New examples of topological solitons in magnetically disordered media

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Two exact solutions corresponding to topological solitons are presented. The first is a domain boundary (kink) moving between two spatially uniform but time-periodic vacuum solutions of the sine-Gordon equation, while the second is a solution of the Landau-Lifshitz equations and is a domain boundary with an internal degree of freedom corresponding to a perturbation that is time-periodic and self-localized in the region of the boundary.

1. In the study of nonlinear phenomena in magnetically ordered media (moving domain boundaries, solitons, and nonlinear spin waves), integrable models associated with the Landau-Lifshitz equation, the sine-Gordon equation, and the nonlinear Schrödinger equation are widely used at the present time (for examples, see, e.g., Ref. 1). A characteristic feature of the most well known objects of investigation—domain boundaries (kinks) and solitons—is the fact that they are essentially nonlinear formations (in particular, self-localized in space) on a background of simple equilibrium states of the medium (e.g., states that are spatially uniform and independent of the time). Envelope solitons, or precession solitons, are not in this sense an essentially new class of objects, since the possibility of isolating simple motion (an oscillating phase or precession) in the initial dynamical equations of the nonlinear medium returns us to the traditional situation.

A number of theoretical and experimental investigations[1–4] associated with the analysis of the formation of solitons and with distinctive features of the motion of domain boundaries against a background of nonlinear spin waves (and also with the analysis of analogous phenomena in distributed Josephson junctions[5]) lead to the necessity of seeking and classifying exact solutions of a more general type, viz., solutions of the domain-boundary (kink) and soliton (breather) type, which are essentially nonlinear formations (transitional layers or regions of self-localization) against a background of states corresponding, e.g., to spatially uniform but time-periodic nonlinear solutions of the fully integrable equations of the medium.

The existence of such a class of solutions for the above-mentioned fully integrable models of magnetically ordered media was noted by us in Ref. 6 and is related to the comparatively simple topological structure of the phase space of the dynamical system generated by a certain class of solutions of different (in their physical nature) fully integrable equations (a more detailed account is given in Ref. 7). The latter circumstance can be used to predict, from already-known types of solutions of some integrable model, their analogs for other models possessing a phase space with the same topological structure.

The construction of spatially self-localized solutions (of the kink and breather type) that tend asymptotically for $x \to \pm\infty$ spatially uniform and time-periodic states of the nonlinear medium (i.e., to a nonzero vacuum) points to the possibility that in the known integrable models there exist new types of solitons, and to the need to analyze features of their interaction just as has been done in the theory of solitons on a background of finite-zone potentials.[8]

2. We shall give a simple example that illustrates both the general statements of Refs. 6 and 7 and the possibility of the construction of soliton solutions of a nonzero vacuum. The well studied equation

$$u_{tt}-u_{xx}+\sin u=0 \tag{2.1}$$

admits the following very simple solution in the class under consideration:

$$u(x,t)=4\,\mathrm{arctg}\,\frac{a\,\mathrm{sn}\,\tau+e^{\Lambda x}}{1+a\,\mathrm{sn}\,\tau e^{\Lambda x}}. \tag{2.2}$$

Here

$$a^2=k\leqslant 1,\quad \Lambda=(1-k)/(1+k)\leqslant 1,\quad \tau=t/(1+k) \tag{2.3}$$

and sn is the Jacobi elliptic sine with modulus k. The solution (2.2) is periodic in time, with a fundamental period T_ω determined by the relation

$$T_\omega=2\pi/\omega=4(1+k)K(k), \tag{2.4}$$

where K is the complete elliptic integral of the first kind, and ω is the fundamental frequency (the parameter with respect to which it is convenient to classify the more complicated solutions). It is obvious that

$$\lim_{x\to-\infty} u(x,t)=u_\omega(t),\qquad \lim_{x\to+\infty} u(x,t)=u_\omega(t)+2\pi, \tag{2.5}$$

