Surface second sound and kinetic phenomena on the interface between solid helium and He II below 1 K

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The spectrum, damping, and interaction of the major modes of macroscopic surface oscillations at an interface (either atom-smooth or atom-rough) between solid helium and He II are investigated with allowance for surface second sound and surface dissipation. The conditions are found for the existence of weakly damped second-sound surface waves on the interfaces of He II with solid helium and with a perfect crystalline dielectric. It is shown that if both the surface and bulk thermal excitations are in the hydrodynamic-oscillation regime the second-sound surface wave is a weakly inhomogeneous bulk second-sound wave in one of the contacting phases. It is pointed out that the surface second sound is a two-dimensional gas of surface thermal excitations only when the surface and bulk excitations are respectively in the hydrodynamic and ballistic oscillation regimes. The interaction and mutual transformation of Rayleigh and crystallization waves on an atom-rough interface are analyzed with allowance for the effective surface mass that leads to the acoustic character of the crystallization-wave spectrum in the high-frequency region. The damping of the crystallization wave is investigated with allowance for the effective surface mass and for the surface dissipative coefficients. The contribution of the thermal crystallization waves having an acoustic spectrum to the surface tension of an atom-rough solid helium–He II interface is found.

The interface between solid and superfluid ^{4}He is an unusual object, since it can be regarded below 1.2 K as either an atom-smooth or an atom-rough surface.[1–5] The difference between the microscopic structures of the two types of surface leads to a difference between the basic types of macroscopic surface oscillations of the interface. At low surface-oscillation amplitude the atom-smooth face remains stable to recrystallization so that the lowest frequencies on such faces are possessed by oscillations in the Rayleigh-Stonely wave that propagates along the solid-liquid interface.[6] On an atom-rough surface there exist two types of low-frequency oscillations: a weakly damped crystallization–melting wave[7,8,3] and a Rayleigh wave that propagates along the free surface of the solid helium.[9,10]

The quanta of the aforementioned surface waves are elementary excitations of the corresponding boundary and contribute to the surface energy, entropy, "normal density," and others. Since bulk second sound can propagate in solid helium below 1 K (Ref. 11), one can posit in this temperature region the existence of weakly damped surface second sound on the solid helium–He II interface. Just as in the case of a free surface of pure He II (Ref. 12), on the interface considered the surface second sound can comprise a secondary wave in a gas surface excitations, accompanied by temperature-temperature oscillations at a practically immobile boundary. The present paper is devoted to calculation, in the hydrodynamic approximation, of the damping and interaction of the major modes of surface oscillations with account taken of the surface second sound and of the surface dissipation on both types of an hcp solid-helium crystal.[1)] Attention is called to the fact that the developed theory of surface second sound on atom-smooth faces of solid helium is applicable also to the case of the corresponding boundary (free or with He II) of a perfect dielectric crystal, in which conditions for propagation of bulk second sound are realized (e. g., LiF and NaF, Ref. 13). An experimental study of surface second sound both in solid helium and in other dielectrics would yield additional information on the thermodynamic and kinetic properties of the boundary.

ANALYSIS OF BOUNDARY CONDITIONS AND BULK EQUATIONS OF MOTION

The spectrum, damping, and interaction of the surface oscillations are obtained by solving the bulk equations of motion for the first second in both media, with boundary conditions imposed on the mass flux, momentum (and quasimomentum), entropy and energy, as well as the condition that the phases be in thermodynamic equilibrium (on atom-rough faces). The boundary conditions for the bulk equations must take into account the transport of the surface entropy, momentum, etc., i. e., allowance must be made for the surface hydrodynamics of the considered boundary.[12–14] We shall discuss the basic equations of surface hydrodynamics and the bulk equations of motion which are necessary to solve the boundary-value problem.

Let the interface coincide with the plane $z = 0$, the crystal occupy the region $z > 0$, and the indices μ, $\nu = 1,2$ number the coordinate axes in the tangent plane. We introduce the quasimomentum per unit surface, i_μ, for which we can use in the linear aproximation the expansion

$$i_\mu = v_{\mu\nu}^{(s)} V_\nu^{(s)}, \tag{1}$$

where $V_\nu^{(s)}$ is the drift velocity of the surface excitations in a Lagrangian coordinate frame fixed in the underformed lattice, $V\mu\nu^{(s)}$ us the tensor of the surface normal density, calculated in the general case in analogy with the bulk phonon-hydrodynamic tensor of the normal density in crystals (see, e. g., § 15 of Ref. 15). In the simplest case of a face that is

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isotropic in its own plane we have $V\mu\nu^{(s)} = V^{(s)}\delta\mu\nu$.

The boundary condition that the quasimomentum flux be continuous takes, with allowance for i_μ, the form

$$\frac{\partial i_\mu}{\partial t}-\frac{\partial \alpha}{\partial x_\mu}+\frac{\partial \tau_{\mu\nu}^{(s)}}{\partial x_\nu}=\Pi_{z\mu}^{(2)}-\Pi_{z\mu}^{(1)}, \tag{2}$$

where α is the surface free energy of the boundary, $\tau\mu\nu^{(s)}$ is the dissipative part of the surface-quasimomentum flux tensor, Π_{ik} is the bulk quasimomentum flux tensor (the phonon-hydrodynamic tensor in a crystal), and the indices 1 and 2 refer to the crystal and liquid, respectively. Surface umklapp processes can be neglected in (2).

The boundary condition of continuity of the entropy flux, with allowance for the transport of the surface entropy σ_s, is given by

$$\frac{\partial \sigma_s}{\partial t}+\frac{\partial}{\partial x_\mu}(\sigma_s V_\mu^{(s)})=S_2(v_{nz}-\dot{H})-S_1(V_z-\dot{\zeta}), \tag{3}$$

where S is the entropy per unit volume, $\mathbf{V}$ is the drift velocity of the phonons in the crystal, $\mathbf{V}n$ is the velocity of the normal motion in He II, and ζ the displacement of the interface from the equilibrium position in the course of recrystallization. On an atom-smooth face at low amplitude of the surface oscillations we have $\zeta = 0$. On an atom-rough boundary the total displacement H consists of the elastic component u_z and of the displacement ζ in the course of the recrystallization, $H = u_z + \zeta$ ($\mathbf{u}$ is the vector of elastic displacement in a solid).

We define the interface, as usual, as satisfying the condition that there be no surface particles: $N_s = 0$.[16] The following expansions are then valid in Eqs. (2) and (3) on the considered interface:

$$\delta\alpha=\frac{\partial\alpha}{\partial T}\delta T_s=-\sigma_s\delta T_s, \qquad \delta\sigma_s=\frac{\partial\sigma_s}{\partial T}\delta T_s, \tag{4}$$

where T_s is the surface temperature.[14]

It is necessary to add to Eqs. (1)–(4) the conditions that connect the surface quantities ($V_\mu^{(s)}$ and T_s) with the corresponding bulk quantities. We write down these conditions together with the conditions for thermodynamic equilibrium of the phases for an atom-rough surface with allowance for the capillary effects and for the fact that the surface dissipative coefficient be finite. These conditions take the form

$$\begin{aligned}
\frac{\Delta\mu}{T}&=-a_{11}\rho_1\dot{\zeta}+a_{12}TS_2(v_{nz}-\dot{H})+a_{13}TS_1(V_z-\dot{\zeta}),\\
\frac{T_2-T_s}{T^2}&=-a_{21}\rho_1\dot{\zeta}+a_{22}TS_2(v_{nz}-\dot{H})+a_{23}TS_1(V_z-\dot{\zeta}),\\
\frac{T_s-T_1}{T^2}&=-a_{31}\rho_1\dot{\zeta}+a_{32}TS_2(v_{nz}-\dot{H})+a_{33}TS_1(V_z-\dot{\zeta}),\\
v_{n\mu}-\dot{u}_\mu-V_\mu^{(s)}&=a_{44}\Pi_{z\mu}^{(2)}+a_{45}\Pi_{z\mu}^{(1)},\\
V_\mu^{(s)}-V_\mu&=a_{54}\Pi_{z\mu}^{(2)}+a_{55}\Pi_{7\mu}^{(1)}.
\end{aligned} \tag{5}$$

