# The Papapetrou and Dixon equations of motion

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It is shown that the well-known equations obtained by Papapetrou for a dipole particle are not equivalent to the initial equations of the original paper and that this explains the existence of unphysical solutions of these equations. The evolution equations obtained by Dixon and Madore are satisfactory. They are used here to study as an example the precession of a Weyl top. It is found that in the case of a multipole particle Dixon's equations using reduced moments are inconsistent with the solution to the evolution problem.

## 1. INTRODUCTION. MULTIPOLE DESCRIPTION OF AN EXTENDED BODY

The motion of a free material particle in an external determined by means of the principle of least action and is described by a goedesic world line. By its definition (Ref. 1, §9), the particle's 4-momentum is tangent to this world line.

This simple solution is valid, however, for a structureless particle when its size is ignored. For an extended body, represented by a timelike world tube in a given space-time, there does not exist a world line that "represents" it, in contrast to the case of a particle. Another difference is that in order to characterize the extended body it is necessary to use a complete set of integral quantities (the first of which is usually the 4-momentum of the body). Such quantities are defined along some arbitrarily fixed world line, and the motion of the extended body is described in terms of the variation of these quantities along it. It is important that the equations which the integral quantities satisfy are based solely on the local conservation law for the energy-momentum tensor of the matter of the body:

$$\nabla_\nu T^{\alpha\nu}=0, \quad \alpha,\nu=0,1,2,3. \tag{1}$$

To form the integral quantities, one can employ very general integral transformations of the tensor $T^{\alpha\nu}$.[2,3] However, in this paper we shall consider the widely used approach[3–12] in which the role of the integral quantities is played by multipole moments of the body of two different types; briefly, they are called $p$ and $\tau$ moments:

$$p^{\nu_1\ldots\nu_n\alpha}(x)=\frac{1}{n!}\int_S \sigma^{\nu_1}(x',x)\ldots\sigma^{\nu_n}\Phi_1{}^{\alpha}{}_{\alpha'}T^{\alpha'\nu'}\sqrt{-g}\,dS_{\nu'}, \tag{2}$$

$$\tau^{\nu_1\ldots\nu_n\alpha\nu}(x)=\frac{1}{n!}\int_S \sigma^{\nu_1}\ldots\sigma^{\nu_n}\Phi_2{}^{\alpha}{}_{\alpha'}\Phi_3{}^{\nu}{}_{\nu'}T^{\alpha'\nu'}\sqrt{-g}\,w^{\mu'}\,dS_{\mu'}. \tag{3}$$

The multipole moments (2) and (3) are determined at the points of the world line $x^\nu(s)$, which can be conveniently called the world line of the "observer." They are calculated by means of spacelike hypersurfaces of a single-parameter family $S(s)$ parametrized by the same parameter $s$ as the observer world line $x(s)$. The points of the hypersurface $S(s)$ of integration are identified by a prime: $x'$. We use the notation system employed in the books of Refs. 13–15, in accordance with which primes distinguishing distinct points are transferred to the coordinate index, i.e., we write $x^{\nu'}$ instead of $x'^{\nu}$ or $T^{\alpha'\nu'}$ instead of $T^{\alpha\nu}(x')$. The vector field $w^\mu$ in formula (3) satisfies the condition

$$w^\mu s_\mu=1, \quad s_\mu=\partial_\mu s(x),$$

in which $s(x)$ is the function that parametrizes the hypersurface: $s(x)=\text{const}$, if $x\in S(s)$.

In order to make the multipole moments tensors at the point $x(s)$, the definitions (2) and (3) contain the tensor two-point (path-independent) functions

$$\Phi_1{}^{\alpha}{}_{\alpha'}(x,x'), \quad \Phi_2{}^{\alpha}{}_{\alpha'}, \quad \Phi_3{}^{\nu}{}_{\nu'},$$

which transport the tensors $T^{\alpha'\nu'}dS_\nu$ and $T^{\alpha'\nu'}w^{\mu'}dS^{\mu'}$ from the points $x'$ to the general point $x$, i.e., associate with them the tensors

$$\Phi_1{}^{\alpha}{}_{\alpha'}T^{\alpha'\nu'}dS_{\nu'}, \quad \Phi_2{}^{\alpha}{}_{\alpha'}\Phi_3{}^{\nu}{}_{\nu'}T^{\alpha'\nu'}w^{\mu'}dS_{\mu'}.$$

