Particle heating by a low-amplitude wave in an inhomogeneous magnetoactive plasma

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The diffusion, in phase space, of particles interacting with a monochromatic wave that propagates at an arbitrary angle to an external nonuniform magnetic field is investigated. It is shown that particles are heated in an inhomogeneous plasma at wave amplitudes much lower than the stochasticity threshold that corresponds to the homogeneous case. The diffusion coefficient is investigated analytically and numerically as a function of the inhomogeneity parameter, of the amplitude, and of the wave propagation angle. It is shown that the diffusion coefficient has a periodic dependence on the inhomogeneity parameter.

1. INTRODUCTION

If the time of resonant interaction of particles with a wave is shorter than the characteristic wave-evolution time, the problem of particle motion in the field of a monochromatic wave can be solved in the given-amplitude approximation. Since the publication of the papers of Mazitov[1] and O'Neil[2] this approach was used in numerous investigations (see Ref. 3 and the literature cited there). In the absence of an external magnetic field, there exists one group of resonant particles for which $\omega = \mathbf{k}\cdot\mathbf{v}$. The nonlinear frequency ω_b is connected with the electrostatic-wave amplitude E by the relation $\omega_b = (eEk/m)^{1/2}$ (Ref. 2). Here ω and k are the frequency and wave vector of the wave, while e and m are the charge and mass of the particle. Allowance for the external magnetic field $\mathbf{B}$ and accordingly for the different cyclotron resonances $\omega - k_\parallel v_\parallel = n\Omega$ ($\Omega = eB/mc$ is the cyclotron frequency and n is an integer that numbers the resonances) introduces into the problem a new parameter, viz., the ratio of the distance between resonances in phase space to the nonlinear width of the resonance. At low wave amplitude, the various cyclotron resonances turn out to be isolated and the particle motion is regular. As the amplitude is increased, the resonances overlap and stochastic motion of the particles in phase space sets in. This phenomenon, which finds numerous applications in both laboratory and cosmic plasma, is the subject of an extensive literature (see Refs. 4 and 5 for citations). The fundamental concept here is the nonlinear-resonance overlap introduced and investigated in detail in Chirikov's papers.[6–8]

It must be emphasized that overlap of high cyclotron resonances still does not mean that the force exerted on the particle by the wave exceeds the torque produced by the external magnetic field. On the contrary, an external magnetic field can play, as before, a decisive role in the particle dynamics. It is still correct to choose as the canonical variables the action and the angle of the unperturbed (without the wave) system, and the problem can be considered for times much longer than the cyclotron period. The opposite situation was investigated in Refs. 9 and 10.

Allowance for the inhomogeneity of the medium (of the density and the external magnetic field) alters qualitatively the character of the resonant interaction between the particles and the wave. The main feature is here the departure of the particle from resonance with the wave, owing to the "disparity" of the resonant velocity and the longitudinal velocity of the particle, as well as the opposite process—transition of the particles from the nonresonant to the resonant region.

Obviously, if the spacing between the resonances exceeds the thermal velocity of the particles it is meaningful to speak of interaction of the particles with the wave only at individual (not more than two) cyclotron resonances that correspond to the lowest absolute value of the particle velocity. For example, in the case of whistlers with $\omega \lesssim \Omega_e$, these are the Čerenkov ($\omega = k_\parallel v_\parallel$) and the first cyclotron ($\omega = k_\parallel v_\parallel + \Omega$) resonances. These were precisely the resonances primarily invoked in the numerous studies (see, e.g., Refs. 3 and 11) of the interaction between whistlers and high-energy electrons in the magnetosphere. We shall base ourselves on the results of these studies when we consider the passage through an isolated resonance.

If the thermal velocity of the particles exceeds the spacing of the resonances, the interaction can involve a large number of resonances. At wave amplitude exceeding the threshold of stochasticity in a homogeneous medium one can expect the inhomogeneity of the magnetic field not to change qualitatively the character of the interaction between the particles and the wave. If, however, the wave amplitude is small, the inhomogeneity of the medium leads to qualitatively new effects compared with the homogeneous case.