The matrix a_{ik} in (5) is positive-definite and symmetric (by virtue of Onsager's symmetry principle for the kinetic coefficients);

$$\Delta\mu=\frac{1}{\rho_1}\left\{\rho_1\sigma_1\delta T_1-\rho_1\sigma_2\delta T_2+\delta\Pi_{zz}^{(2)}\left(\frac{\rho_1}{\rho_2}-1\right)-\tilde{\alpha}_{\mu\nu}\frac{\partial^2 H}{\partial x_\mu\,\partial x_\nu}+M_s\ddot{\zeta}\right\} \tag{6}$$

is the difference between the chemical potentials (per unit mass) of the contiguous phases; σT is the deviation of the temperature from the equilibrium value (on the melting curve); $\delta\Pi_{zz}^{(2)} = \delta p_2 + \tau_{zz}^{(2)}$, where σp_2 is the deviation of the pressure in the liquid and $\tau_{zz}^{(2)}$ is the dissipative part of the tensor of the momentum-flux in He II; σ is the entropy per unit mass, ρ is the mass density ($S = \rho\sigma$),

$$\tilde{\alpha}_{\mu\nu}=\alpha\delta_{\mu\nu}+\partial^2\alpha/\partial\varphi_\mu\partial\varphi_\nu$$

is the surface rigidity tensor (we assume hereafter for simplicity that $\tilde{\alpha}_{\mu\nu} = \tilde{\alpha}\delta_{\mu\nu}$; M_s is the effective surface mass of the atom-rough boundary[17–19];

$$-\rho_1\dot{\zeta}\equiv\rho_1(\dot{u}_z-\dot{H})=j_z-\rho_2\dot{H} \tag{7}$$

is the flux of matter through the boundary. In the continuity equation (7), $j = \rho_n\mathbf{V}_n + \rho_s\mathbf{V}_s$ is the mass-flow vector in He II, while δ_n and δ_s are the densities of the normal and superfluid components in the He II.

In connection with the boundary conditions (5), we call attention to the following circumstance. If surface excitations are neglected (for example, in a crystallization wave), the system (5) reduces to (see Refs. 20–23):

$$\begin{gathered}
\Delta\mu/T=aJ_M+b\bar{J}_E,\\
(T_2-T_1)/T^2=bJ_M+c\bar{J}_E, \qquad v_{n\mu}-\dot{u}_\mu-V_\mu=d\Pi_{z\mu},
\end{gathered} \tag{5a}$$

where $J_M = -\rho_1\dot{\zeta}$ is the mass flux,

$$\bar{J}_E=TS_2(v_{nz}-\dot{H})=TS_1(V_z-\dot{\zeta})$$

is the energy flux referred to the equilibrium chemical potential,

$$\Pi_{z\mu}=\Pi_{z\mu}^{(1)}=\Pi_{z\mu}^{(2)}$$

is the quasimomentum flux through the boundary,

$$a=a_{11}, \quad b=a_{12}+a_{13}, \quad c=a_{22}+2a_{23}+a_{33}, \quad d=a_{44}+2a_{45}+a_{55}. \tag{5b}$$

The physical meaning of the surface dissipative coefficients is clear from the form of (5a):

$$\rho_1 T\left(a-\frac{b^2}{c}\right)=\left.\frac{\Delta\mu\rho_1}{J_M}\right|_{\Delta T=0}\equiv\frac{1}{K}$$

is the reciprocal kinetic growth coefficient of the atom-rough boundary;

$$cT^2=\left.\frac{\Delta T}{\bar{J}_E}\right|_{J_M=0}\equiv R_K$$

is the Kapitza thermal resistance of the interface; $d > 0$ is the surface dissipative coefficient and is inversely proportional to the coefficient of passage of the phonons through the boundary.[22,23] A detailed kinetic-theory calculation of the growth coefficient K (and of the crossover coefficient b) of an atom-rough solid helium—He II interface is given in Refs. 22 and 23, and the Kapitza thermal resistance of such an interface was investigated in Refs. 9, 19, 24, and 25. There is at present no kinetic-theory calculation of all the components of the matrix a_{ik} of the system (5). Therefore, on the basis of the available estimates of a, b, c, and d and of relations (5b), we assume hereafter that

$$a_{11}=a\sim\rho_n c_L/\rho_1{}^2 T, \quad |a_{12}|\sim|a_{13}|\sim|b|\sim(\rho_1 c_L T)^{-1}, \tag{5c}$$
$$a_{22}\sim|a_{23}|\sim a_{33}\gtrsim c\sim(\alpha_{L_s}S_2 c_L T^2)^{-1},$$
$$a_{44}\sim|a_{45}|\sim a_{55}\gtrsim d\sim(\alpha_{L_s}\rho_n c_L)^{-1},$$

where c_L is the speed of sound in the liquid, α_{L_s} is the coefficient of transition of the phonons from the liquid into the solid, with $\alpha\sim(T/\Theta)^2$ in the temperature region $T\lesssim 0.2$ K (Refs. 9, 19, 24) and Θ the Debye temperature.

The boundary conditions on the normal stresses σ_{zi} in a solid follow from the continuity of the momentum flux through the boundary:

$$\sigma_{zz}+\Pi_{zz}^{(2)}=-g\left(\frac{\partial^2 u_z}{\partial x_\nu{}^2}+\frac{\partial^2\zeta}{\partial x_\nu{}^2}\right)+P_s(\ddot{u}_z+\ddot{\zeta}), \tag{8}$$

$$\sigma_{z\mu}+\Pi_{z\mu}^{(2)}=-(g+h)\frac{\partial^2 u_\mu}{\partial x_\nu{}^2}+P_s\ddot{u}_\mu. \tag{9}$$

In Eqs. (8) and (9) are taken into account the capillary effects in the boundary conditions of elasticity theory[26,27] in the high-symmetry-face approximation; $g_{\mu\nu}=g\delta_{\mu\nu}$ is the surface-stress tensor; $h=h_{11}$ is one of the excess surface elastic moduli; P_s is the density of the excess surface mass. Note that the surface mass P_s that enters in the mechanical-equilibrium conditions (8) and (9) is an independent characteristic of the surface, together with the effective surface mass M_s that enters in the thermodynamic-equilibrium conditions (5) and (6). The effective mass M_s can be determined, for example, by the connected hydrodynamic mass of the delocalized growth steps and of the kinks on them and the atom-rough solid helium–He II interfaces (see Ref. 17). At the same time, the excess mass P_s on the considered interfaces is determined by the excess number N_s of the surface particles: $P_s=mN_s$, where m is the helium-atom mass. With the interface so defined, this quantity vanishes identically ($P_s=0$), whereas the effective surface mass M_s has a nonzero value that can be estimated from experimental data (see Refs. 18 and 19).[2)]

The set of boundary conditions (1)–(9) enables us to determine the spectrum, damping, and interaction of all types of long-wave surface oscillations with account taken of the surface second sound and of the surface dissipation on both types of solid helium-He II interface. This set of equations can also be used in boundary value problems of anomalous reflection of sound and mutual transformation of first and second sounds[21,28,29] on atom-rough faces below 1 K, when the solid He can no longer be regarded as a thermal insulator.

We precede the consideration of the boundary-value problems by a brief analysis of the bulk equations of motion in both phases. In the liquid phase the equations of motion are described by two-velocity hydrodynamics of superfluid liquids.[30] Motion in a solid is described by the equations of elasticity-theory[31] and of phonon hydrodynamics.[15]

In both media we can neglect as small the thermal expansion that determines the interaction between the volume first and second sounds.