These functions satisfy the natural condition $\Phi^{\alpha}_{\beta'}\to\delta^{\alpha}_{\beta}$ as $x'\to x$ and are called translators. The papers of various authors[4–6,9,16] differ in the type of translators employed.[1)] For example, Oliver[16] uses the translator of parallel transport along geodesics. It is denoted in Ref. 16 by $g^{\alpha}_{\alpha'}$. The same translator was used by Dixon in Ref. 5. However, in the series of his papers of Refs. 6–8 the role of $\Phi_1$, $\Phi_2$, $\Phi_3$ is played by Dixon's translators $K$ and $H$ (see also Ref. 10). The simplest translator was used by Papapetrou.[4] His translator is simply equal to the Kronecker delta in the employed coordinate system. The general theory of two-ponit functions is presented in Ref. 17, where they are determined by means of Taylor-type expansion series. For example, the parallel-transport translator from the point $x'=x+\Delta x$ to the point $x$ with allowance for the first two terms of the expansion has the form

$$g^{\alpha}{}_{\beta'}=\delta^{\alpha}{}_{\beta'}+\delta^{\beta}{}_{\beta'}\Gamma^{\alpha}{}_{\beta\nu}\Delta x^\nu.$$

The two-point vector $\sigma^\nu(x',x)$ in Eqs. (2) and (3) is usually constructed by differentiating Synge's world function $\Omega(x',x)$ (Ref. 18, Chap. 2):

$$\sigma^\nu(x',x)=-g^{\mu\nu}\partial_\mu\Omega(x',x).$$

The difference between the $p$ and $\tau$ moments is due to the fact that information about the tensor field $T^{\alpha\nu}(x)$ is lost because of the scalar multiplication $T^{\alpha\nu}dS_\nu$ employed in the formation of the $p$ moments. In contrast, the $\tau$ moments contain all information about the field $T^{\alpha\nu}$. More-

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over, they are independent of each other unless restrictions are imposed on the tensor $T^{\alpha\nu}$. Use of the conservation law (1) leads to a countable set of relations between the $p$ and $\tau$ moments, regarded as functions of $s$, and the derivatives of the $p$ moments along the observer's world line.[11,2,3] These relations are the dynamic equations of the extended body. In the present paper, they are given for the Newtonian case (see Sec. 5).

## 2. EQUATIONS OF MOTION OF A DIPOLE PARTICLE

A body for which the multipole moments of order greater than $N$ are ignored is called a multipole particle of $N$th order. Allowance for only a finite number of moments is equivalent to the energy-momentum tensor of the body being regarded as a singular function whose support is the observer's world line $x^\nu(s)$, since any body with a world tube of extended cross section possesses an infinite number of nonvanishing moments.

The multipole particle of first order has often been considered in the literature. It is, in particular, the subject of Ref. 4. An important property of such a particle is the possibility of eliminating for it all the $\tau$ moments and the symmetric part $p^{(\alpha\nu)}$ of the momentum moment from the dynamic equations given above, so that these equations are expressed solely in terms of the momentum $p^\alpha$ and the "spin" $p^{[\alpha\nu]}$. The particle is therefore called a dipole particle. However, two different systems of equations (dipole particle dynamics) obtained as the result of such elimination are known.

1) Papapetrou's equations

$$\frac{D}{ds}\left(p^\beta v_\beta u^\alpha+2v_\beta\frac{D}{ds}p^{[\alpha\beta]}\right)+R^{\ \ \ \alpha}_{\beta\mu\nu}u^\beta p^{[\mu\nu]}=0, \tag{4}$$

$$\frac{D}{ds}p^{[\alpha\beta]}+u^\alpha v_\nu\frac{D}{ds}p^{[\beta\nu]}-u^\beta v_\nu\frac{D}{ds}p^{[\alpha\nu]}=0. \tag{5}$$

These equations were obtained in Ref. 4. are given in Ref. 19 (§20.6, problem 40.8), and are widely used in the literature. Here, $u^\alpha=dx^\alpha/ds$, and $v_\nu$ is an arbitrary vector defined on the observer's line and satisfying the single condition $v_\nu u^\nu=1$.