The onset of particle diffusion in phase space on passage through many resonances in an inhomogeneous magnetic field was considered for laboratory and magnetospheric conditions in Refs. 12–14. Notwithstanding the different character of the magnetic-field inhomogeneity (sinusoidal modulation of the field in a tokamak and monotonic variation of the field in dipole geometry), the physical results are quite similar. The studies cited, however, were confined to purely transverse wave propagation. It must be noted that in a certain sense this is a degenerate case, for the resonance conditions are determined here only by the particle position and not by its velocity, and the nonlinear frequency of the oscilla-

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tions at resonance is proportional to the wave amplitude E (and not to $E^{1/2}$). Propagation at an arbitrary angle was considered in Ref. 15, but only for the case of strong nonlinearity, which admits of an analytic description. There is thus at present no complete picture of particle diffusion at arbitrary wave-propagation angles and arbitrary relation between the inhomogeneity and the nonlinearity. This question is the subject of the present paper.

2. EQUATIONS OF MOTION

Consider an electrostatic wave propagating at an arbitrary (variable in space) angle to an external nonuniform magnetic field $\mathbf{B}$. We express the potential of the wave in the form

$$\Phi=\Phi_0 \sin\left(\int \mathbf{k}\, d\mathbf{r}-\omega t\right) \tag{2.1}$$

and assume that the magnetic field $\mathbf{B}(\mathbf{r})$ and the wave vector $\mathbf{k}(\mathbf{r})$, both of which vary in space, lie in the same plane. Under magnetospheric conditions this corresponds to wave propagation in a meridional plane.

It is known that where the adiabatic approximation is valid the canonically conjugate variables are the longitudinal particle velocity $v_\parallel$ and its coordinate s along the external magnetic field, the transverse adiabatic invariant $I=v_\perp^2/2\Omega$, and the gyrophase φ. Here $v_\perp$ is the modulus of the transverse velocity of the particle. The Hamiltonian that describes the particle motion is the total particle energy expressed in terms of the canonical variables

$$H=\frac{v_\parallel^2}{2}+I\Omega+\frac{e\Phi_0}{m}\sin\left[\int^s k_\parallel(s')\,ds'+k_\perp\left(\frac{2I}{\Omega}\right)^{1/2}\sin\varphi-\omega t\right], \tag{2.2}$$

where $k_\parallel$ and $k_\perp$ are the longitudinal and perpendicular components of the wave vector. The momenta in (2.2) are normalized to the particle mass. It must be borne in mind that the quantities $k_\parallel$, $k_\perp$, and Ω are functions of the coordinate s. From (2.2) we have the equations of motion

$$\frac{ds}{dt}=v_\parallel,\quad \frac{dv_\parallel}{dt}=-I\frac{d\Omega}{ds}+\varepsilon k_\parallel\cos\zeta,$$
$$\frac{d\varphi}{dt}=\Omega-\frac{\varepsilon k_\perp^2}{\mu\Omega}\sin\varphi\cos\zeta,\quad \frac{dI}{dt}=\varepsilon\mu\cos\varphi\cos\zeta, \tag{2.3}$$

where

$$\varepsilon=-\frac{e\Phi_0}{m},\quad \mu=k_\perp\left(\frac{2I}{\Omega}\right)^{1/2},\quad \zeta=\int k_\parallel\,ds+\mu\sin\varphi-\omega t. \tag{2.4}$$

Following the standard procedure of separating the resonances in particle-wave interaction, we write the Hamiltonian (2.2) in the form

$$H={}^1/_2 v_\parallel^2+I\Omega-\varepsilon\sum_{n=-\infty}^{\infty} J_n(\mu)\sin\xi_n, \tag{2.5}$$

where $J_n(\mu)$ is a Bessel function of order n, and

$$\xi_n=\int^s k_\parallel(s')\,ds'+n\varphi-\omega t. \tag{2.6}$$

From (2.5) we get an equation for ξ_n (J_n' is the derivative of J_n with respect to μ):

$$\frac{d\xi_n}{dt}=u_n-\frac{\varepsilon n k_\perp^2}{\mu\Omega}\sum_n J_n'(\mu)\sin\xi_n,\qquad u_n=k_\parallel v_\parallel+n\Omega-\omega. \tag{2.7}$$