The general solution of bulk equations of motion of He II for the velocity of the normal component can be represented in the form

$$\mathbf{v}_n=\mathbf{v}_{nl}+\mathbf{v}_{nL}+\mathbf{v}_{nt}, \tag{10}$$

where $\mathbf{v}_{nl}$ is the velocity of the normal motion in a longitudinal first-sound wave, $\mathbf{v}_{nL}$ is the velocity in a longitudinal second-sound wave, and $\mathbf{v}_{nt}$ is the velocity in a transverse viscous wave:

$$\rho_n\dot{\mathbf{v}}_{nt}=\eta_2\Delta\mathbf{v}_{nt}, \quad \operatorname{div}\mathbf{v}_{nt}=0,$$

where η_2 is the first-viscosity coefficient of He II.

Let the x axis be directed along the wave vector $\mathbf{k}$ of the surface wave and let ω be the wave frequency. In a liquid occupying the half-space $z<0$ the relations between v_{nx} and v_{nz} in the surface waves are then given by

$$v_{nlx}=kp_2'/\omega\rho_n, \quad v_{nlz}=-i\gamma_{l2}p_2'/\omega\rho_n, \tag{11}$$

$$v_{nLx}=\frac{\rho_s\sigma_2}{\rho_n}\frac{kT_2'}{\omega}, \quad v_{nLz}=-i\frac{\rho_s\sigma_2}{\rho_n}\frac{\gamma_{L2}T_2'}{\omega}, \tag{12}$$

$$v_{ntz}=-ikv_{ntx}/\gamma_{t2}, \tag{13}$$

where T_2' and p_2' are the oscillations of the temperature and pressure in the wave,

$$\gamma_{l2}=(k^2-\omega^2/c_L{}^2)^{1/2}, \quad \gamma_{L2}=(k^2-\omega^2/u_2{}^2)^{1/2}, \quad \gamma_{t2}=(-i\omega\rho_n/\eta_2)^{1/2},$$

c_L and u_2 are the velocities of the first and second sounds in He II with allowance for their viscous damping,[30] and $U_{t2}=(-i\omega\eta_2/\rho_n)^{1/2}$ is the velocity of the viscous wave in He II (the condition $\omega^2\gg U_{t2}{}^2k^2$) must be satisfied for weakly damped oscillations).

The solutions of the equations of the elasticity-theory for the sound oscillations and of phonon hydrodynamics of an isotropic solid can also be resolved into purely longitudinal and transverse components:

$$\mathbf{u}=\mathbf{u}_l+\mathbf{u}_t, \tag{14}$$

with the following relations between u_x and u_z in the surface wave on the crystal ($z>0$)

$$u_{lx}=-\frac{ik}{\varkappa_l}u_{lz}, \quad u_{tx}=-\frac{i\varkappa_t}{k}u_{tz}, \tag{15}$$

where

$$\varkappa_l=(k^2-\omega^2/c_l{}^2)^{1/2}, \quad \varkappa_t=(k^2-\omega^2/c_t{}^2)^{1/2},$$

c_l and c_t are the velocities of the longitudinal and transverse first sound in the solid.

For the drift velocity of phonons in a crystal we have similarly

$$\mathbf{V}=\mathbf{V}_l+\mathbf{V}_t, \tag{16}$$

where $\mathbf{V}_l$ and $\mathbf{V}_t$ are the second-sound wave drift velocities in a longitudinal wave and in a transverse viscous wave described by

$$\nu_1\dot{\mathbf{V}}_t+\frac{TS_1{}^2}{\varkappa_1}\mathbf{V}_t=\eta_1\Delta\mathbf{V}_t, \quad \operatorname{div}\mathbf{V}_t=0.$$

Here ν_1 is the normal density of the phonon gas (the coefficient of proportionality of the quasimomentum per unit volume to the phonon drift velocity), η_1 is the phonon-hydrodynamic viscosity of the crystal, and $\varkappa_1$ is the thermal conductivity. The phonon-hydrodynamics equations take into account additional terms that describe umklapp processes (the crystal anisotropy is disregarded for simplicity).

The relations between V_x and V_z in the surface wave are

$$V_{lx}=\frac{ikS_1}{i\omega\nu_1-TS_1^2/\varkappa_1}T_1', \quad V_{lz}=\frac{-\gamma_{l1}S_1}{i\omega\nu_1-TS_1^2/\varkappa_1}T_1', \tag{17}$$

$$V_{tz}=ikV_{tx}/\gamma_{t1}, \tag{18}$$

where T_1' is the oscillation of the temperature in the wave,

$$\gamma_{l1}=\left[k^2-\left(\omega^2+i\omega\frac{TS_1^2}{\varkappa_1}\right)\Big/u_1^2\right]^{1/2}, \quad \gamma_{t1}=\left(k^2-\frac{\omega^2}{u_{t1}^2}\right)^{1/2},$$

u_1 is the velocity of the bulk second sound in the crystal, with account taken of the velocity damping due to the viscosity of the crystal phonon gas,[15] and

$$u_{t1}=[\eta_1\omega^2(i\omega\nu_1-TS_1^2/\varkappa_1)^{-1}]^{1/2}$$

is the velocity of the viscous phonon wave in the crystal. Just as in a liquid, the existence of weakly damped oscillations of the drift velocity in a solid calls for satisfaction of the condition $\omega^2 \gg u_{t1}^2 k^2$.

Note that in these boundary-value problems there is no need to write down the solution for the superfluid motion velocity $\mathbf{v}_s$ in He II since $\mathbf{v}_s$ does not enter explicity in the conditions (1)–(9), and all the necessary relations between the normal and superfluid components in the bulk of the He II are contained in Eqs. (10)–(13).

SURFACE WAVES ON ATOM-ROUGH BOUNDARIES

We shall use Eqs. (1)–(18) to find the spectrum and damping of surface waves on an atom-rough solid helium–He II interface. Solution of the boundary-value problem yields three types of long-wave surface oscillation.

The dispersion equation for coupled oscillations in a weakly damped crystallization–melting wave ($\zeta \gg u_z$) and in an elastic surface wave ($u_z \gg \zeta$), without allowance for dissipation, is

$$\left[\omega^2(\rho_1-\rho_2)^2+\rho_2\left(k^2-\frac{\omega^2}{c_L^2}\right)^{1/2}(M_s\omega-\tilde{\alpha}k^2)\right]\left\{\left(2k^2-\frac{\omega^2}{c_t^2}\right)^2 -4k^2\left[\left(k^2-\frac{\omega^2}{c_t^2}\right)\left(k^2-\frac{\omega^2}{c_l^2}\right)\right]^{1/2}-\frac{\omega^2(g+h)k^2}{\rho_1c_t^4}\left(k^2-\frac{\omega^2}{c_t^2}\right)^{1/2}\right\}$$
$$=\frac{\omega^4}{c_t^4}\left(k^2-\frac{\omega^2}{c_l^2}\right)^{1/2}\left\{[\tilde{\alpha}\rho_2+g(\rho_1-\rho_2)]k^2 +M_s\frac{\rho_2}{\rho_1}\left[gk^2\left(k^2-\frac{\omega^2}{c_L^2}\right)^{1/2}-\rho_2\omega^2\right]\right\}. \tag{19}$$