2) The equations obtained by Dixon[5] and Madore[9]:

$$\frac{D}{ds}p^\alpha+R^{\ \ \ \alpha}_{\beta\mu\nu}u^\beta p^{[\mu\nu]}=0, \tag{6}$$

$$\frac{D}{ds}p^{[\alpha\beta]}+u^{[\alpha}p^{\beta]}=0. \tag{7}$$

These equations are simple and, very importantly, evolutionary equations in the sense that they enable one, given the momentum $p^\alpha$ and the spin $\partial^{[\nu\alpha]}$, to calculate the derivatives of these quantities.

Comparing the systems of Eqs. (4)–(5) and (6)–(7), we readily see that Papapetrou's equations are a consequence of Eqs. (6) and (7). Indeed, contracting Eq. (7) with the covariant vector $v_\alpha$, we find for the momentum the expression

$$p^\nu=p^\alpha v_\alpha u^\nu+2v_\alpha\frac{D}{ds}p^{[\nu\alpha]}, \tag{8}$$

and then, substituting (8) in (6), we obtain (4). With regard to Eq. (5), it simply reflects the fact that $Dp^{[\nu\alpha]}$ in Eq. (7) is a simple bivector, i.e., $u^{[\mu}Dp^{[\nu\alpha]]}=0$. Contraction of this equation with $v_\mu$ gives Eq. (5).

The here-indicated transition from the system (6)–(7) to the system (4)–(5) shows that these systems are inequivalent. Indeed, the system (4)–(5) is much weaker than the system (6)–(7). Dixon pointed out[5] that the system (4)–(5) necessarily includes the second derivative of $u^\alpha$ or $p^{[\nu\alpha]}$ and, therefore, admits a specification of the position, velocity, spin, and acceleration of the particle, whereas physically the acceleration cannot be specified independently.

## 3. CRITICISM OF THE DERIVATION OF PAPAPETROU'S EQUATIONS

It is of interest to see how Eqs. (4) and (5) were obtained in Ref. 4.

A distinctive feature of Papapetrou's approach in this paper is the use of the simplest translator, having $\delta$-function form in the employed coordinate system, as $\Phi^\alpha_{1\alpha'}$, $\Phi^\alpha_{2\alpha'}$, $\Phi^\nu_{3\nu'}$ in (2), (3). The momentum moments $p^\alpha$, $p^{\nu\alpha}$ and the expression for the second-order moment, which is equal to zero for a dipole particle, $p^{\nu_1\nu_2\alpha}=0$, that correspond to such a choice of the translators are covariantly differentiated in Ref. 4 along the observer world line, again with respect to the $\delta$ translator. After this, in agreement with the standard method for obtaining evolution equations in the dynamics of an extended body,[2,3,9–12] Green's theorem and the conservation law (1) are used.

The resulting equations

$$\frac{d}{dx^0}p^\alpha=-\Gamma_{\beta\nu}{}^\alpha\tau^{\beta\nu}-\tau^{\mu\beta\nu}\partial_\mu\Gamma_{\beta\nu}{}^\alpha, \tag{9}$$

$$\frac{d}{dx^0}p^{\nu\alpha}=-u^\nu p^\alpha+\tau^{\alpha\nu}-\Gamma_{\beta\mu}{}^\alpha\tau^{\nu\beta\mu}, \tag{10}$$

$$0=-u^{(\nu}p^{\nu_1)\alpha}+\tau^{(\nu_1\nu)\alpha} \tag{11}$$

are given in Ref. 4 under the numbers (3.2), (3.3), (3.4). They are indisputable and serve there as the initial equations for the procedure for eliminating the $\tau$ moments and $p^{(\nu\alpha)}$. However, the elimination of these moments from the system (9)–(11) and its transformation into the system (4)–(5) is made in Ref. 4 by a long chain of inequivalent transformations, and we shall not reproduce them here, since this elimination can be readily done by using a coordinate system locally geodesic at the points of the observer's world line (the generality of the results is not restricted by doing so). In such coordinates,

$$\Gamma^\alpha{}_{\beta\nu}=0,\quad u^\mu\partial_\mu\Gamma^\alpha{}_{\beta\nu}=0$$

and antisymmetrization of Eq. (10) immediately leads to Eq. (7), and not (5). Further, using the symmetry $\tau^{\nu_1\nu\alpha}=\tau^{\nu_1(\nu\alpha)}$, we can express $\tau^{\nu_1\alpha\nu}$ from (11) as follows:

$$\tau^{\nu_1\alpha\nu}=u^{\nu_1}p^{(\nu\alpha)}+u^\nu p^{[\nu_1\alpha]}+u^\alpha p^{[\nu_1\nu]}. \tag{12}$$

