave function describing a uniform incompressible electron fluid with an occupancy of the first Landau level $\nu = 1/m$, where m is any odd number. He argued that excitations in such a system carry fractional charge, equal to $\pm e/m$, and finite energy. Haldane[16] and Halperin[17] expressed the idea that these excitations could in turn reconstitute the incompressible Laughlin fluid for certain values of ν. They associated these states with the appearance of steps on the Hall characteristic $\sigma_{xy}(H)$ at the values $\sigma_{xy} = (p/q)\sigma_0$, where p and q are whole numbers, with q odd.

Only interelectron interaction was considered by Laughlin, by Haldane and by Halperin. Besides, it was apparent that in the case of fractional quantization, the interaction with impurities was considerable. Without it, the steps on the $\sigma_{xy}(H)$ curve would not arise.

The aim of the present work is to take account simultaneously both of the interaction between electrons and of the interaction between electrons and impurities. Laughlin's idea of excitations with fractional charge and Landau's concept of dissipationless motion of a quantum fluid enable us to justify in a unified fashion both the integral and the fractional quantization of the Hall conductivity.

We shall find exact functions of the ground state and rigorously calculate σ_{xy} for a simplified model of the interaction between electrons. The simple physical picture which arises is easily generalized for more realistic situations. A short communication of the results of the present work has been published earlier.[18]

2. DESCRIPTION OF THE METHOD

We shall regard the interactions of electrons both with impurities and with one another as small compared with the separation of the Landau levels. This enables us to confine

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ourselves to a discussion of only the first Landau level. If interaction is neglected and if $\nu < 1$, the states of this level are multiply degenerate. The interaction of electrons with one another and with impurities lifts the degeneracy totally or partially. The problem consists in finding the ground state for a given whole number of particles N and number of impurities N_i. It is assumed that the distribution of impurities is random and fixed. We will characterize the external magnetic field by the number of flux quanta $2S$ through the area occupied by the system. This value of $2S$ determines the number of states in the first Landau level. We use the axial gauge in which the one-electron wave function is of the form

$$\psi_k(z) = z^k \exp(-|z|^2/4), \tag{1}$$

where $z = x + iy$, x and y are the electron coordinates and the magnetic length is taken equal to unity. The integer k numbers the states of the first Landau level and ranges from zero to $2S$. Physically this limitation on k means the electrons do not go outside the limits of a circle of radius equal to $2S$, where they are confined by a compensating charge. The limitation on the value of k thus follows from the boundary conditions.

The many-electron wave function $\psi(z_1,\ldots,z_N)$ should be an antisymmetrical linear combination of the products of one-electron wave functions. It follows from Eq. (1) that the most general form of $\psi(z_1,\ldots,z_N)$ is

$$\psi(z_1,\ldots,z_N) = P(z_1,\ldots,z_N)\exp\left(-\sum_{l=1}^{N}|z_l|^2/4\right), \tag{2}$$

where $P(z_1,\ldots,z_N)$ is an antisymmetric polynomial of degree not more than $2S$ in each of the variables.

We replace the Coulomb interaction between electrons by a short-range repulsive pair potential $V(r)$. Its radius of action a is assumed small compared with the magnetic length l_H We represent this potential in the form of a series in the small parameter $(a/l_H)^2$:

$$V(r) = V_0\delta(\mathbf{r}) + V_1\Delta\delta(\mathbf{r}) + V_2\Delta^2\delta(\mathbf{r}) + \ldots, \tag{3}$$

where $V_n \sim \int V(r) r^{2n} d^2r$. Because of the Pauli principle, the leading term in the expansion of Eq. (3) does not contribute to the energy of the electrons.

We will also consider the interaction between electrons and impurities as short-range. The potential produced by the k th impurity is

$$W(\mathbf{r}) = W_k\delta(\mathbf{r}-\mathbf{r}_k). \tag{4}$$

The value of W_k will be considered random and as taking positive as well as negative values. We will divide impurities into weak and strong, depending on the relation between $|W_k|$ and V_1.