where

$$u_\omega(t)=4\,\mathrm{arctg}\,(a\,\mathrm{sn}\,\tau). \tag{2.6}$$

Thus, according to (2.5) and (2.6) the solution (2.2) is a generalization of the well known special solution—the kink (topological soliton)—and determines the transitional layer between two spatially uniform "vacuum" states corresponding to time-periodic solutions of Eq. (2.1). As $\omega^2\to 1$ and, correspondingly $k\to 0$, the solution (2.2) degenerates into a kink:

$$\lim_{\omega^2\to 1} u(x,t)=4\,\mathrm{arctg}\,e^x, \tag{2.7}$$

linking the spatially uniform vacuum states $u=0$ and $u=2\pi$. By virtue of the Lorentz invariance of Eq. (2.1) the solution (2.2) admits an obvious generalization to the case of motion with constant velocity S:

$$u(x,t)=4\mathrm{arctg}\,\Phi,\quad \Phi=\frac{a\,\mathrm{sn}\,\tilde\tau+e^{\tilde\Lambda\tilde x}}{1+a\,\mathrm{sn}\,\tilde\tau e^{\tilde\Lambda\tilde x}}. \tag{2.8}$$

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Here,

$$\tilde{\tau}=(t+Sx)/(1+k)(1-S^2)^{1/2},\quad \tilde{x}=x-St, \tag{2.9}$$

and the characteristic size $\tilde{\Lambda}^{-1}$ of the transitional layer is determined by the relation

$$\tilde{\Lambda}=(1-k)/(1+k)(1-S^2)^{1/2}. \tag{2.10}$$

The difference of the functionals

$$E[u(x,t)]-E[u_\omega(t)]=\mathscr{E}(\omega),$$
$$E[u]=\int_{-\infty}^{\infty} dx\,(^1/_2u_t^2+{}^1/_2u_x^2+1-\cos u) \tag{2.11}$$

determines the energy of the topological soliton (2.2) relative to the spatially uniform vacuum state (2.6). Straightforward calculations show that

$$\mathscr{E}(\omega)=8\Lambda=8(1-k)/(1+k)\leqslant 8. \tag{2.12}$$

Here the modulus $k\leqslant 1$ of the elliptic function is connected with the frequency ω of the vacuum oscillations by the relation (2.4). As $\omega^2\to 1$ the energy (2.12) of the topological soliton tends to the energy of the kink (zero-vacuum topological soliton), which is equal to $\mathscr{E}(1)=8$. With decrease of the frequency of the vacuum oscillations, and increase (in accordance with (2.4) and (2.6)) of their amplitude, the nonzero-vacuum topological soliton (2.12) decreases.

Regarding the topological soliton (2.2) as an elementary formation with energy (or rest mass) dependent on the frequency of the vacuum oscillations, we arrive at the conclusion that such an elementary formation can be characterized by an "internal" degree of freedom. In this sense the nonzero-vacuum topological soliton is analogous to the traditional zero-vacuum breather, which, as an elementary formation, possesses an internal degree of freedom. Taking into account that quantization of the breather states of a nonlinear field leads to a definite branch of the mass spectrum of the elementary formations, we must expect that the solution of the problem of the quantization of nonzero-vacuum soliton states (primarily, topological solitons) can lead to a new branch of the mass spectrum of the sine-Gordon equation.

From the point of view of the theory of dynamical systems, both the solutions (2.2) and (2.7) are singular—namely, they are solution of the separatrix type. Here, according to Ref. 6, upon decrease of the parameter $\omega^2\leqslant 1$ there appear not only the singular solution (2.2), which bifurcates at $\omega^2=1$ from the spatially uniform solution (2.7), but also new, more-complicated singular solutions (the analogs of the well known solutions of the breather or wobbly-kink type), which are generated when the parameter ω^2 passes through the bifurcation values

$$\omega_n{}^2=1/n^2,\quad n=2,3,\ldots\,. \tag{2.13}$$

Unlike its analogs, this new class of singular solutions is doubly asymptotic (as $x\to\pm\infty$) to the spatially uniform and time-periodic solution of the type $u_{n\omega}(t)$, i.e., to the solutions (2.6) with ω replaced by $n\omega$ in the dispersion relations (2.3) and (2.4).