After this, substitution of the expression (12) in (9) with allowance for the fact that the coordinates are locally geodesic gives Eq. (6) and not (4).

Thus, the initial equations of the paper Ref. 4 lead to the system of equations (6)–(7), whereas Eqs. (4)–(5) are weaker consequences of them and do not describe the behavior of the particle.

## 4. EXAMPLE OF THE USE OF THE SYSTEM OF EQUATIONS (6)–(7)

It should be noted that Eqs. (6)–(7), like the system (4)–(5), are not by themselves equations of motion of the particle, since they do not impose any restrictions on the "observer" world line representing the particle. Instead, they make it possible to calculate $p^\alpha(s)$ and $p^{[\nu\alpha]}$ from initial values $p^\alpha(0)$ and $p^{[\nu\alpha]}(0)$ for arbitrary specification of this line. The moments $p^{(\nu\alpha)}(s)$, $\tau^{\alpha\nu}(s)$, $\tau^{\nu_1\alpha\nu}(s)$, that ensure such "motion" can be determined after a calculation on the basis of the initial equations (9)–(11).

A simple three-dimensional analogy of such a situation is a stressed fiber of arbitrary shape in equilibrium in a curved space under the influence of a mechanical load consisting of a set of forces and couples applied at the end. This set is equivalent to the momentum and spin specified at the end of the world line.

This arbitrariness for a real particle in space-time is eliminated by the condition of the mass-energy being positive definite. It can be shown that if this condition is to hold in the case of a dipole particle it is necessary to satisfy the natural requirement that the momentum $p^\alpha$ be tangent to the particle's world line, as is the case for a monopole (structureless) particle. If this condition is adopted, $p^\alpha \propto u^\alpha$, the system of equations (6)–(7) makes it possible, given initial data, to calculate the complete world line of the particle, which is not a geodesic when $R^{\ \alpha}_{\beta\mu\nu} \neq 0$. However, for such a particle we always have $Dp^{[\nu\alpha]} = 0$, i.e., there is parallel transport of the spin of the particle along its world line.

It is such behavior of the spin that leads to Weyl precession of an orbiting top (Ref. 1, §106, problem 4) which we calculate as an example, ignoring the deviations of the world line from a geodesic.

In Schwarzschild coordinates, the equation of a geodesic for $r = \text{const}$, $\theta = 90°$ leads to the relation

$$\Gamma_{00}{}^1 dt^2 = -\Gamma_{33}{}^1 d\varphi^2.$$

The components of the parallel-transported vector $A^\alpha$ acquire the increments

$$dA^0 = -\Gamma_{10}{}^0 A^1 dt,$$

$$dA^1 = -\Gamma_{00}{}^1 A^0 dt - \Gamma_{33}{}^1 A^3 d\varphi, \qquad dA^3 = -\Gamma_{13}{}^3 A^1 d\varphi.$$

Eliminating $dt$, $A^0$, $A^3$ from these four equations, we substitute the well-known expressions for the connection coefficients in the Schwarzschild coordinates:

$$\Gamma_{00}{}^1 = c^2 r_g (r - r_g)/2r^3, \quad \Gamma_{10}{}^0 = r_g/2r(r - r_g),$$

$$\Gamma_{33}{}^1 = -(r - r_g), \quad \Gamma_{13}{}^3 = 1/r.$$

We then obtain the equation

$$\frac{d^2 A^1}{d\varphi^2} + \left(1 - \frac{3}{4}\frac{r_g}{r}\right) A^1 = 0,$$

which shows that after one revolution the vector initially directed tangentially to the orbit is turned through the angle $3\pi r_g/2r$, in agreement with the result obtained in Ref. 1 in a different manner.