In the long-wave region $k \ll (\rho_1-\rho_2)^2/M_s\rho_2$ Eq. (19) describes a crystallization wave with a "capillary" dispersion law,[7] with account taken of the surface mass of the atom-rough boundary[17] and the compressibility of the solid and liquid:

$$\omega^2\left\{(\rho_1-\rho_2)^2+k\left[M_s\rho_2+\frac{\tilde{\alpha}\rho_2+g(\rho_1-\rho_2)}{2c_t^2(1-c_t^2/c_l^2)}+\frac{\tilde{\alpha}\rho_2}{2c_L^2}\right]\right\}=\tilde{\alpha}k^3\rho_2, \tag{20}$$

and an elastic surface Rayleigh wave ($\omega = c_R k$) on the free surface of a solid with allowance for capillary effects:

$$\left(2k^2-\frac{\omega^2}{c_t^2}\right)^2-4k^2\left[\left(k^2-\frac{\omega^2}{c_t^2}\right)\left(k^2-\frac{\omega^2}{c_l^2}\right)\right]^{1/2}$$
$$=\frac{\omega^2}{c_t^4}\left\{\frac{(g+h)k^2}{\rho_1}\left(k^2-\frac{\omega^2}{c_l^2}\right)^{1/2} +\frac{1}{(\rho_1-\rho_2)^2}\left[(\tilde{\alpha}\rho_2+g(\rho_1-\rho_2))k^2-M_s\frac{\rho_2^2}{\rho_1}\omega^2\right]\right\}. \tag{21}$$

In the short-wave region $k \gg (\rho_1-\rho_2)^2/M_s\rho_2$ Eq. (19) describes a crystallization wave with an acoustic spectrum and with velocity

$$c_M=(\tilde{\alpha}/M_s)^{1/2} \tag{22}$$

and an elastic surface wave whose velocity is determined from the equation

$$\left(2-\frac{c^2}{c_t^2}\right)^2-4\left[\left(1-\frac{c^2}{c_t^2}\right)\left(1-\frac{c^2}{c_l^2}\right)\right]^{1/2}$$
$$+\left(\frac{c}{c_t}\right)^4\left(1-\frac{c^2}{c_l^2}\right)^{1/2}\left(1-\frac{c^2}{c_L^2}\right)^{-1/2}$$
$$\times\frac{\tilde{\alpha}\rho_2+g(\rho_1-\rho_2)-M_s(\rho_2^2/\rho_1)c^2}{\rho_2(\tilde{\alpha}-M_sc^2)}=0. \tag{23}$$

The velocity of the elastic surface wave on the solid helium—He II interface in the short-wave region is close (both at $c_M < c_R$ and at $c_M > c_R$) to the velocity c_{RS} of the Rayleigh-Stonely wave on a solid-liquid interface; the latter velocity is determined from the equation[6]

$$\left(2-\frac{c_{RS}^2}{c_t^2}\right)^2-4\left[\left(1-\frac{c_{RS}^2}{c_t^2}\right)\left(1-\frac{c_{RS}^2}{c_l^2}\right)\right]^{1/2}$$
$$+\frac{\rho_2}{\rho_1}\left(\frac{c_{RS}}{c_t}\right)^4\left(1-\frac{c_{RS}^2}{c_l^2}\right)^{1/2}\left(1-\frac{c_{RS}^2}{c_L^2}\right)^{-1/2}=0. \tag{24}$$

We note that, as can be seen from (21), the influence of the capillary effects on the Rayleigh-wave velocity in the long-wave region is enhanced (by a factor $\rho_1^2/(\rho_1-\rho_2)^2 \approx 10^2$) compared with the case of the free surface of a solid ($\zeta_2 = 0$). Thus, the capillary effects decrease the velocity of the Rayleigh wave (as the frequency is increased) at $c_M < c_R$ and increase it at $c_M > c_R$. This means that "intersection" and mutual transformation of the crystallization wave and of the elastic surface wave takes place only at $c_M > c_R$ (see Fig. 1). In other words, a sufficiently large effective surface mass M_s (i. e., $c_M < c_R$) can substantially alter the character of the interaction of crystallization and Rayleigh waves, compared with the case $M_s = 0$ (Ref. 0). Čerenkov emission of transverse sound into the crystal does not lead to substantial dissipation of the short-wave acoustic crystallization wave at $c_M > c_R$ (since $\zeta \gg |\mathbf{u}|$ in the crystallization wave). In the limit $M_s = 0$ we have $c_M - c_l$, where c_l is the velocity of the longitudinal sound in the crystal (the maximum velocity of elastic waves in the system).

The velocity of a Rayleigh wave on the free surface of solid helium is $c_R = 245$ m/s (see the Appendix). According to the available experimental data, $\tilde{\alpha} \approx 0.3$ erg/cm^2 (Refs. 1, 3, and 5) and $M_s \approx 2 \cdot 10^{-10}$ g/cm^2 (Ref. 19). At $k > k_1 \approx 8 \cdot 10^6$ cm^{-1} the crystallization wave acquires therefore an acoustic spectrum and a velocity $c_M \approx 300$ m/s. In the wave-number region $k \sim k_0 \approx 5 \cdot 10^6$ cm^{-1} is located on an arbitrary intersection point of the "bare" branches 1 and 2 (dashed lines in the figure). In this wavelength region we have the strongest interaction and mutual transformation of the Rayleigh and crystallization waves, while at $k \gg k_0$ the velocity of the crystallization wave exceeds that of the

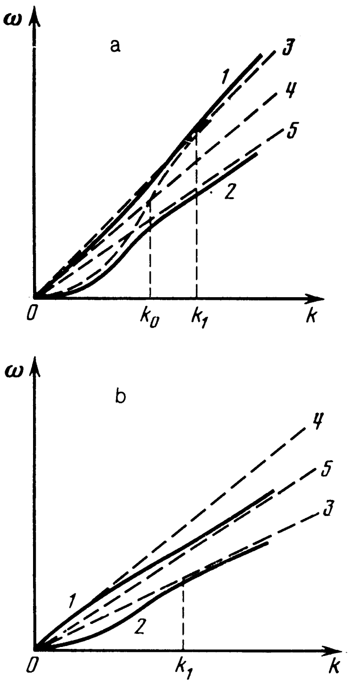

FIG. 1. Dispersion curves $\omega(k)$ of a surface elastic wave (solid curve 1) and a crystallization wave (solid curve 2) on the interface of solid helium and He II: a) at $c_M > c_R$—presence of resonance and mutual transformation of the surface waves; b) at $c_M < c_{RS}$—absence of mutual transformation of the waves, where $c_M = (\tilde{\alpha}/M_s)^{1/2}$ (curve 3) is the velocity of the high-frequency crystallization having an acoustic spectrum, while c_R and c_{RS} (curves 4 and 5) are the velocities of the Rayleigh wave and the Rayleigh-Stonely wave.

Rayleigh wave. We note that this value of c_M is only an estimate (in view of the scatter of the experimental data, the anisotropies of and M_s (Refs. 3 and 19), and other factors).

We write down also the spectrum of the bare crystallization wave with allowance for damping due to viscosity and to the surface dissipation coefficients a_{ik} (5) and (5a). Using Eqs. (12) and (17), we introduce the thermal impedances Z_1 and Z_2 of the contiguous

$$Z_1^{-1} = -TS_1^2\gamma_{l1}/(i\omega\nu_1 - TS_1^2/\varkappa_1), \tag{25}$$

$$Z_2^{-1} = iTS_2\rho_s\sigma_2\gamma_{L2}/\rho_n\omega \tag{26}$$

for the solid and liquid, respectively (according to the definition of the thermal impedances we have $TS_1V_z = \delta T_1/Z_1$, $TS_2v_{nLz} = -\delta T_2/Z_2$). At sufficiently low temperatures, $T < 0.5$ K, where an anomaly of the Kapitza jump is observed on an atom-rough interface of solid helium and He II (Refs. 9, 19, 24, 25), and the heat transport is both phases is in the second-sound regime or by ballistic phonons, the thermal resistance $R_K = cT^2$ of the boundary is large compared with the thermal impedances of the contiguous phases: $R_K \gg Z_1$, $R_K \gg Z_2$. In this case we can neglect the left-hand side of the second equation in (5a) and obtain for the spectrum of the "bare" low-frequency crystallization wave (with allowance for the effective surface mass) the expression

$$\omega^2\left[M_s + \frac{(\rho_1-\rho_2)^2}{\rho_2 k}\right] - \tilde{\alpha}k^2$$

$$= -i\omega\left[\rho_1^2T\left(a - \frac{b^2}{c}\right) + \frac{4}{d} + 4(\eta_1+\eta_2)k\right]. \tag{27}$$

The surface and bulk dissipation mechanisms are, naturally, additive.