3. THE GROUND STATE OF AN IMPURITY-FREE SYSTEM

We start with a simplified problem in which there are no impurities. This problem was solved by the present authors[19] and by Trugman and Kivelson.[20] We limit ourselves to the terms in Eq. (3) proportional to V_1. Any wave function of the form

$$\psi = Q(z_1,\ldots,z_N)\prod_{j<k}^{N}(z_j - z_k)^3 \exp\left(-\sum_{l=1}^{N}|z_l|^2/4\right) \tag{5}$$

then gives a zero and, consequently, minimum possible energy. Here $Q(z_1,\ldots,z_N)$ is a symmetrical polynomial. Let s be the leading power in the polynomial in any of the variables. The maximum power of any z in Eq. (5) is equal to $s + 3(N-1)$. The inequality

$$s + 3(N-1) \leqslant 2S \tag{6}$$

follows from the boundary conditions we have taken.

Since $s \geqslant 0$, it follows from Eq. (6) that wave functions of the form of Eq. (5) are only possible if $\nu \leqslant 1/3$ is satisfied. In particular, for $\nu = 1/3$ or, more exactly, on satisfaction of the relation

$$3(N-1) = 2S \tag{7}$$

s goes to zero while the polynomial $Q(z_1,\ldots,z_N)$ goes to a constant value. For this particular value of the occupancy, the ground state is nondegenerate and it corresponds to the wave function ψ_L proposed by Laughlin:[15]

$$\psi_L = \prod_{j<k}^{N}(z_j - z_k)^m \exp\left(-\sum_{l=1}^{N}|z_l|^2/4\right), \tag{8}$$

where $m = 3$. In actual fact, any other wave function corresponding to the same values of N and S should have only simple zeros when corrdinates of two particles coincide. In the opposite case, these zeros should have been at least triple because of the Pauli principle. The wave function would then coincide with ψ_L. The operation on a wave function with simple zeros with a Hamiltonian of the form

$$H_1 = {}^1/_2 V_1 \sum_{j\neq k}\Delta\delta(\mathbf{r}_j - \mathbf{r}_k) \tag{9}$$

does not transform it to zero. The energy of the ground state per particle for any pair interaction $V(r)$ is of the form

$$\varepsilon = \int V(r) K(r)\, d^2r, \tag{10}$$

where $K(r)$ is the two-particle correlation function. This latter is obtained from $|\psi(z_1,\ldots,z_N)|^2$ by integrating over all variables except two. If the zeros of $\psi(z_1,\ldots,z_N)$ are simple, then $K(r)$ has a double zero for $r = 0$. The Laplacian eliminates this double zero so that the energy of such a state is strictly positive.

Laughlin showed that the wave fucntion of Eq. (8) describes a spacially homogenous incompressible electron fluid. He made use of a formal analogy between the quantity $|\psi_L|^2$ and a Gibbs distribution of a two-dimensional one-component plasma. Haldane[16] proposed another approach. He considered electrons on a sphere of large radius. A uniform magnetic field was produced by a monopole situated at the center of the sphere. According to the Dirac condition, the magnetic flux $2S$ through the sphere should be a whole number. The analog of the one-electron function of Eq. (1) over the sphere is of the form

$$\psi_k = u^{2S-k}v^k, \qquad (11)$$
$$u=\cos(\theta/2)\exp(i\varphi/2), \quad v=\sin(\theta/2)\exp(-i\varphi/2),$$

where θ and φ are angular spherical coordinates. The quantities u and v are the components of a spinor with spin 1/2. The quantities ψ_k can be regarded as spin functions of magnitude S, which has a projection $S-k$ along the z axis. A spin of magnitude S can be associated with each electron, directed to that point on the sphere where the electron is situated. The analogy between the electron wave functions in a magnetic field and the spin wave functions was first found by Peres.[21] The analog of the Laughlin wave function for the sphere is of the form

$$\psi_L = \prod_{j<k}^{N} (u_j v_k - u_k v_j)^m. \qquad (12)$$

The uniform electron density on the sphere is evident in this case, since any factor in the product of Eq. (12) describes a pair of particles with zero total spin.