## 5. USE OF DIXON'S REDUCED MOMENTS

For the study of a multipole particle of high order or an extended body of general type, the problem of extended-body dynamics is understood differently in the present paper and in Dixon's well-known series of papers.[6–8] For a clear comparison of the two approaches, we consider a static space-time with Euclidean hypersurfaces $S(s)$ in which $s(x) = x^0$ in the chosen coordinate system and of all the Christoffel symbols $\Gamma^\alpha_{\beta\nu}$ the only nonvanishing ones are $\Gamma^a_{00}$, which play the part of the components of the gravitational force[2)]:

$$f^a = -\Gamma_{00}{}^a, \quad a, b = 1, 2, 3,$$

with $\partial_0 f^a = 0$. This situation in fact corresponds to the Newtonian case chosen by Dixon to illustrate his method.[8] The dynamic equations mentioned at the end of the Introduction decouple in this case into time and space parts. They are given in Ref. 8, and we reproduce them without derivation:

$$\frac{d}{dx^0} p^{a_1\ldots a_n 0} = -u^{(a_n} p^{a_1\ldots a_{n-1})0} + p^{(a_1\ldots a_n)}, \tag{13}$$

$$\frac{d}{dx^0} p^{a_1\ldots a_n a} = -u^{(a_n} p^{a_1\ldots a_{n-1})a} + t^{(a_1\ldots a_n)a} + \frac{m!}{n!(m-n)!} \sum_{m=n}^{\infty} \partial_{a_m\ldots a_{n+1}} f^a p^{a_1\ldots a_m 0}. \tag{14}$$

Here, $t^{a_1\ldots a_n ab}$ are the moments of the momentum flux density obtained from the third-rank part of the energy-momentum tensor:

$$t^{a_1\ldots a_n ab} = \frac{1}{n!} \int_S (x'-x)^{a_1} \ldots (x'-x)^{a_n} \delta_{a'}{}^a \delta_{b'}{}^b \hat{T}^{a'b'}\, dS_{0'}.$$

Here, $(x'-x)^a = x^{a'}\delta^a_{a'} - x^a$, and $\hat{T}^{ab}$ denotes the tensor density of weight $+1$ and replaces $T^{ab}\sqrt{-g}$, $a, b = 1, 2, 3$. Equations (13) and (14) make it possible to calculate all derivatives of the momentum moments of the body ($p$ moments) along the observer's world line at a certain time[3)] if the $p$ and $t$ moments are given at this time.

On the other hand, complete information about the body is contained in the set of (independent) $p$ and $t$ moments, so that the field of the energy-momentum tensor $T^{\alpha\nu}$ can be recovered from these moments. Therefore, for complete calculation of the behavior of the body in time it is necessary to specify the initial values of the $p$ moments and the values of the $t$ moments along the complete observer world line, since the variations of these last moments, which describe, in particular, the mechanical stresses in the body, cannot be predicted on the basis of the conservation law (1).

In contrast to this evolutionary description of the extended body by a system to equations of the type (13)–(14), Dixon's idea is that to describe the behavior of the body fully one should use, not the independent moments

$$p^{a_1\cdots a_n\alpha}(0), \quad t^{a_1\cdots a_n ab}(x^0),$$

which make it possible to solve the evolution problem by means of Eqs. (13)–(14), but the so-called reduced moments, all specified along the line $x^{\nu}(s)$. In the case considered, these moments are

$$p^{a_1\ldots a_n0}(x^0),\quad p^{a}(x^0),\quad p^{a_1\ldots a_{n-1}[a_na]}(x^0),\quad t^{a_1\ldots[a_{n-1}[a_na]b]}.$$