At higher temperatures $T \gtrsim 0.6$–0.7 K the heat transport in the crystal at low wave frequencies $\omega \ll TS_1^2/\varkappa_1 v_1$ is by diffusion through heat conduction. In this case the thermal impedance Z_1 of the solid phase increases appreciably and can exceed both the thermal impedance Z_2 of He II and the thermal resistance R_K of the boundary: $Z_1 \gg Z_2$, $Z_1 \gg R_K$. At the same time, just as at low temperatures, we have $Z_2 \ll R_K$ because of the nondissipative (convective) heat transport in He II. It can be readily seen from (3) and (5a) that at this relation between Z_1, Z_2, and R_K we have on the interface $\delta T_2 \ll \delta T_1$, $V_z \ll \zeta$, $V_{nz} \sim \zeta$ and we obtain for the spectrum of the initiating low-frequency crystallization wave

$$\omega^2\left[M_s + \frac{(\rho_1-\rho_2)^2}{\rho_2 k}\right] - \tilde{\alpha}k^2$$

$$= -i\omega\left[\rho_1^2T(a + 2bT\sigma_1 + cT^2\sigma_1^2) + 4\eta_2 k\frac{(S_2-S_1)^2}{S^2}\right]. \tag{28}$$

The contributions of the viscosity of the phonon gas and of the thermal conductivity of the crystal to the damping of the crystallization wave are much longer in this temperature region than that on He II (since $V \ll v_{nz} \sim \dot{\zeta}$). The surface jump of the tangential velocities (allowance for the fact that the coefficient d in (5a) is finite) makes likewise no noticeable contribution to the crystallization-wave damping in this temperature region (in contrast to the surface temperature jump, which is governed by the quantity $R_K = cT^2$).

It can be seen from (27) and (28) that in the hydrodynamic regime the contributions of the He II viscosity and of the phonon gas of the crystal to the crystallization-wave damping are small compared with the contribution from the surface dissipative coefficients a,b,c, and d by factors $kl_2 \ll 1$, $kl_N \ll 1$ (l_2 is the phonon mean free path in He II and l_N is the mean free path of the phonons due to the normal processes in the crystal). The crystallization-wave damping on account of the kinetic growth coefficient $K^{-1} = \rho_1 T(a - b^2/c)$ in the form (27) is preserved also in the ballistic regime $kl_2 \gg 1$, $kl_N \gg 1$ (see Ref. 23). Thus, the damping of the low-frequency crystallization wave has the same characteristic form $\operatorname{Im}\omega \propto k$ (Ref. 7) both in the hydrodynamic and in the ballistic regime, and the transition from one regime to the other (e. g., change of temperature) should not influence the damping substantially.

Absorption of a crystallization wave of given frequency increases monotonically with increasing temperature, in contrast, e. g., to the case of bulk first sound in He II or in a perfect solid dielectric, when the transition from the ballistic to the hydrodynamic regime determines the maximum on the absorption curve.[15,30] In the temperature region $T \lesssim 0.5$ K we have $\operatorname{Im}\omega \propto T^4$ (phonon region), and at $T \gtrsim 0.5$ K we have $\operatorname{Im}\omega \propto \exp(-\Delta/T)$ (roton region, with Δ the roton gap).[7] The frequency dependence of the damping of long-wave crystallization wave of the type $\operatorname{Im}\omega \propto k$ causes the quality factor of the oscillations to increase with increasing frequency (at a fixed temperature. In fact, in that frequency region where the crystallization wave has an acoustic character ($k > k_1$, see Fig. 1), we have from (27) and (28)

$\operatorname{Im}\omega_2 = \gamma(T)$, where the function $\gamma(T)$ has the form described above. This means that at such a temperature the high-frequency crystallization oscillations of the boundary can be weakly damped. Such oscillations are, for example, the thermal crystallization oscillations at $T \gtrsim 1$ K. Therefore the contribution of the thermal crystallization oscillations (with the acoustic dispersion law $\omega_2 = c_M k$) to the surface free energy of an atom-rough surface at $T \gtrsim 1$ K, viz.,

$$\delta\alpha_2 = -\zeta(3)T^3/2\pi\hbar c_M^2, \tag{29}$$

is of the same order as the contribution of the Rayleigh waves with $\omega_1 = c_R k$ (Ref. 10). Allowance for the contribution (29) can decrease the discrepancy between the experimentally observed and theoretically predicted temperature dependence of the interfacial tension of an atom-rough interface between solid helium and He II above 1 K (Ref. 32).

The third type of surface waves on an atom-rough solid helium—He II interface are coupled oscillations of surface and bulk excitations, accompanied by temperature oscillations but with the interface practically immobile. To analyze the spectrum of such oscillations we introduce the thermal impedance Z_s of the surface excitations, defined by

$$\frac{1}{Z_s} = iT\left(\omega\frac{\partial\sigma_s}{\partial T} - \frac{k^2}{\omega}\frac{\sigma_s^2}{\nu_s}\right). \tag{30}$$

At low temperatures, when the characteristic wave number k_T of the thermal surface excitations is smaller than k_0 or k_1 (see Fig. 1) we have

$$\begin{gathered}
\nu_s = \frac{5T^{5/3}}{18\pi\hbar^{2/3}}\left[\frac{(\rho_1-\rho_2)^2}{\tilde{\alpha}\rho_2}\right]^{4/3}\Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right),\\
\sigma_s = \frac{7T^{4/3}}{12\pi\hbar^{4/3}}\left[\frac{(\rho_1-\rho_2)^2}{\tilde{\alpha}\rho_2}\right]^{2/3}\Gamma\left(\frac{7}{3}\right)\zeta\left(\frac{7}{3}\right),\\
u_s^2 = \frac{\sigma_s^2}{\nu_s}\frac{\partial T}{\partial\sigma_s} = \frac{63}{40}\frac{\Gamma(7/3)\zeta(7/3)}{\Gamma(5/3)\zeta(5/3)}\left[\frac{\tilde{\alpha}\rho_2}{(\rho_1-\rho_2)^2}\frac{T}{\hbar}\right]^{2/3}.
\end{gathered} \tag{31}$$

Here u_s is the velocity of the surface second sound (in the gas of the crystallization-wave quanta), and increases with increasing temperature like $u_s \propto T^{1/3}$, just as on the free surface of pure He II (Ref. 12). At higher temperatures, when $k_T > k_0, k_1$, we have

$$\nu = \frac{3\zeta(3)}{2\pi\hbar^2}\frac{T^3}{c^4} \qquad \sigma_s = \frac{3\zeta(3)}{2\pi\hbar^2}\frac{T^2}{c^2}, \qquad u_s = \frac{c}{\sqrt{2}}, \tag{32}$$

where $c \approx \min(c_M\ c_{RS}) = c_{RS} \approx 200$ m/s is the velocity of the Rayleigh-Stonely wave on the solid helium–He II interface (see the Appendix). In this temperature region the surface second sound propagates in the gas of the quanta of the Rayleigh-Stonely waves, and its velocity does not depend on the temperature (just as on an atom-smooth face). The surface second sound velocity reaches a constant value ≈ 140 m/s at $T = T_0 \approx 0{,}6$–$0{,}8$ K.