For $\nu < 1/3$, the energy of the ground state remains equal to zero, while the degree of degeneracy increases together with the quantity

$$q \equiv s_{max} = 2S - 3(N-1). \qquad (13)$$

We now consider the case of $\nu > 1/3$. In this case it is not possible to construct the wave function in the form of Eq. (5) for the reason indicated above. All the zeros of the wave function, if the electron coordinates coincide will therefore be simple, while the energy corresponding to such a state is positive. We will show that on addition of one particle to the system with occupancy $\nu = 1/3$ [see Eq. (7)], the energy of the ground state changes by a finite amount. It is convenient for this to make use of the spherical representation of Haldane. In this representation, any pair interaction that depends only on the distance between the particles can be represented in the form

$$\hat{V} = \sum_{I=0}^{2S} V_{2S-I}\hat{P}_I, \qquad (14)$$

where $\hat{P}_I$ is the projection operator for the state of a pair of spins with total spin I. The total spin $I = 2S$ corresponds to a pair of electrons lying at one and the same point on the sphere. In Eq. (3) the term $V_0\delta(r)$ corresponds to it. This value of the total spin is forbidden by the Pauli principle. The value of the spin $I = 2S-1$ is allowed and the term $V_1\Delta\delta(\mathbf{r})$ in Eq. (3) corresponds to it. The equality of the energy of the Laughlin state to zero indicates that there is no pair with total spin $2S-1$. However, the energy of the ground state is already not zero for $N = 2S/3 + 2$. Consequently, in this state there is only one pair with moment $2S-1$. The minimum value of the energy of the ground state is not less than V_1.

We showed that the derivative of the energy of the ground state $E(N)$ with respect to the number of particles has a finite discontinuity for an occupancy $\nu = 1/3$. Using the symmetry between particles and holes, the following relation can be obtained:

$$E(N) = E(2S-N) + 4(N-S)V_1. \qquad (15)$$

It follows from this that the derivative dE/dN at $\nu = 2/3$ has such a discontinuity. The inclusion of the next term in the expansion of Eq. (3) $V_2\Delta^2\delta(\mathbf{r})$ only changes the value of the jump of the derivative dE/dN at the points $\nu = 1/3$, $2/3$ insignificantly. Similar changes arise on including the term $V_3\Delta^3\delta(\mathbf{r})$. The energy of the ground state is equal to zero for $0 \leqslant \nu \leqslant 1/5$. In the interval $1/5 < \nu < 1/3$ the degeneracy is partially lifted. The energy of the ground state in this range is of the order of $V_3 N$. The Laughlin wave function with $m = 5$ is the accurate wave function of the ground state for $\nu = 1/5$. For an occupancy $\nu = 1/3$ corrections of order V_3/V_1 appear in the Laughlin function with $m = 3$. These corrections lead to a weak separation of the triple zeros. The discontinuity in the derivative dE/dN at $\nu = 1/3$ changes weakly, by an amount of the order of V_3. In return, a new jump in the derivative dE/dN appears at $\nu = 1/5$; its magnitude is of the order of V_3. On including further terms of the series, a succession of jumps in the derivative dE/dN appears at the points $\nu = 1/m$ and identical jumps for $\nu = 1 - 1/m$.

4. EXCITATIONS IN AN IMPURITY-FREE SYSTEM

We shall consider a state in which $2S = 3(N-1) + 1$, i.e., the flux increased by one quantum relative to the Laughlin state with $\nu = 1/3$. The wave function of the ground state with the given values of S and N has the form

$$\psi = \prod_{i=1}^{N} (z_i - \zeta)\psi_L(z_1, \ldots, z_N), \qquad (16)$$

where ζ is an arbitrary complex number.

Such a wave function was proposed by Laughlin,[15] and is exact in the model with short-range interaction in those cases when the wave functions ψ_L for $\nu = 1/m$ are exact. Laughlin showed that the wave function of Eq. (16) describes a quasi-hole with charge $|e|/m$ positioned at the point ζ. To show this he used the aforementioned analogy with the statistics of a one-component plasma.