The solution (2.8) given above corresponds to motion, with a constant velocity, of a topological soliton "frozen into" a nonzero vacuum. The question naturally arises of the existence of topological solitons moving with a constant velocity relative to a nonzero vacuum. Such solutions can be constructed, e.g., by the method of Darboux transformations, if as the " bare" solution we use the solution (2.6). Using the general relations given in Ref. 9 one can show that the solution can be represented in the form

$$u(x,t)=4\,\mathrm{arctg}\,\Psi,$$
$$\Psi=\frac{\Psi_0(t-\delta)\pm\exp\{\Lambda(x-Vt-X(t))\}}{1\pm\Psi_0(t+\delta)\exp\{\Lambda(x-Vt-X(t))\}}. \tag{2.14}$$

Here,

$$\Psi_0{}^{-1}=k^{1/2}\,\mathrm{sn}\left(\frac{t}{1+k},k\right),\quad \frac{2\pi}{\omega}=4(1+k)K(k), \tag{2.15}$$

$$\Lambda^2=1-\varkappa^2+\frac{1}{4}\left(\lambda-\frac{1}{\lambda}\right)^2,\quad \varkappa^2=\frac{4k}{(1+k)^2}, \tag{2.16}$$

$$V=\frac{1-\lambda^2}{1+\lambda^2}\frac{\Pi(n,\varkappa)}{K(\varkappa)},\quad n=\frac{4\lambda^2\varkappa^2}{(\lambda^2+1)^2}, \tag{2.17}$$

$$\mathrm{sn}(\delta,\varkappa)=\frac{\lambda^2-1}{2\lambda\Lambda}, \tag{2.18}$$

where λ is the parameter of the Darboux transformation, $X(t)$ is a time-periodic function characterizing the oscillations of the position of the front of the topological soliton in the comoving (with velocity V) reference frame (the explicit form of this function is not important for our purposes), and π is a complete elliptic integral of the third kind.

We shall discuss the principal characteristics of the topological nonzero-vacuum soliton (2.14). For $\lambda=\pm 1$ we find that $V=0$, $X(t)\equiv 0$, $\delta=0$, and the solution (2.14) degenerates into the stationary nonzero-vacuum soliton (2.2). By direct calculations one can show that the energy (given by the relations (2.11)) of the topological soliton (2.14) is

$$\mathscr{E}=8\Lambda. \tag{2.19}$$

Consequently, the relations (2.16), (2.17), and (2.19) after elimination of the auxiliary parameter λ determine the dependence of the energy $\mathscr{E}(V,\omega)$ on the "average" velocity V of the motion of the front of the topological soliton and on the frequency ω (or amplitude) of the oscillations of the nonzero vacuum. It is not difficult to convince oneself that for $\omega^2\lesssim 1$ and, correspondingly, $\varkappa^2\ll 1$ the relation (2.19) leads to

$$\mathscr{E}(V,\omega)=\frac{8}{(1-V^2)^{1/2}}\left(1-\frac{1}{2}\varkappa^2\right)+O(\varkappa^4). \tag{2.20}$$

Consequently, the effect of the oscillations of the position of the front of the topological soliton on the functional dependence of the energy on the velocity is manifested only in the next terms of the expansion of the energy in the characteristic amplitude of the oscillations of the nonzero vacuum. In the general case the relations (2.15)–(2.17) determine the functional dependence of the energy (or parameter Λ) on the velocity V and oscillation frequency ω in the form of the transcendental equation

$$^1/_8\mathscr{E}(V,\omega)\equiv\Lambda=m(\varkappa^2,v^2)/(1-v^2)^{1/2}. \qquad (2.21)$$

Here

$$m(\varkappa^2,\ v^2)=[1-\varkappa^2(1-v^2)]^{1/2}\geqslant 0, \qquad (2.22)$$

$$V=v\Pi[(1-v^2)\varkappa^2,\ \varkappa]/K(\varkappa). \qquad (2.23)$$

The relation (2.22) determines the dependence of the effective mass of the topological soliton on the velocity and effective amplitude $\sim\varkappa$ of the oscillations of the nonzero vacuum, while the relation (2.23) determines the dependence $V=V(v,\varkappa)$. Finally, we note that the relation (2.18)

$$\operatorname{sn}(\delta,\varkappa)=\pm v[1-\varkappa^2(1-v^2)]^{-1/2}$$

determines the relative shift, equal to 2δ, in the phase of the nonzero-vacuum states as $x\to\pm\infty$.