Taking expressions (25), (26), and (30) for the bulk and surface thermal impedances into account, we obtain the following dispersion relation for coupled surface temperature oscillations (neglecting the bulk viscosities η_1 and η_2)$^{3)}$:

$$\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{(a_{22}+2a_{23}+a_{33})T^2}{Z_1Z_2} = \frac{1}{Z_s}\left[1 + \frac{a_{33}T^2}{Z_1} + \frac{a_{22}T^2}{Z_2} + \frac{(a_{22}a_{33}-a_{23}^2)T^4}{Z_1Z_2}\right]. \tag{33}$$

The connection between the temperature-oscillation amplitudes is obtained from the relations

$$\delta T_1\left(\frac{a_{23}+a_{33}}{Z_1} + \frac{1}{T^2}\right) = \delta T_2\left(\frac{a_{22}+a_{23}}{Z_2} + \frac{1}{T^2}\right) = \delta T_s\left(\frac{a_{23}}{Z_s} + \frac{1}{T^2}\right). \tag{34}$$

Note that the estimates (5c) for the matrix elements a_{ik} were not used in the derivation of (33) and (34).

To solve (33) we must take the following circumstance into account. The region of applicability of expressions (33) and (34) is bounded by the conditions of the hydrodynamic regime of oscillations for bulk and surface excitations: $kl_2 \ll 1$, $kl_N \ll 1$, $\omega\tau_{Ns} \ll 1$, where τ_{Ns}^{-1} is the frequency of the surface normal collisions. In a crystal, the phonon mean free path l_N decreases with increasing temperature in proportion to T^{-5}, in He II in the region of phonon-phonon collisions ($T \lesssim 0.5$ on the melting curve) the phonon mean free path is $l_2 \propto T^{-9}$ for four-phonon processes[30] and $l_2 \propto \exp(\Delta/T)$ in the region of the phonon-roton collisions ($T \gtrsim 0.5$ K). The normal processes in a surface-excitation gas are three-phonon processes (since the spectrum is a decaying one). For the lifetime τ_ω of the quanta of the crystallization waves (with "capillary" dispersion law) of frequency ω we have $\tau_\omega \propto k^{-5} \propto \omega^{-10/3}$ (see Ref. 33). From this we obtain at temperatures $T < T_0$ a surface normal-collision frequency $\tau_{Ns}^{-1} \propto T^{10/3}$. At $T > T_0$, when the thermal surface excitations are the quanta of the acoustic Rayleigh-Stonely waves, just as on atom-smooth faces, we have $\tau_{Ns}^{-1} \propto T^4$. From the foregoing estimates of the limiting frequencies and from expressions (30)–(32) it follows that the inequality $Z_s \gg R_K$ holds in the entire region of applicability of the dispersion equation (33). When solving Eq. (33), just as in the analysis of the damping of crystallization waves, a distinction must be made between the two limiting cases $R_K \gg Z_1$ and $R_K \ll Z_1$.

In the case $R_K \gg Z_1, Z_2$ (low-temperature region, the heat is transported in the crystal in the second-sound regime), Eq. (33) has two different roots corresponding to weakly damped waves. The first root

$$\frac{1}{Z_2} = -\frac{1}{R_K}\left[1 - \frac{Z_1}{R_K}\right] \tag{35}$$

corresponds to $\delta T_2 \gg \delta T_1, \delta T_s$. At $R_K = \infty$ and $d = \infty$ [see (5) and (5a)], meaning absence of heat exchange and quasimomentum exchange between the media, the solution (35) describes a homogeneous bulk second-sound wave propagating in the He II along the interface ($v_{nz} = 0$) and satisfying the boundary condition that there be no heat or quasimomentum flux through the interface: $\Pi_{z\mu} = 0$. At large but finite boundary thermal resistance, Eq. (35) describes a weakly inhomogeneous bulk second-sound wave propagating in a liquid at a small glancing angle φ to the interface: In He II ($z < 0$) the temperature distribution in the wave is of the form

$$\begin{gathered}
\delta T_2 = \delta T_2^0 \exp\,(ikx - i\omega t + \gamma_{L2}z),\\
\gamma_{L2} = \left(k^2 - \frac{\omega^2}{u_2^2}\right)^{1/2}
\end{gathered} \tag{36}$$

$$= -\frac{i\omega\rho_n}{R_K T S_2 \rho_s \sigma_2} - \frac{\omega^2}{(k^2-\omega^2/u_1^2)^{1/2}} \frac{\rho_n \nu_1}{S_2 \rho_s \sigma_2 S_1^2 R_K^2}$$

the glancing angle is $\varphi = |\gamma_{L2}|/k \ll 1$); in the crystal ($z>0$) we have

$$\delta T_1 = \delta T_1^0 \exp(ikx - i\omega t - \gamma_{l1} z), \quad \gamma_{l1} = (k^2-\omega^2/u_1^2)^{1/2} > 0.$$

It can be easily verified that the damping of the wave in question is determined mainly by bulk dissipation and is close to the damping of bulk second sound in He II, $\operatorname{Im}\omega \sim \eta_2 k^2/\rho_n$. The surface dissipation (finite R_k^{-1}) leads only to a relatively small increase of the surface-wave velocity relative to that of the bulk wave:

$$\omega = u_2 k\left[1 + \frac{1}{2}\left(\frac{\rho_n u_2}{R_K T S_2 \rho_s \sigma_2}\right)^2\right],$$

and to the onset of weak inhomogeneity of the depth distribution of the amplitude of the temperature oscillations in the wave [see (36)].

The second root of (33) at $R_K \gg Z_1, Z_2$

$$\frac{1}{Z_1} = -\frac{1}{R_K}\left[1 - \frac{Z_2}{R_K} - \frac{(a_{23}+a_{33})^2 T^4}{R_K Z_s}\right] \tag{37}$$

corresponds to a weakly inhomogeneous bulk second-sound wave propagating in the crystal ($\delta T_1 \gg \delta T_2, \delta T_s$) at a small glancing angle with the boundary:

$$\delta T_1 = \delta T_1^0 \exp(ikx - i\omega t - \gamma_{l1} z),$$

$$\gamma_{l1} \equiv \left(k^2 - \frac{\omega^2}{u_1^2}\right)^{1/2}$$

$$= -\frac{i\omega\nu_1}{R_K T S_1^2} + \frac{(a_{23}+a_{33})^2 T^4}{R_K^2 T S_1^2} \frac{\partial\sigma_s}{\partial T}(\omega^2 - u_s^2 k^2) \ll k,$$

$$\omega = u_1 k[1 + {}^1/_2 (u_1 \nu_1 / T S_1^2 R_K)^2].$$

The damping of the wave considered is also close to the damping of the bulk second sound (inside the crystal in this case), see Ref. 15:

$$\operatorname{Im}\omega \sim \eta_1 k^2/\nu_1 + T S_1^2/\varkappa_1 \nu_1.$$

At low temperatures, when the heat is transported in both media in the second-sound regime and $R_K \gg Z_1, Z_2$, the surface (more accurately, the weakly inhomogeneous near-surface) second-sound wave has thus a velocity and damping (in the weak-damping region) that are close to the velocity and damping of the corresponding bulk second-sound wave.

If $Z_1 \gg R_K \gg Z_2$ (higher temperature, and the heat is transported in the crystal by thermal conduction), Eq. (33) has one root

$$1/Z_2 = 1/Z_s - 1/Z_1, \tag{38}$$

which corresponds to $\sigma T_1 \approx \sigma T_2 \approx \sigma T_s$. In this case (e. g., at $\omega \ll T S_1^2/\varkappa_1 v_1$, $k \ll (\omega C_1/\varkappa_1)^{1/2}$, where C_1 is the heat capacity per unit volume of the crystal), Eq. (38) takes the form

$$\left(k^2 - \frac{\omega^2}{u_2^2}\right)^{1/2} = \frac{\rho_n}{\rho_s \sigma_2 S_2}\left[\omega^2 \frac{\partial\sigma_s}{\partial T} - k^2 \frac{\sigma_s^2}{\nu_s} + \frac{1+i}{\sqrt{2}} \frac{\omega}{T} (\omega \varkappa_1 C_1)^{1/2}\right]. \tag{38a}$$

This equation describes a weakly inhomogeneous second-sound surface wave propagating in He II ($\omega \approx u_2 k$) and weakly damped (in the considered frequency range) because of the thermal conductivity of the crystal and the viscosity of the He II:

$$\operatorname{Im}\omega \sim \varkappa_1 C_1 k^2/C_2^2 + \eta_2 k^2/\rho_n,$$

where C_2 is the heat capacity per unit volume of He II. It can be seen from (38a) that allowance for the surface excitations causes, together with the thermal conductivity of the crystal, a weak inhomogeneity of the depth distribution of the amplitude of the temperature oscillations in the wave.