The magnitude of the charge of the quasiparticles is the key factor of the theory. We will, therefore, give another derivation of the magnitude of the charge of a quasi-hole, proposed by Arovas and Schrieffer.[22] We shall consider a large closed path Γ, inside which there are n electrons. After the passage of a quasi-hole along this path in an anticlockwise direction, the phase of the wave function ψ of Eq. (16) changes by $-2\pi n$. On the other hand, it follows from gauge invariance that the phase change of the wave function on the passage of a particle with charge e^* along the path Γ in a counterclockwise direction is equal to $(e^*/\hbar c)\Phi$, where Φ is the magnetic flux through the area bounded by the path Γ. For an occupancy $\nu = 1/m$, m flux quanta $\Phi_0 = 2\pi\hbar c/e$ occur. We now equate the flux changes obtained by the two different means $-2\pi n = (e^*/\hbar c)nm\Phi_0 = 2\pi n m e^*/e$. It follows from this that $e^* = |e|/m$. A concrete form of the quasi-hole wave function was used in this derivation. We do not know the exact form of the quasi-electron wave function

and cannot write the exact wave functions for other values of ν, for example $\nu = p/q$ with $p > 1$.

It is useful to draw a general conclusion not based on specific properties of the wave function. Suppose that for some rational concentration $\nu = p/q$ the wave function of the ground state is nondegenerate. We shall change the number of magnetic flux quanta by ± 1. Let the additional or lacking flux quantum be associated with a local change in electron density. As any electron encircles the localized quantum, the phase of the wave function changes by $\pm 2\pi$. It follows from this through the reciprocity, that as the localized quantum encircles round an electron, the phase changes by $\mp 2\pi$. By repeating the considerations given above, we obtain that the charge associated with a localized flux quantum is equal to $\mp e\nu$. In particular, for $\nu = 1/m$ we obtain the already known result.

The situation is more complicated when $\nu = p/q$. In this case a new quasiparticle is formed by the change in the number of flux quanta and the number of particles or quasiparticles by ∓ 1. Let us consider the simplest case $\nu = 2/3$. We shall define the quasi-electron as the result of an increase in the number of flux quanta and the number of particles by one. Such a quasiparticle has charge $e/3$. The same result could be obtained starting from the symnmetry between states with occupancy ν and $1 - \nu$.

For $\nu = 4/3$, one Landau level is occupied and it is obvious in advance that the charge of a quasiparticle is $\mp 1/3$, as for the first Landau level. In our scheme these quasiparticles are obtained simultaneously by a reduction or increase in the number of electrons and in the number of flux quanta by one.

As a more complicated example we shall consider an occupancy $\nu = p/(mp \pm 1)$, where m is an odd and p is an even integer. According to the proposal of Haldane[16] and of Halperin,[17] such occupancies are obtained from the Laughlin states with $\nu = 1/m$ if quasi-holes or quasi-electrons form states similar to the Laughlin state with intrinsic concentration $1/p$. New quasiparticles with the minimum possible charge are obtained simultaneously by an increase in the number of quanta and of quasi-electrons with charge e/m by one. The charge of the new quasiparticle is equal to $\pm e/m(mp \pm 1)$. In particular, for an occupancy $\nu = 2/5$ it is equal to $e/15$, while for $\nu = 2/7$ the magnitude is $e^* = e/21$.

In a similar way, the minimum charge can be obtained of an excitation for a step of the general form $\nu = p/q$. The result is as the following: the number ν can be decomposed into a continuous fraction in which the first denominator is m, an odd number, while the remainder are even. Cutting off this fraction at the k th step, we obtain the convergent $\nu_k = p_k/q_k$, where all q_k are odd whole numbers. Let $\nu_k = \nu = p/q$. Then

$$|e^*/e| = 1/q_{n-1}q_n. \tag{17}$$

The quasiparticles described, above with fractional charge, arising for rational concentrations, relate to an inter-electron potential of general form. In the case of a short-range potential of the special form of Eq. (9), singularities arise only near $\nu = 1/3$ and $2/3$, while the charges of the quasiparticles are equal to $\pm e/3$.

We shall consider the excited states for $\nu = 1/3$, i.e., $2S = 3(N - 1)$. It is natural to suggest that they consist of one quasi-electron and one quasi-hole. We have already seen that for the potential of Eq. (9) the energy of a quasi-hole is equal to zero. The energy of a quasi-electron is, apparently, equal to some positive constant of the order of V_1. We cannot exclude the existence of an exciton, i.e. of a bound state of a quasi-hole and a quasi-electron, but in this case too the gap in the spectrum will be of the order of V_1.