The vanishing of u_n is the condition of the nth cyclotron resonance. For particles that satisfy this condition, the nth term in the sum (2.5) is a slowly varying quantity, while the nearest terms oscillate at a frequency Ω. If the characteristic value of $d\xi_n/dt$ in the resonance region is larger than or of the order of Ω, the resonances overlap and, according to Chirkov's criterion, the motion of the particles is stochastic. The inverse inequality

$$\max\left(\Delta u_n,\ \frac{\varepsilon n k_\perp^2 J_n'(\mu)}{\mu\Omega}\right)\ll\Omega, \tag{2.8}$$

where Δu_n is the change of u_n in the resonance region, corresponds to the case of isolated resonances. When the conditions (2.8) are satisfied, the motion of the particles in the resonance region can be described by the averaging method,[16] i.e., only the slowly varying nth term need be retained in the Hamiltonian (2.5). It is important that, owing to the variation of the quantities $k_\parallel$, Ω, and $v_\parallel$, a particle in an inhomogeneous medium cannot remain infinitely long at the nth resonance with the wave, and inevitably goes over from one resonance to another. To describe the particle motion over a sufficiently long time it is therefore necessary to start from the exact Hamiltonian (2.2). We consider now the equation of motion for the nth cyclotron resonance, assuming that condition (2.8) is met. In this case the motion is described by the Hamiltonian

$$H={}^1/_2 v_\parallel^2+I\Omega-\varepsilon J_n(\mu)\sin\left(\int^s k_\parallel ds'+n\varphi-\omega t\right). \tag{2.9}$$

Since the variables φ and t enter in (2.9) only in the form of the combination $n\varphi-\omega t$, the quantity

$$\varkappa_n=nH-\omega I \tag{2.10}$$

is an integral of the motion and allows us to make the problem one-dimensional. It suffices for this purpose to choose the coordinate s as the new independent variable, to make ξ_n the new phase, and to eliminate from the equations of motion the quantity $v_\parallel$ with the aid of the integral (2.10). As a result we get for the canonical variables (I,ξ_n) equations that are specified by the Hamiltonian

$$\mathscr{H}_n=(k_\parallel I-nv_0)-\frac{\varepsilon nJ_n}{v_0}\sin\xi_n,$$
$$v_0=\left\{\frac{2}{n}\left[\varkappa_n+I(\omega-n\Omega)\right]\right\}^{1/2}\operatorname{sign}v_\parallel. \tag{2.11}$$

We drop hereafter the subscript n of the phase ξ_n. At $k_\parallel\neq 0$ and under some additional conditions (see below) the Hamiltonian (2.11) can be reduced to standard form, i.e., to a sum of a kinetic and a potential energy. It is convenient for this purpose to change to a new independent variable τ, which is a single-valued function of s:

$$\tau=\int^s \frac{k_\parallel^2 ds'}{nv_R}, \tag{2.12}$$

where v_R is the resonant value of the longitudinal velocity of the particle

$$v_R=(\omega-n\Omega)/k_\parallel, \tag{2.13}$$

after which the Hamiltonian take the form

$$\mathscr{H}=\mathscr{H}_0-\frac{\varepsilon n^2 v_R J_n(\mu)}{k_\parallel^2 v_0}\sin\xi,\quad \mathscr{H}_0=\frac{n v_R}{k_\parallel^2}(k_\parallel I-n v_0). \tag{2.14}$$

The vanishing of the derivative $\partial\mathscr{H}_0/\partial I$ determines the resonant value of I as a function of the variable τ and of the parameter $\varkappa_n$:

$$k_\parallel+\frac{n\Omega-\omega}{v_0}=0\quad\text{yields}\quad I=I_R=\frac{n v_R^2-2\varkappa_n}{2(\omega-n\Omega)}. \tag{2.15}$$

At $I=I_R$ we have $v_0=v_R$. Expanding now $\mathscr{H}_0$ near I_R up to quadratic terms and substituting the value $I=I_R$ in the second term of (2.14) we get

$$\mathscr{H}=\frac{1}{2}(I-I_R)^2-\frac{\varepsilon n^2 J_n(\mu_R)}{k_\parallel^2}\sin\xi. \tag{2.16}$$