Among them,

$$p^{a_1\ldots a_n0},\quad p^{a},\quad p^{a_1\ldots a_{n-1}[a_na]}$$

are related by exactly four equations, while the remainder are completely arbitrary. The reduced moments together with their derivatives make it possible to determine all the remaining moments on the basis of the relations (13) and (14) as follows. The quantities $p^{a_1\ldots a_n0}(x^0)$ and $p^a(x^0)$, specified arbitrarily subject to the conditions

$$p^0(x^0)=\text{const},\qquad \frac{d}{dx^0}p^{a0}=-u^ap^0+p^a, \tag{15}$$

satisfy Eq. (13), determining the symmetrized moments $p^{(a_1\ldots a_n)}(x^0)$ but without restricting the freedom of the antisymmetric combinations $p^{a_1\ldots[a_{n-1}a_n]}(x^0)$. In their turn, the quantities $p^{a_1\ldots a_n}(x^0)$, formed from $p^{(a_1\ldots a_n}(x^0)$ and $p^{a_1\ldots(a_{n-1}a_n)}(x^0)$ and subject to the conditions

$$\frac{d}{dx^0}p^a=\sum_{m=0}^{\infty}p^{a_1\ldots a_m0}\,\partial_{a_m\ldots a_1}f^a, \tag{16}$$

$$\frac{d}{dx^0}p^{[a_1a]}=-u^{[a_1}p^{a]}+m\sum_{m=1}^{\infty}\partial_{a_m\ldots a_2}f^{[a}p^{a_1]a_2\ldots a_m0},$$

satisfy Eq. (14), determining the symmetric combinations $t^{(a_1\ldots a_n)a}(x^0)$ but leaving the combinations $t^{a_1\ldots[a_{n-1}[a_na]b]}(x^0)$ free. The relations (15) and (16) are equations that connect the reduced moments.[4)]

Our analysis shows the inconsistency of describing the body by reduced moments with specification of the initial state of the body, as foreseen by the formulation of the evolution problem. As functions of $x^0$, the reduced moments, in contrast to the moments $t^{a_1\ldots a_nab}(x^0)$, determine the symmetric combinations $p^{(a_1\ldots a_n)}(x^0)$, in particular on the initial hypersurface, and prevent arbitrary specification of the initial state. This circumstance remains fully valid for a curved space too, for which the analogs of Eqs. (15) and (16) are Dixon's well-known "equations of motion" (4.9) and (4.10) of Ref. 8:

$$Dp^{\alpha}=-R^{\alpha}_{\beta\mu\nu}p^{[\mu\nu]}\,dx^{\beta}+\varphi^{\alpha}(J^{\nu_1\ldots\nu_n},\ldots), \tag{17}$$

$$Dp^{[\mu\nu]}=-dx^{[\mu}p^{\nu]}+\psi^{\mu\nu}(J^{\nu_1\ldots\nu_n},\ldots), \tag{18}$$

where $\varphi^{\alpha}$ and $\psi^{\mu\nu}$ are known tensor functions of the reduced moments $J^{\nu_1\ldots\nu_n}(n=4,5,6\ldots)$.

Conversely, if the initial state of the body is given as a set of moments (including reduced ones) for the initial hypersurface, the relations (17) and (18) make it possible to determine only the derivatives of the momentum $p^{\alpha}$ and the antisymmetrized momentum moment $p^{[\nu\alpha]}$. Determination of the derivatives of the remaining moments is impossible, and therefore it is impossible to calculate the behavior of the body in the future. When Dixon's method of describing the body is used, one would need for such calculation the derivatives of all the reduced moments at the initial point, but Dixon's approach does not give them.

At the same time, Dixon's approach to the dynamics of an extended body makes it possible to calculate the internal stresses of the body that give rise to its preassigned motion. Indeed, the incompletely determined system of equations (15)–(16) allows in general a freedom in the specification of the functions $p^0=\text{const}$, $p^a(x^0)$, $p^{[ab]}(x^0)$ compatible with the existence of the solution $p^{a_1\ldots a_n0}(x^0)$ of this system. Addition to such a solution of the remaining arbitrary reduced moments

$$p^{a_1\ldots[a_na]}(x^0)\quad(n\geqslant 2),\qquad t^{a_1\ldots[a_{n-1}[a_na]b]}(x^0)$$

leads to a complete description of the body. In the case of curved space, this program is even made easier in view of the dynamical influence of the correlations of the stresses of the body with inhomogeneities of the space.[20]

[1)]Of course, this has no essential influence on the results of the studies.
[2)]In such an approach, the metric of space is not considered.
[3)]The geometrical image of the moment of time is the hypersurface $S(s)$, so that specification of the moment of time is equivalent to specifying $s=x^0$.
[4)]From the set of reduced moments one can eliminate $p^a$, reducing thereby the number of equations relating them to three.


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