5. THE GROUND STATE AND THE SPECTRUM OF A SYSTEM WITH IMPURITIES

If the inter-electron interaction is given by Eq. (9), then in the absence of impurities and for $\nu < 1/3$ the ground state of the electron system is degenerate. Impurities reduce the degeneracy. For impurities repelling an electron, for which $W_k > 0$, conditions are favorable for the electron wave function to go to zero at those points where impurities are situated. We shall consider the state with the following wave function:

$$\psi = \prod_{\substack{1 \leqslant j \leqslant N \\ 1 \leqslant k \leqslant N_+}} (z_j - \zeta_k)\psi_L(z_1, \ldots, z_N), \tag{18}$$

where $\zeta_k = x_k + iy_k$ is the complex coordinate of the k th impurity, N_+ is the number of repulsive impurities. The energy of this state is zero to first order in $(a/l_H)^2$. That energy of electrons which do not interact with one another is zero in the field of point impurities was first found by Baskin *et al.*[23] The wave function of Eq. (18) can only be constructed for a small number of repulsive impurities $N_+ < q = 2S - 3(N - 1)$. In the case of a strong inequality, the ground state remains multiply degenerate. For $N_+ = q$, the ground state described by Eq. (18) is nondegenerate.

The wave function of Eq. (18) allows of a simple interpretation: a Laughlin quasi-hole with charge $|e|/3$ is localized at each repulsive impurity. The degeneracy at $N_+ < q$ is due to the fact that the number of quasi–holes exceeds the number of impurities and the energy is independent of the position of the extra quasi-holes.

The most intersting case is when $N_+ > q$. The q quasi-holes present are localized at the q impurities in a way as to achieve the minimum energy. We consider the impurities to be weak ($W_k < V_1$), so that they do not destroy the Laughlin part of the wave function. The appearance of pairs of quasiparticles with subsequent localization of a quasi-hole at an impurity is energetically unfavorable.

We assume that $N_+ \ll N$. The wave function of the ground state can then be written in the form

$$\psi = \prod_{\substack{1 \leqslant j \leqslant N \\ 1 \leqslant k \leqslant q}} (z_j - \zeta_k)\psi_L(z_1, \ldots, z_N). \tag{19}$$

We have assigned numbers from unity to q to those impurities at which holes are positioned. The energy of such a state is

$$E = \sum_{q < k \leqslant N_+} W_k \rho(\zeta_k), \tag{20}$$

where $\rho(\zeta)$ is the electron density at point ζ. The electron density becomes zero where holes are located. At a distance of the order of l_H from the hole, $\rho(\zeta)$ becomes a constant approximately equal to $N/2S$. Values of the density at those impurities at which there is not a hole go into the sum of Eq. (20). Since the mean distance between impurities is much greater than l_H, $\rho(\zeta_k)$ in Eq. (20) can be replaced by $N/2S$. We then obtain

$$E \approx \frac{N}{2S} \sum_{q<k\leqslant N_+} W_k, \tag{21}$$

in the ground state, holes are localized at those impurities at which W_k is the largest possible.

In the interval of values $0 \leqslant q \leqslant N_+$, the ground state is thus nondegenerate. Its wave function has the form of Eq. (19). Weak impurities with small concentration, attracting electrons, do not alter this result.

For an occupancy $\nu > 1/3$, the explicit form of the wave function in the presence of impurities is unknown, but the physical picture can be drawn by analogy with the case of $\nu < 1/3$. We imagine the ground state as a state with $|q|$ quasi-electrons localized at attracting impurities. Such a state is nondegenerate if the number of attracting impurities $N_- \geqslant |q|$.