According to Ref. 10, Eq. (2.23) can be written in the form

$$\frac{V}{v}=1+\frac{1}{v\Lambda}\left[E(\varphi/\alpha)-\frac{E(\alpha)}{K(\alpha)}F(\varphi/\alpha)\right], \qquad (2.24)$$

$$\sin\alpha=\varkappa,\qquad \varphi=\arcsin(1-v^2)^{1/2}.$$

On the basis of the representations (2.23) and (2.24) it is not difficult to convince oneself that

$$\lim_{\varkappa\to 0}\frac{V}{v}=1,\quad \lim_{\varkappa\to 1}\frac{V}{v}=1+\frac{(1-v^2)^{1/2}}{v^2}\ln\frac{(1+v)^{1/2}+(1-v)^{1/2}}{(1+v)^{1/2}-(1-v)^{1/2}}$$

$$\lim_{v\to 1}\frac{V}{v}=\frac{E(\varkappa)}{(1-\varkappa^2)K(\varkappa)}\geqslant 1,$$

$$\frac{V}{v}\sim 1+\left[1-\frac{E(\varkappa)}{K(\varkappa)}\right](1-v^2)\quad \text{for}\quad v^2\to 1.$$

The equation

$$2E(\varkappa)=K(\varkappa)$$

determines a critical value $\varkappa=\varkappa_{\rm cr}\sim 0.91$ ($\alpha_{\rm cr}\sim 65.5°$) such that for $\varkappa>\varkappa_{\rm cr}$ motions of the topological soliton with velocities $V>1$ become possible.

Characteristic dependences of the curves $V=V(v)$ on the amplitude (determined by the parameter $\varkappa=\sin\alpha$) of the oscillations of the nonzero vacuum are given in Figure 1. By virtue of the relations (2.21) the energy of the topological soliton is finite for all values $v^2<1$. We draw attention to the fact that the function $v=v(V)$ becomes two-valued for $\varkappa>\varkappa_{\rm cr}$.

Thus, the sine-Gordon equation admits the extraction of a new class of singular solutions, corresponding to distributions of the kink and breather type on a background of spatially uniform and time-periodic solutions.

We are aware that an attempt to construct nonzero-vacuum soliton solutions on the basis of perturbation theory was made in Ref. 4. However, no exact nonzero-vacuum soliton solutions were obtained. One should expect that the use of the very simple exact solutions (2.2) and (2.14) as "bare" solutions in various methods in the modern theory of fully integrable field equations will make it possible to investigate features of interaction processes of elementary nonzero-vacuum formations. According to Refs. 6 and 7 one should expect that solutions of the nonzero-vacuum topological-soliton type can also be discovered for other fully integrable field equations (e.g., for the Landau-Lifshitz equations).

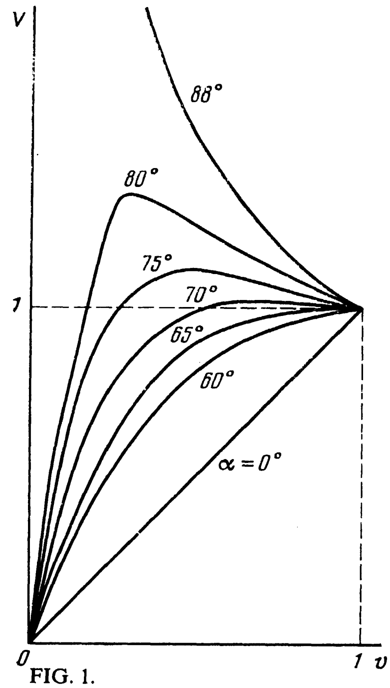

FIG. 1.