The region of applicability of (2.16) is determined by the inequalities

$$k_\parallel\Delta(I-I_R)/n v_R\ll 1,\quad \Delta(\mu-\mu_R)\ll 1, \tag{2.17}$$

where $\Delta(I-I_R)$ and $\Delta(\mu-\mu_R)$ are the variations of the corresponding quantities in the resonance region. The explicit expressions for the inequalities (2.17), as well as for (2.8), depend on the relation between the nonlinearity and inhomogeneity, and will be discussed below. It must be emphasized that I_R in (2.16) is a function of the "time" τ, so that in a number of cases it is more convenient to replace I by another momentum, u:

$$u=I-I_R. \tag{2.18}$$

Obviously, u is proportional to the deviation of the particle longitudinal velocity from the resonant value v_R and differs therefore only by a factor from u_n (2.7). Transforming to the variable u, we obtain in place of (2.16), in accordance with the general formulas for canonical transformations,[17] a Hamiltonian in the form

$$\mathscr{H}'=u^2/2-\beta\sin\xi+a\xi \tag{2.19}$$

and the corresponding equations of motion

$$d\xi/d\tau=u,\quad du/d\tau=\beta\cos\xi-a, \tag{2.20}$$

where

$$\beta=\frac{\varepsilon n^2 J_n(\mu_R)}{k_\parallel^2},\quad a=\frac{dI_R}{d\tau}=\frac{n^2}{k_\parallel^3}\left(\frac{1}{2}\frac{dv_R^2}{ds}+I\frac{d\Omega}{ds}\right). \tag{2.21}$$

3. PASSAGE THROUGH AN ISOLATED RESONANCE

The particle motion described by Hamiltonian (2.19) with constant or slowly varying parameters was discussed in many papers (see, e.g., Ref. 18). Using the results of these papers, we obtain expressions for the change of the momentum I on passage through an isolated resonance. These expressions are needed to estimate the coefficients of the diffusion that sets in on passage through many cyclotron resonances.

The Hamiltonian (2.19) corresponds to a potential energy $U=a\xi-\beta\sin\xi$. The case $|\beta/a|>1$ corresponds to weak inhomogeneity; U has in this case potential wells in which phase-trapped particles move. At $|\beta/a|<1$ there are no potential wells and trapped particles; this corresponds to the case of strong inhomogeneity. An essentially new factor in our analysis is allowance for the oscillations of the quantity β. As a result of these oscillations, the value of β goes through zero, and the potential U degenerates into a straight line, so that all the trapped particles leave the potential wells.

We begin the analysis with the case of weak inhomogeneity, assuming $|\beta/a|\gg 1$. According to Refs. 11 and 18 the change of the momentum I as a result of passage of an untrapped particle through resonance (corresponding to reflection of the particle from the potential) is equal to

$$\Delta I_{UT}=-\frac{8}{\pi}|\beta_R|^{1/2}\operatorname{sign}a, \tag{3.1}$$

where the subscript R denotes the value of β at the instant of reflection (exact resonance). For trapped particles we have at the same time

$$\overline{dI/d\tau}=a, \tag{3.2}$$

which follows directly from (2.18) and (2.21) when it is recognized that $\bar{u}=0$. The changes of the momentum I of untrapped and trapped particles are of opposite sign. We calculate now the average and mean squared changes of I as β varies from zero to its maximum and then back to zero. Taking into account the definition (2.21) of β and the asymptotic behavior of the Bessel function at $\mu>n\gg 1$, we note that in this case μ_R changes by π. The corresponding time $\Delta\tau$ can be easily obtained from the equation [see (2.4) and (2.21)]

$$d\mu_R/d\tau=k_\perp^2 a/\mu\Omega, \tag{3.3}$$

so that

$$\Delta\tau=\pi\mu\Omega/k_\perp^2|a|. \tag{3.4}$$

Since I is constant far from resonance, the rate of entry of the particles into the resonance region is determined by $dI_R/d\tau$ and is equal to $|a|$. Consequently, the phase space of the particles that interact resonantly with the wave during the time $\Delta\tau$ is $|a|\Delta\tau$. We denote by Γ the effective phase volume of the trapped particles[18]:

$$\Gamma=(8/\pi)|\beta|^{1/2}, \tag{3.5}$$

and by Γ_T its maximum value. If the phase volume of the trapped particle increases, then the fraction of the particles which is proportional to $\dot\Gamma=d\Gamma/d\tau$ is trapped by the wave, and the remainder is reflected from the potential so that the capture and reflection probabilities are respectively

$$W_T=\dot\Gamma/|a|,\quad W_r=(|a|-\dot\Gamma)/|a|. \tag{3.6}$$

Equations (3.6) hold at $|a|-\dot\Gamma>0$; in the opposite case, all the particles would be captured. It can be easily verified that (2.8) and (2.17) lead to $|a|\gg\dot\Gamma$. If, however, the phase space decreases, a fraction of the trapped particles goes over into the untrapped region, and all the untrapped particles are reflected from the potential. The phase volume of the particle in the trapped region is thus Γ_T, and the mean time is $\Delta\tau/$

2. The change of momentum of such particles is then

$$\langle \Delta I_T\rangle = a\frac{\Delta\tau}{2} - \frac{8}{\pi}\langle|\beta|^{1/2}\rangle\,\mathrm{sign}\,a, \tag{3.7}$$

where $\langle|\beta|^{1/2}\rangle$ is the mean value of $|\beta|^{1/2}$ and account is taken of the fact that on leaving trapped region the particle experiences a momentum change that corresponds to its reflection from the potential [see (3.1)]. For trapped particles, the phase volume is $(|a|\Delta\tau - \Gamma_T)$, and the mean value of the particle momentum is given by (3.1) but with the substitution $|\beta_R|^{1/2}\to\langle|\beta|^{1/2}\rangle$. Taking the foregoing into account, we obtain by elementary calculations for the average and mean squared change of the momentum the expressions

$$\langle \Delta I\rangle = \frac{1}{|a|\Delta\tau}[\langle\Delta I_{UT}\rangle(|a|\Delta\tau-\Gamma_T)+\langle\Delta I_T\rangle\Gamma_T]=0, \tag{3.8}$$

$$\langle \Delta I^2\rangle = \frac{1}{|a|\Delta\tau}[\langle\Delta I_{UT}^2\rangle(|a|\Delta\tau-\Gamma_T)+\langle\Delta I_T^2\rangle\Gamma_T]$$

$$=\frac{64}{\pi^2}\langle|\beta|\rangle\frac{|a|\Delta\tau-\Gamma_T}{\Gamma_T}, \tag{3.9}$$

where $\langle|\beta|\rangle$ is the mean value of $|\beta|$.

We proceed now to calculate $\langle\Delta I\rangle$ and $\langle\Delta I^2\rangle$ in the case of strong inhomogeneity, when $|\beta/a|\ll 1$. In this case all the particles are untrapped, and passage through resonance corresponds to their reflection from the effective potential U. According to (2.20) the phase ξ in the vicinity of the resonance can be represented at $|\beta/a|\ll 1$ as

$$\xi=\xi_R - a\tau^2/2. \tag{3.10}$$

Substituting (3.10) in the equation $dI/d\tau = \beta\cos\xi$ that follows from (2.16) and integrating over to $d\tau$, we obtain

$$\Delta I=\beta_R\left(\frac{2\pi}{|a|}\right)^{1/2}\cos\left[\xi_R-\frac{\pi}{4}\,\mathrm{sign}\,a\right], \tag{3.11}$$

where ξ_R is the value of the phase at the reflection point. Since ξ_R depends on the gyrophase φ via the term $n\varphi$, it follows from (3.11) that $\langle\Delta I\rangle = 0$ and

$$\langle\Delta I^2\rangle=\pi\langle\beta^2\rangle/|a|. \tag{3.12}$$

We have calculated above the average and mean squared changes of the particle momentum on passage through an isolated resonance. It follows from (2.7) that a transition of a particle between resonances corresponds to a change of u_n by Ω. From the obvious equality