We shall consider the nature of the excitation spectrum in the range $-N_- < q < N_+$, within which there are $|q|$ localized quasiparticles in the ground state. In a large system, an arbitrarily close configuration can be found at a sufficiently large distance for any impurity configuration. The ground state is therefore separate from the nearest excited state by an energy interval inversely proportional to the dimensions of the system. The spectrum is quasi-continuous. However, in order to obtain the nearest excited state from the ground state, it is necessary to interchange a large number of quasiparticles from single impurities to others. For this, it is essential to overcome a macroscopically large potential barrier. The nearest excited state is not reachable from the ground state by tunneling. On the other hand, it is possible to transpose a single quasiparticle from impurity to impurity. Such excitations are separated from the ground state by a gap. The gap can be made small if a quasiparticle is displaced by a large distance. This situation is reminiscent of that which arises in the Mott hopping conductivity. Finally, an excitation with a quasiparticle breaking away from an impurity is possible. A gap equal to the minimum binding energy of a quasiparticle to an occupied impurity νW_q corresponds to such an excitation. For $q = N_+$ or $q = -N_-$, this gap goes to zero.

6. THE HALL CURRENT

Suppose that a uniform electric field **E** acts on the electrons, perpendicular to a magnetic field **H**. In the absence of impurities this field would lead to a drift of the whole system of interacting electrons with a velocity

$$\mathbf{v} = c[\mathbf{E} \times \mathbf{H}]/H^2. \tag{22}$$

The Hall conductivity is then given by the classical formula

$$\sigma_{xy} = Nec/H. \tag{23}$$

In order to understand how impurities alter this result, we go over to a frame of reference moving with the velocity of Eq. (22). In this system the electric field is zero. The electrons forming the Laughlin fluid are, therefore, stationary in this coordinate system. All the impurities, including the impurities capturing quasiparticles, travel with velocity $-\mathbf{v}$.

A moving bare impurity cannot drag with it the electron fluid, at least at sufficiently small velocities. In fact for such a drag a quasi-electron–quasi-hole pair must be created in the fluid, which requires a finite energy Δ_0 (in the model considered $\Delta_0 \sim V_1$). One of the quasiparticles must then be associated with the impurity while the other must be stationary. Association of a quasiparticle with an impurity decreases the reduced mass of the impurity since it reduces the interaction of the impurity with the electron fluid. The effective mass of a bound quasiparticle can thus be negative. We shall denote the binding energy of a quasiparticle with an impurity by Δ_1. The criterion for a nondissipative motion of an impurity through the quantum fluid, by analogy with the Landau criterion for a superfluid liquid is:

$$\Delta_0 - \Delta_1 > -m^* v^2/2, \tag{24}$$

where m^* is the effective mass of a bound quasiparticle. If $m^* > 0$, then a critical velocity exists at which, if exceeded, spontaneous pair production and dragging of charge by impurities set in. In the model with the short-range potential of Eq. (9), the effective mass of a quasi-hole is $-m_e/3$, where m_e is the electron mass.

We shall now consider a moving impurity on which a quasiparticle was localized in the ground state. The simplest way of exciting this complex is to detach the quasiparticle to the free state. For this it is necessary to surmount a gap Δ_1 equal to the binding energy of a quasiparticle with an impurity. The criterion for such a process to occur is of the form

$$\Delta_1 < m^* v^2/2. \tag{25}$$

Processes of spontaneous creation of a pair with removal of a quasiparticle from an impurity turn out to be mutually exclusive.

Another means of an impurity-quasiparticle complex can be excited is by hopping of a quasiparticle to another impurity. The kinetic energy does not change in such a jump. The interaction energy of a quasiparticle with the impurities can only increase, since the initial state was the ground state. At absolute zero such an excitation is, therefore, impossible.

Until the velocity v exceeds the critical velocity, all the quasiparticles thus move together with the impurities capturing them, giving a current

$$\mathbf{j}_q = -e^* |q| \mathbf{v}. \tag{26}$$

We now go over to the laboratory coordinate system. If there were no captured quasiparticles, the current in it would be equal to $Ne\mathbf{v}$. Taking account of the captured quasiparticles changes the total current which becomes equal to

$$\mathbf{j} = Ne\mathbf{v} + \mathbf{j}_q = e(N + |e^*/e| q)\mathbf{v}. \tag{27}$$

Near the occupancy $\nu = 1/m$ the following relations are satisfied

$$|e^*/e| = 1/m, \tag{28}$$

$$q = 2S - m(N-1). \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (27) we obtain in the limit of large N

$$j = e\nu 2S/m. \tag{30}$$

Using the well-known formula for the number of states at the Landau level $2S = eH/2\pi\hbar c$ and Eq. (22) for ν, we find

$$\sigma_{xy} = e^2/m2\pi\hbar. \tag{31}$$

Equation (27) can be interpreted in a somewhat different way. In the ground state, $|q|$ quasiparticles are captured by impurities and do not take part in the drift motion. In view of the law of conservation of the charge, the moving electron fluid changes its charge by an amount $-e^*q$.