3. As the next example we shall consider the fully integrable system of Landau-Lifshitz equations and exhibit a new singular solution corresponding to a domain wall with internal degrees of freedom—a solution of the "wobbly domain wall" type, analogous to the wobbly kink[11,12] for the sine-Gordon equation. The fully integrable system of Landau-Lifshitz equations for the spherical variables $\theta(s,t)$ and $\varphi(x,t)$ of the unit vector of the magnetic moment $\mathbf{m}(x,t)$ has the form[1]

$$\frac{\partial^2\theta}{\partial x^2}-\left[1+\varepsilon\cos^2\varphi+\left(\frac{\partial\varphi}{\partial x}\right)^2\right]\sin\theta\cos\theta=\frac{\partial\varphi}{\partial t}\sin\theta,$$

$$\frac{\partial}{\partial x}\left(\sin^2\theta\frac{\partial\varphi}{\partial x}\right)+\varepsilon\sin^2\theta\cos\varphi\sin\varphi=-\frac{\partial\theta}{\partial t}\sin\theta. \qquad (3.1)$$

Here ε is the parameter of the biaxial ferromagnet. Using the method of Hirota,[13] one can show that the time-periodic solution satisfying the asymptotic conditions

$$\lim_{x\to-\infty}\theta(x,t)=0,\qquad \lim_{x\to+\infty}\theta(x,t)=\pi \qquad (3.2)$$

is determined by the relations

$$\operatorname{tg}\frac{\theta}{2}e^{i\varphi}=\frac{g_1(x,t)+g_3(x)}{1+f_2(x,t)}. \qquad (3.3)$$

Here,

$$g_1(x,t)=A_1\exp(\alpha x+i\omega t)+A_2\exp(\alpha x-i\omega t)+Be^{\beta x},$$

$$g_3(x)=-\frac{\omega^2(\alpha-\beta)^2}{\varepsilon\alpha^2(\alpha+\beta)^2}A_1A_2B\exp(2\alpha+\beta)x, \qquad (3.4)$$

$$f_2(x,t)=-\frac{\omega^2}{\varepsilon\alpha^2}A_1A_2e^{2\alpha x}+\frac{\alpha-\beta}{\alpha+\beta}B^*[A_1\exp\{(\alpha+\beta)x+i\omega t\}$$

$$+A_2\exp\{(\alpha+\beta)x-i\omega t\}],$$

$$A_1={}^1/_2\varepsilon a,\quad A_2=(\alpha^2-1-\omega-{}^1/_2\varepsilon)a.$$

Furthermore, the characteristic precession frequency ω and parameter α are connected by the dispersion relation

$$(\alpha^2-1)(\alpha^2-1-\varepsilon)=\omega^2. \qquad (3.5)$$

Finally, $B = b$ and $\beta^2 = 1 + \varepsilon$ for the asymptotic behavior of the solution that is typical for a domain wall of the Néel type, while $B = ib$ and $\beta^2 = 1$ for the asymptotic behavior of the solution that is typical for a domain wall of the Bloch type. We note that a and b are arbitrary parameters, one of which can be associated with a shift along a spatial variable. According to Ref. 6, for the dynamical system generated by time-periodic solutions of Eqs. (3.1) the singular solutions (3.3) determine a one-parameter family of heteroclinic trajectories that are doubly asymptotic (as $x \to \pm \infty$) to the singular points $\theta = 0$ and $\theta = \pi$. it is clear that to these solutions there correspond domain walls (in the asymptotic sense of Bloch or Néel domain walls) associated with a spatially self-localized and time-periodic soliton-type perturbation of the magnetic moment. The solutions (3.3) extend the class of time-periodic and spatially self-localized solutions first constructed in Ref. 14 to the case of states with nonzero topological charge. Singular solutions of the type (3.3) may be of interest in connection with the problem of the motion of a domain wall in an external field with allowance for damping. Such perturbations destroy the integrability of the Landau-Lifshitz equations and lead to a dependence of the total energy of such a formation on the position of the time-periodic and spatially self-localized perturbation relative to the domain wall. Consequently, this opens up the possibility of the appearance of new channels of dissipation of the energy of a moving domain wall with increase of the external field, these channels being associated with the excitation of soliton-type internal degrees of freedom of the wall.

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