$$du=dI-a\,d\tau,$$

which follows from (2.18) and (2.21) we find that the average time (in units of τ) of transition of a particle between resonances is $\delta\tau = |n\Omega/k_\parallel^2 a|$. Account was taken here of the relation $u = nu_n/k_\parallel^2$ between u_n and u, as well as of the equality $\langle\Delta I\rangle = 0$. The relation obtained yield estimates of the particle-diffusion coefficients in phase space, which determine the rate of particle heating: $D\sim\langle\Delta I^2\rangle/\delta\tau$. These questions are discussed in greter detail in the sections that follow.

4. TRANSITION TO POINCARÉ MAPPING

The preceding section dealt with the change of the transverse adiabatic invariant of a particle on passing through an isolated resonance. Our main purpose, however, is to investigate the passage of particles through many cyclotron resonances and the ensuing particle diffusion in phase space. In this case, as already noted, we must start from the exact equations (2.3).

Particle-wave interaction described by Eqs. (2.3) is effective if the quantity ζ has on the cyclotron circle stationary-phase points defined by the equation

$$\mu\Omega\cos\varphi+k_\parallel v_\parallel-\omega=0. \tag{4.1}$$

Obviously, stationary-phase points exist under the condition $\mu>\nu$, where

$$\nu=(\omega-k_\parallel v_\parallel)/\Omega. \tag{4.2}$$

Furthermore, under the condition $\Omega\gg\varepsilon k_\perp^2/\mu\Omega$, which we assumed satisfied, the influence of the wave on the particle dynamics becomes manifested only over many cyclotron periods. In this case we can transform from the differential equations (2.3) to the Poincaré mapping, which establishes the connection between the quantities at the points $\varphi = n\pi$. This procedure was carried out for the case of perpendicular wave propagation in the homogeneous and inhomogeneous cases in Refs. 19 and 12, respectively. Using a similar approach for the case of an arbitrary wave-propagation angle, we obtain a model mapping that is described by the following system of difference equations:

$$\begin{gathered}\rho_{m+1}=\rho_m+A_1\sin[\rho_m-(-1)^m\theta_m],\\ \theta_{m+1}=\theta_m+\pi\nu_{m+1}+(-1)^mA_1\sin[\rho_m-(-1)^m\theta_m],\\ \nu_{m+1}=\nu_m-A_2\sin[\rho_m-(-1)^m\theta_m]+\alpha,\end{gathered} \tag{4.3}$$

where the subscript m denotes the values of the corresponding quantities at $\varphi = m\pi$. The dimensionless Larmor radius $\rho = k_{\perp 0}(2I/\Omega_0)^{1/2}$ plays here the role of the momentum, θ is the generalized phase, and the quantity ν is defined in (4.2). The amplitude A_1 and A_2 are of the following orders of magnitude:

$$A_1\sim(2\pi)^{1/2}\varepsilon\frac{k_\perp^2\nu}{\rho^{1/2}\Omega^2},\quad A_2\sim(2\pi)^{1/2}\varepsilon\frac{k_\parallel^2}{\mu^{1/2}\Omega^2}. \tag{4.4}$$

The parameter α, which has the meaning of the phase acceleration, is the dimensionless analog of the quantity a (2.21):

$$\alpha=(\pi k_\parallel^4/\nu^2\Omega^2)\,a_{n=\nu}. \tag{4.5}$$

The subscript $n = \nu$ in (4.5) means that after differentiating with respect to s it is necessary to replace n by ν. It follows from (4.2) that the resonance conditions coincide with the equality $\nu = n$, in which case the difference $\theta_{m+2} - \theta_m$ is equal to an integer multiple of 2π (if no account is taken of the term $\sim A_1$).