The current of quasiparticles is determined by Eq. (26) only if the following condition is satisfied

$$-N_- \leqslant q \leqslant N_+. \tag{32}$$

This equation gives the maximum possible width of the step on the Hall characteristic for point impurities.

For $m = 1$, Eq. (31) gives the integral quantum Hall effect. In a number of works[24-26] the existence of steps on the Hall characteristic of a system of electrons interacting only with impurities was demonstrated with the help of the Kubo formua and some special relations for the matrix elements of the one-electron operators in a magnetic field. Our approach enables us to avoid these complicated calculations and to obtain an analogous result for the fractional quantum Hall effect.

We shall now consider the occupancy ν in the close vicinity of some rational ν_0 with an odd denominator. In the state with occupancy ν there is some number q of excess of lacking flux quanta compared with the nondegenerate state with occupancy ν_0:

$$q = 2S - N/\nu_0. \tag{33}$$

As was shown in Sec. 4, a charge $e\nu_0$ is associated with each excess flux quantum. Suppose that all these charges, however they are shared among the quasiparticles, are localized at impurities. The moving part of the charge has, because of the conservation law, the value

$$Ne + q\nu_0 e = 2S\nu_0 e. \tag{34}$$

In this situation σ_{xy} takes the value

$$\sigma_{xy} = \nu_0 e^2/2\pi\hbar. \tag{35}$$

The boundaries of the step are, in general, determined by conditions different from Eq. (32). This is connected with the fact that for an arbitrary ν_0, the number of quasiparticles coincides with $|q|$. For example, for the case considered by us $\nu_0 = p/(mp \pm 1)$, the number of quasiparticles is equal to $mp|q|$.

7. WHAT DEPENDS ON THE MODEL?

The main question which still arises concerning impurity-free systems is: how are the occupancies $\nu = 1/m$ selected? For short-range interelectron repulsion, the answer to this question is clear. In this case that wave function is energetically favorable which has the maximum order of zero upon coincidence of the electron coordinates. A zero of the mth order is only possible for an occupancy $\nu = 1/m$. It is just for this reason that the points $\nu = 1/m$ are the ground states. As was shown in Sec. 3, the ground state of an impurity-free system at these points is nondegenerate and is separated from the excited states by an energy gap. These properties are evidently preserved in a certain range of the parameter (a/l_H) of the order of unity, i.e. for potentials of finite radius. On the other hand, in a strong magnetic field, a Coulomb potential can in some sense be considered as a potential of finite action radius $\sim l_H$. The point is that a direct Coulomb interaction between electrons is compensated by their interaction with the positively charged background, while the remaining part of the interaction is determined by the wave-function overlap which decreases exponentially with the square of the distance between the particles. One can easily be convinced of this by considering the exact expression for the electron interaction energy:

$$\varepsilon = e^2 \int K(r) \frac{d^2r}{r} - e^2 \int \frac{d^2r}{(r^2+d^2)^{1/2}}, \tag{36}$$

where d is the distance from the electron layer to the layer of compensating charge.

Equation (36) can be rewritten in the form

$$\varepsilon = e^2 \int (K(r)-1) \frac{d^2r}{r} + e^2 \int \left(\frac{1}{r} - \frac{1}{(r^2+d^2)^{1/2}} \right) d^2r. \tag{37}$$

The first term in Eq. (37) contains the exponentially decreasing factor $K(r) - 1$; only this term depends on the form of the wave function.

The spectial nature of the states with $\nu = 1/m$ for an arbitrary interaction can be understood as follows. As has been shown, the addition of one flux quantum leads to the appearance of a localized charge $-e/m$. It is natural to expect that m quanta and one electron with a charge e form a bound state. The addition to the system of one electron and m quanta can therefore give a nondegenerate ground state with the same occupancy, not differing from the former.