Thus, under conditions when both the surface and the bulk excitations are in the hydrodynamic regime, the bulk thermal impedances "shunt" the surface impedance, and the surface second-sound wave constitutes a weakly inhomogeneous bulk second-sound wave in one of the contiguous media. However, the estimates presented for the mean free paths (or the lifetimes) of the bulk and surface excitations show that at sufficiently low temperatures (and high frequencies) conditions are realizable under which the surface excitations can be in the hydrodynamic regime, $\omega\tau_{Ns} \ll 1$, and the bulk ones in the ballistic regime, $\omega\tau_N \gg 1$, $\omega\tau_\perp \gg 1$, where τ_N is the time of the normal phonon collisions in the crystal and $\tau_\perp$ is the time of the transverse relaxation in He II. This is possible because at low temperatures ($T \ll \Theta$) the frequency of the normal collisions in a surface-excitation gas ($\tau_{Ns}^{-1} \propto T^{10/3}$ at $T < T_0$ and $T < T_0$, $\tau_{Ns}^{-1} \propto T^4$ at $T > T_0$) is higher than in a crystal ($\tau_N^{-1} \propto T^5$) and than the rate $\tau_\perp^{-1} \propto T^5$ transverse relaxation in He II. In such an "intermediate"regime the interaction between the surface and bulk excitation is substantially weakened compared with the fully hydrodynamic regime considered above. These conditions correspond in fact to "freezing out" of the bulk thermal excitations compared with the surface ones, when a normal-motion velocity and a normal component exist only on the surface (see Ref. 12), meaning that one can neglect the right-hand sides in the boundary conditions (2) and (3). In the intermediate regime the surface second sound is therefore a thermal wave in a gas of surface excitations: it is practically nondispersive ($\omega = u_s k$), and its damping is determined mainly by the purely two-dimensional processes of scattering of surface excitations (by the surface viscosity η_s):$\operatorname{Im}\omega \approx \eta_s k^2/v_s$ (2). This mechanisms limits from above the frequency region where weakly damped second sound exists: $\omega^{-1}\operatorname{Im}\omega \sim \eta_s \tau_{Ns}$.

The interaction between the crystallization wave and the surface second sound is also determined essentially by the temperature and frequency regions. This interaction is a minimum at the lowest temperatures of the intermediate regime of the surface second sound (owing to the weak interaction between the bulk and surface excitations) and a maximum in the highest-temperature region $Z_1 \gg R_K \gg Z_2$, where it is determined by the heat of crystallization $T(\sigma_2 - \sigma_1)$ (the interaction in the hydrodynamic region $R_K \gg Z_1, Z_2$ is determined also by the crossover elements of the matrix a_{ik} (5)).

SURFACE WAVES ON ATOM-SMOOTH INTERFACES

Consider an atom-smooth interface that is isotropic in its own plane (such as (0001) for an hcp solid-helium single crystal). To find the spectrum and damping of the surface waves on such an interface we use Eqs. (1)-(25), in which we put $\zeta = 0$ and disregard the first equation of the system

(5). The solution of the boundary-value problem yields two types of long-wave surface oscillations that do not interact with each other in the absence of thermal expansion.

The first type of oscillation is an isothermal elastic Rayleigh-Stonely surface wave $\omega = \omega_1(k)$ on a solid-liquid interface, calculated with allowance for capillary effect. The dispersion equation for this wave is

$$4k^2\left[\left(k^2-\frac{\omega^2}{c_l^2}\right)\left(k^2-\frac{\omega^2}{c_t^2}\right)\right]^{1/2}-\left(\frac{\omega}{c_t^2}-2k^2\right)^2$$

$$-\frac{\rho_2\omega^4}{\rho_1 c_t^4}\left(k^2-\frac{\omega^2}{c_L^2}\right)^{-1/2}\left(k^2-\frac{\omega^2}{c_l^2}\right)^{1/2}=\frac{\rho_2\omega^2}{(k^2-\omega^2/c_l^2)^{1/2}}$$

$$\times\left\{k^2-\left[\left(k^2-\frac{\omega^2}{c_l^2}\right)\left(k^2-\frac{\omega^2}{c_t^2}\right)\right]^{1/2}\right\}\frac{(g+h)k^2}{\rho_1^2 c_t^4}-\frac{\omega^2}{\rho_1 c_t^4}$$

$$\times\left\{gk^2\left(k^2-\frac{\omega^2}{c_l^2}\right)^{1/2}+(g+h)k^2\left(k^2-\frac{\omega^2}{c_t^2}\right)^{1/2}\right\}. \quad (24a)$$

The right-hand side of (24a) describes in the linear approximation the influence of capillary effects on the Rayleigh-Stonely wave [see (24)]. Capillary effects, generally speaking, lead only to weak dispersion of the velocity of the wave in question (the relative change of the velocity is of the order of $gk/\rho_1 c_t^2 \sim ak$, where a is the interatomic distance), just as in the case of a Rayleigh wave on a free surface of a solid ($\rho = 0$).

Let us estimate the velocity c_{RS} of the Rayleigh-Stonely wave. When the parameters

$$\rho_2=0.91\rho_1, \qquad c_l=\left(\frac{c_{11}}{\rho_1}\right)^{1/2}=460\ \text{m/s},$$

$$c_t=\left(\frac{c_{44}}{\rho_1}\right)^{1/2}=255\ \text{m/s}, \quad c_L=360\ \text{m/s}$$

are used, solution of (24) yields $c_{RS} \approx 192$ m/s (c_{ik} are the elastic moduli of the crystal. The velocity c_{RS} can be estimated more accurately by regarding solid helium as a strongly anisotropic hexagonal crystal in which the various elastic moduli differ rather strongly[34]: $c_{44}, c_{13} \ll c_{11}, c_{33}$ (for example $c_{44}^2/c_{11}c_{33} \approx 0{,}07$, $c_{13}^2/c_{11}c_{33} \approx 0{,}05$). Such a calculation (see the Appendix) yields the more accurate value $c_{RS} \approx 205$ m/s for the Rayleigh-Stonely wave velocity on an atom-smooth solid-helium–He II interface.

The second type of surface oscillation of the considered boundary constitute a surface second-sound wave $\omega = \omega_2(k)$, in which the temperature oscillates while the lattice sites remain immobile ($\mathbf{u} = 0$). The dispersion equation for the surface second sound is given by (33) in the notation in (25), (26), and (30). Expressions (32) are valid for v_s, σ_s and $u_s = c_{RS}/\sqrt{2}$. On atom-smooth faces, just as on atom-rough ones, we have apparently $R_K \gg Z_1$, Z_2 is the heat transport in both contiguous phases is in the second-sound regime (and $Z_1 \gg R_K \gg Z_2$, if the heat is transported in the crystal by diffusion). Therefore all the conclusions concerning the character of the surface second-sound wave, which follow from the solution of the dispersion equation (33) [see Eqs. (35)–(38)], remain in force also in the case of an atom-smooth interface between solid helium and He II. Thus, under conditions of diffusive heat transport in the crystal, at sufficiently high values of the thermal conductivity $\varkappa_1$ ($T \gtrsim 0{,}8K$), a deeply penetrating second-sound surface wave $\omega \approx u_2 k$ can propagate in the He II on the considered interface. A similar weakly damped surface-second sound wave can exist also on the interface of He II with a perfect crystalline dielectric, in which a transition from diffusive to wave propagation of the heat is observed, such as LiF or NaF (Ref. 13).