We write down now the conditions (2.8) and (2.17) in terms of the dimensionless parameters A_1, A_2, and α. We note first that a transition in the case of purely transverse propagation is determined by the inequality $A_2\ll A_1^2$, and the criteria of weak and strong inhomogeneity take respectively the forms $\max(A_1^2, A_2) > \alpha$ and $\max(A_1^2, A_2) < \alpha$. Next, in

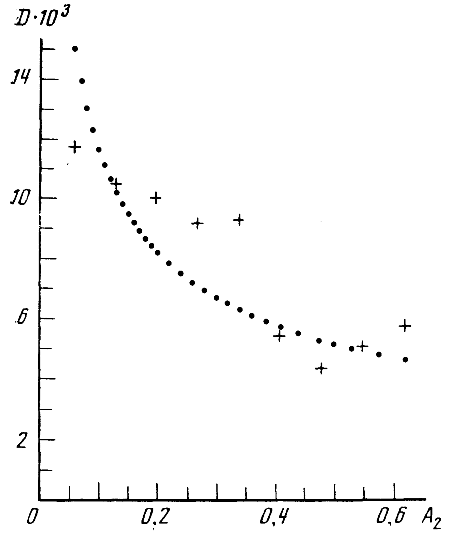

FIG. 1. Diffusion coefficient D vs the amplitude A_1 at $A_1 = 0.17$, $\alpha = 0.011$.

the case of weak inhomogeneity the isolated-resonances conditions (2.8) take the form $A_1 \ll 1$, $A_2^{1/2} \ll 1$, and the possibility of transforming to a Hamiltonian in the standard form (2.16), (2.19) [i.e., the condition (2.17)] is set by the inequality $A_1 \ll A_2^{1/2}$. In the case of strong inhomogeneity these conditions take the respective forms $A_1 \ll 1, |\alpha|^{1/2} \ll 1$ and $A_1 |\alpha|^{1/2} \ll A_2$. Using the definitions (2.4) and (4.4), we can easily rewrite these inequalities in terms of physical dimensional parameters.

5. PARTICLE DIFFUSION ON PASSAGE THROUGH MANY RESONANCES

To ascertain the character of particle motion over long times corresponding to passage through many cyclotron resonances, we solved the system (4.3) numerically. In each variant, the parameters A_1, A_2, and α were assumed for simplicity to be constants. The system (4.3) has then the obvious integral of motion

$$\nu_m - \alpha m + (A_2/A_1)\rho_m = \text{const}, \tag{5.1}$$

whose conservation was monitored in the course of the computation. We considered 400 particles uniformly distributed at the initial instant of time ($m = 0$) along a straight line $\rho = \theta$ in an interval $0 < \theta < 2\pi$. The total number of steps was 6000 and for every 200 steps we calculated the average and mean squared changes of the momentum over all the particles. For the mean square we use hereafter the notation

$$\Delta = \langle (\rho - \rho_0)^2 \rangle. \tag{5.2}$$

We note first that in all the variants considered the average momentum change $\langle \rho - \rho_0 \rangle$ is close to zero [in accord with (3.8) and (3.12)], and the mean squared change Δ increases in proportion to the number m of the steps. This linearity of $\Delta(m)$ holds for both strong and weak inhomogeneity. One can speak in this case of particle diffusion in phase space, with a diffusion coefficient

$$D = \Delta/m. \tag{5.3}$$

It follows from the numerical calculations, in particular, that at $\alpha \ll A_2$ the mean squared change of the momentum on passage through a cyclotron resonance does not change on α [see (3.4) and (3.9)].

Note that all the analytic estimates of Sec. 4 pertain to the case of isolated resonances, i.e., max $(A_1, A_2^{1/2}, |\alpha|^{1/2}) \ll 1$, whereas the region of applicability of the mapping (4.3) is much wider and is determined by the conditions

$$A_1 \ll \nu/\mu^{1/2}, \quad A_2 \ll \mu^{1/2}, \quad |\alpha| \ll \nu^{1/2}. \tag{5.4}$$

We shall therefore study the mapping (4.3) and the diffusion coefficient D associated with it in the entire range of the inhomogneity parameter α, confining ourselves at the same time to the values of the amplitudes A_1 and A_2 below the threshold of stochasticity in a homogeneous medium, i.e., $A_1 \ll 1$ and $A_2^{1/2} \ll 1$.