Corrections to the wave function of the ground state, associated with transitions to another Landau level, are of the relative order of magnitude $V/\hbar\omega_c$, where V is the characteristic size of the interelectron interaction and ω_c is the cyclotron frequency. The Laughlin wave functions are distorted, but the nondegeneracy of the ground state and the gap in the excitation spectrum remain in a certain range of values of $V/\hbar\omega \sim 1$.

So far we have only considered weak impurities which do not destroy the Laughlin part of the wave function. We shall discuss the influence of strong impurities. Suppose that there are N_+ weak and N^s_+ strong impurities in the system, repelling electrons, and also N_- and N^s_- strong attractive impurities. The strong impurities partially destroy the

Laughlin wave function for $\nu = 1/m$, giving rise to dissipation of $\max(N^s_+, N^s_-)$ quasi-electron–quasi-hole pairs. The N^s_+ quasi-holes and N^s_- quasi-electrons are localized at the strong impurities. The excess quasiparticles which result from the breakup of pairs of number equal to $|N^s_+ - N^s_-|$, settle at the weak impurities. The Hall conductivity σ_{xy} for $\nu = 1/m$ will be equal to $(1/m)(m^2/2\pi\hbar)$ only for the case when the number of weak impurities of the corresponding sign is sufficient for localization of the $|N^s_+ - N^s_-|$ excess quasiparticles. This does not mean that a step of size $\sigma_{xy} = (1/m)(e^2/2\pi\hbar)$ on the Hall characteristic disappears in the case when the number of weak impurities is less than $|N^s_+ - N^s_-|$. Strong impurities can only lead to a shift in this step. The new boundaries of the step are determined by the inequality [cf. Eq. (32)]

$$-N_- - N_-^s + N_+^s \leqslant q \leqslant N_+ + N_+^s - N_-^s. \tag{38}$$

The inequalities (38) give only the maximum possible size of the steps. Firstly, it is essential to take account of the competition of the Laughlin states associated with other rational occupancies ν. Secondly, impurities are not, in general, point impurities and can capture not one but several quasiparticles.

In the more general case, a random potential acting on the electrons, and not impurities, can be considered. The quasi-electrons and quasi-holes will then be localized in the wells and humps of this potential. The number of quasiparticle states in the random potential, and correspondingly the width of a step, depends on its correlation properties.

CONCLUSION

We will discuss the possibilities of an experimental verification of the theory. The most interesting is the feasibility of experimentally finding fractional quasiparticle charges. We consider the characteristic energies for electron systems with real Coulomb interaction. The energy necessary for breakup of a quasi-electron–quasi-hole pair is of the order of magnitude

$$\varepsilon_1 \sim e^{*2}/\varkappa l_H^*, \tag{39}$$

where $\varkappa$ is the permittivity.

This quantity plays the same role as does the parameter V_1 for short-range forces. In particular, ε_1 determines the maximum possible binding energy of a quasiparticle with a weak impurity. Impurities with such binding energy are filled with quasiparticles in the first place. Equation (39) determines the edge of the absorption band in the middle of the step on the Hall characteristic. By comparing the absorption edge for steps with different ν, the ratio of the effective charges of the excitations can be found. For this purpose it is sufficient to note that $\varepsilon_1 \sim e^{*5/2}$. For example, for two close values $\nu = 1/3$ and $\nu = 2/5$, the values of ε_1 differ 56-fold.

It is interesting to note that ε_1 is related to the average electron Coulomb energy $\varepsilon_0 = e^2/\nu c l_H$ as $(e^*/e)^{5/2}$. For example, for a step with $\nu = 1/3$, this ratio is 0.06. This result does not agree badly with other numerical estimates of the activation energy.[15,27] The magnitude of ε_1 also determines the temperature up to which one or another step will exist.

It was found experimentally[28] that when the magnetic field is changed σ_{xy} deviates from its value on a step, but then rapidly returns to it. Such a behavior agrees with our picture of localized quasiparticles, in that σ_{xy} is determined only by their overall number. The distribution of quasiparticles over the impurities can be non-equilibrium and varies as a function of the external conditions. At the instant of the rearrangement, the value of σ_{xy} will change, but after the quasiparticles have again been localized, the value of σ_{xy} must resume its previous value.

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