The conclusion that a temperature wave can propagate in a two-dimensional gas of surface excitations under the intermediate-regime conditions (relative to bulk excitations) is valid also for the case of atom-smooth solid helium–He II interfaces, just as for the interfaces with a perfect dielectric crystal. In the case of the interface of He II with a dielectric crystal it must be recognized that for He II the sound velocity and the density are much lower than for ordinary crystals. Therefore the slowest surface elastic wave on such an interface will be a Stonely wave with velocity close to that of sound in a liquid (see Ref. 6): in (32) we must put $c = c_L$ and accordingly $U_s = c_L/\sqrt{2}$. On the other hand, in the case of a crystal-vacuum interface we have $c = c_R$, $u_s = c_R/\sqrt{2}$, where c_R is the velocity of a Rayleigh wave on the free surface of the crystal.

The essential conditions under which weakly damped second sound can propagate in a solid dielectric is that the crystal boundary be smooth enough and perfect. The characteristic dimension δ of the roughness should be less than the wavelength λ_T of the thermal surface excitations: $\delta \ll \lambda_T \sim \hbar c/T$. On atom-smooth faces at $T \ll \Theta$ this situation is certainly realized. The perfection of the boundary is determined also by the condition $\tau_{Ns} \ll \tau_{Us}$, where τ_{U_s} is the lifetime of the surface excitation due to processes with nonconservation of the quasimomentum, such as surface umklapp processes.

CONCLUSION

The spectrum, damping, and interaction of the major types of macroscopic surface oscillations on an interface between solid helium and He II were investigated with account taken of surface second sound and of surface dissipation. Both types of boundary below 1.2 K were considered—atom-rough and atom-smooth.

If both the surface and bulk excitations are in the hydrodynamic oscillation regime, the bulk thermal impedance "shunt", the impedance of the surface excitations, and the surface second-sound wave constitutes a weakly inhomogeneous bulk second-sound wave in one of the contiguous media. Under certain conditions, weakly inhomogeneous (deeply penetrating) surface second-sound waves exist on the interfaces of He II with solid helium and with a perfect dielectric crystal. An intermediate case can exist, in which the surface and bulk excitations are respectively in hydrodynamic and ballistic oscillation regimes. Under these conditions the interaction between the surface and bulk excitations is weakened and surface second sound constitutes a temperature wave in a two-dimensional surface-excitation gas, has no dispersion, and its damping is determined by two-dimensional scattering of surface excitations (by surface viscosity).

In investigations of the interaction between Rayleigh and crystallization waves, account must be taken of the effective surface mass M_s of the atom-rough boundary[17–18], which leads to an acoustic spectrum of the crystallization wave (with velocity $c_M \approx (\tilde{\alpha}/M_s)^{1/2}$) in the high-frequency region. Noticeable interaction between the crystallization and Rayleigh waves occurs only at $C_M > C_{RS} \approx 200$ m/s. The available experimental data on the surface rigidity $\tilde{\alpha}$ (Refs. 1, 3 and 5) and on the surface mass M_s (Ref. 19) lead to the estimate $C_M \approx 300$ m/s. Interaction and mutual transformation of the Rayleigh and crystallization waves can therefore be expected in the wave-number region $k \approx 5 \cdot 10^6$ cm^{-1}. Damping of the crystallization wave, due to the surface dissipative coefficients and to the bulk viscosity of the contiguous phases, has been investigated with allowance for the effective surface mass. At temperatures $T \gtrsim 1$ K the contribution of thermal acoustic-spectrum crystallization waves are of the same order as that of Rayleigh wave and of the surface tension of an atom-rough interface between solid helium and He II.

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APPENDIX

Surface elastic wave in a strongly anisotropic hexagonal crystal

The elastic moduli of the hcp phase of solid helium are greatly unequal:

$$c_{44},\ c_{13} \ll c_{11},\ c_{33}. \qquad (A.1)$$

These inequalities identify solid helium as a strongly anisotropic crystal. In view of the presence in the equations of the the theory of a small elasticity parameter, the properties of the surface waves in such crystals can be calculated analytically, just as in the case of an isotropic solid.

Consider a plane of type (0001) in a strongly anisotropic hexagonal crystal bordering on a liquid. The dispersion relation for a surface wave polarized in the sagittal plane (Rayleigh-Stonely wave) is

$$\left(1-\frac{\rho_1 c_{RS}^2}{c_{44}}\right)^{1/2}=\frac{\rho_1 c_{RS}^2}{(c_{11}c_{33})^{1/2}}+\frac{\rho_2 c_{RS}^2}{(c_{44}c_{33})^{1/2}}\left(1-\frac{c_{RS}^2}{c_L^2}\right)^{-1/2}. \qquad (A.2)$$

Using the elastic moduli of the hcp phase of solid helium (with maximum molar volume),[34] we find from (A.2) that $c_{RS} \approx 205$ m/s.

The Rayleigh-wave velocity c_R actually realized on a free surface of solid helium an atom-rough solid helium–He II interfaces, can be obtained from (A.2) by putting $\rho_2 = 0$:

$$c_R^2=(c_{44}/\rho_1)(1-c_{44}^2/c_{11}c_{33}). \qquad (A.3)$$

The numerical value is $c_R \approx 246$ m/s. Note that for a strongly anisotropic crystal we have in Eqs. (A.2) and (A.3) one small parameter $c_{44}^2/c_{11}\,c_{33} \approx 0{,}07$, as well as another $c_{13}^2/c_{11}\,c_{33} \approx 0{,}05$ that does not enter in the equations. Using (A.3) and the bulk equations of motion we can find the penetration depth δ of a Rayleigh wave on a (0001) surface of solid helium:

$$\delta=\lambda(c_{33}/c_{44})(c_{11}/c_{44})^{1/2}\approx 8\lambda.$$

Here λ is the wavelength. Recall for comparison that $\delta \approx \lambda$ in an isotropic solid. The dispersion equation for the velocity of a surface wave propagating along (0001) on faces of the type $(10\bar{1}0)$ or $(11\bar{2}0)$ can be obtained from (A.2) and (A.3) by interchanging in them c_{11} and c_{33}. In particular, for the velocity of a Rayleigh wave on a free surface of a crystal we get (A.3), while for the depth of penetration of the Rayleigh wave we obtain an equation close to it:

$$\delta=\lambda(c_{11}/c_{44})(c_{33}/c_{44})^{1/2}=7\lambda.$$

On faces intermediate between (0001) and $(10\bar{1}0)$ the velocity and penetration depth of a Rayleigh wave with sagittal plane ZX are close to the cited values. Thus, Rayleigh waves penetrate deeply in a strongly anisotropic solid.

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[1] The content of the present paper was reported at the 23rd All-Union Conference on Low-Temperature Physics, Tallin, 1984.

[2] Note that if the interface is differently defined, with both surface masses M_s and P_s different from zero, all the observed physical quantities must depend on a combination of these two masses that is invariant to small shifts of the interface (in a direction perpendicular to itself), see Ref. 26. It can thus be shown that the coefficient of capillary passage of low-frequency sound through an atom-rough solid helium–He II interface is determined at $T = 0$ by the invariant surface parameters $\tilde{\alpha} + g(\rho_1 - \rho_2)/\rho_2$ and $M_s + P_s(\rho_1 - \rho_2)/\rho_2$.

[3] At $Z_1 = \infty$ Eq. (33) describes a surface second-sound wave on a free boundary of He II.

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