At $|\alpha| \ll 1$ the diffusion coefficient is determined by two factors: the mean squared change of the momentum on passage through one resonance, and the number of resonance passed per unit time, equal to $|\alpha|$. In this case the analysis of Sec. 4 yields estimates for the diffusion coefficients in the strong and weak inhomogeneity limits:

$$\begin{aligned} &D \sim A_1 |\alpha| / A_2^{1/2}, \quad \alpha \ll A_2 < 1, \quad A_1 \ll A_2^{1/2}, \\ &D \sim A_1^2/2, \quad A_2 \ll \alpha < 1, \quad A_1 |\alpha|^{1/2} \ll A_2. \end{aligned} \tag{5.5}$$

Numerical calculations show that the dependence of the diffusion coefficient on the amplitudes A_1 and A_2 agrees well with the analytic formulas (5.5). The nontrivial $D(A_2)$ dependence for the case of weak inhomogeneity is shown in Fig. 1. The dotted curve is a plot of the relation $D = 3.7 \cdot 10^{-3} A_2^{-1/2}$ that approximates the results of the numerical calculations (crosses) by least squares. For the chosen parameter values, (5.5) leads to $D \approx 1.9 \cdot 10^{-3} A_2^{-1/2}$.

Proceeding to discuss the dependence of the diffusion coefficient on the inhomogeneity parameter α, we note first that D is a periodic function of α with period 2, as follows directly from (4.3). In addition, it can be seen from the results of Sec. 3 that at $|\alpha| \ll 1$ both the mean squared change of the momentum on passage through an isolated resonance [(3.9), (3.12)] and the time of motion of the particle between resonances (equal to $|\alpha|^{-1}$ in dimensionless variables) are independent of the sign of α. The diffusion coefficient is therefore an even function of α. The numerical calculation shows that this property of the diffusion coefficient is

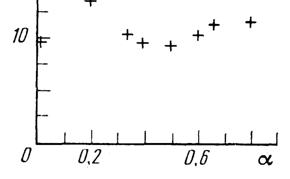

FIG. 2. Diffusion coefficient D vs the inhomogeneity parameter α.

not restricted by the condition $|\alpha| \ll 1$. The $D(\alpha)$ dependence in the interval $0 < \alpha < 1$ is shown for $A_1 = 0.17, A_2 = 0.11$ in Fig. 2. Note the approximate symetry of D about $\alpha = 0.5$, which suggests, together with the parity, an approximate periodicity of the diffusion coefficient in α with a period 1 (besides the obvious rigorous period 2).

We have thus shown that when particles interact with a monochromatic wave in an inhomogeneous plasma, particle heating (diffusion in phase space) takes place even at wave amplitudes below the stochasticity threshold in a homogeneous plasma, and is due to passage through many cyclotron resonances. This interpretation is meaningful at $|\alpha| \ll 1$. With increasing $|\alpha|$ the number of resonances passed per unit time increases. At $|\alpha| \gtrsim 1$ the foregoing interpretation becomes meaningless, whereas the mapping (4.3), as well as the general adiabaticity conditions $v_\parallel d\Omega/ds \ll \Omega^2$, remain in force. As shown above, the diffusion coefficient acquires periodicity when the inhomogeneity parameter is increased, so that the heating rate is determined by mod $(\alpha,1)$. At variable α one can therefore speak of an average diffusion coefficient in an inhomogeneous medium.

The main results of this study, which pertain to diffusion and heating of the particles, apply if the inequalities $\mu > \nu \gg 1, v_t \gg \Omega/k_\parallel$ are satisfied (v_1 is the thermal velocity). In dimensional variables the first condition is of the form

$$k_\perp v_\perp/\Omega > (\omega - k_\parallel v_\parallel)/\Omega \gg 1,$$

which ensures a large number of cyclotron resonances with a noticeable interaction amplitude equal to $\varepsilon J_n(\mu)$. The second condition ensures the presence of an appreciable number of particles that interact effectively with the wave. It should be noted that the analysis in Secs. 2 and 3 is not subject to these conditions.

From among the applications of the foregoing results we point out, for example, ion heating by lower-hybrid waves in tokamaks, proton spilling by VLF waves, interaction of electrons with upper hybrid waves in the magnetosphere, and others.

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