Motion in rapidly oscillating resonant fields and vortical Raman phenomena

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The behavior of particles and other entities (a broad class of systems) in rapidly oscillating fields is analyzed for the case in which there is some set of high-Q vibrational modes which are resonant in the field and which are coupled in a nonlinear way with slow motions. These resonances give rise to an intense and essentially irreversible energy exchange with the high-frequency field. Some characteristic features of the renormalization of the effective elasticity and of the decay of the slow motions in multimode resonant fields are derived in the lowest order of nonlinearity as functions of the rate at which the resonances are crossed, of the spectrum of these resonances, and of the nature of the excitation. In particular, there is a vortical field reaction generated along with a dissipative action in motions in multimode resonant fields. The vortical action may be the predominant factor in the irreversible energy exchange with the field. The vortical motion driven by a "strong" wave, in contrast with other known effects of the stimulated-Raman-scattering type, is excited efficiently during a pumping of the field energy either up or down the spectrum. It does not fit into the conventional interpretation in terms of wave decay and coalescence processes. The analysis is illustrated with some examples, one of which demonstrates the principle of a new "parametric motor," different from that discussed by Papaleksi (Collected Works, Vol. 1, Izd. Akad. Nauk SSSR, Moscow, 1948).

INTRODUCTION

Under conditions such that the characteristic frequencies of interacting fast motions ("fields") and relatively slow motions are widely spaced, it is frequently possible to describe interaction effects in terms of an effective potential corresponding to the average forces exerted by the high-frequency field on the states of the slow system. According to the concept advanced by Hertz,[2] the potential energy can in general be regarded as being of kinetic origin, i.e., as representing the energy of "latent" oscillatory motions. The average action on a system of given fields which are oscillating at a high frequency is, under certain simplifying assumptions, also equivalent to an effective potential which is equal to the kinetic energy of the high-frequency "jarring" of the system,[3] as is illustrated in a graphic way by a pendulum with a rapidly vibrating suspension point.[4] A concept related to the effective potential is the "quasienergy."[5] On the basis of no more than the very fact that an effective potential exists and certain general properties of this potential, it is frequently possible to draw conclusions about the overall picture of events under various conditions, without resorting to detailed calculations. For this reason, the concept of an effective potential has proved useful in a variety of fields in physics.

However, a qualitatively new picture of events, which is frequently very different from the pictures specified by these representations, arises when an interaction with a rapidly oscillating field efficiently generates nonpotential forces and builds up (or, on the contrary, suppresses) slow motions. The nonpotential forces are by no means necessarily an effect of the next-higher approximation in the small parameter, with which the concept of a *rapidly oscillating* field is related. The manifestations of these forces are extremely significant and varied when there is a nonlinear coupling of the slow motions with high-Q, high-frequency vibrational systems ("resonant media") which are excited near resonant frequencies. Such situations are quite common, and we are led to ask whether, again in such cases, we can see general features without resorting to detailed calculations. In the present paper we take up this question at the most elementary level.

As an introduction to interactions of this type we will discuss the problem of the sharp increase in the mobility of particles in rapidly oscillating resonant fields. Indeed, it was a study of the mechanisms for increases in mobility which was a particular motivation for the present analysis. In this connection, and also from the methodological standpoint, the present study is related to my earlier paper.[6]

The mobility of various kinds of particles (bodies, atoms, inclusions, domain walls, dislocations, and so forth) may increase substantially in rapidly oscillating fields which are at resonance with high-frequency vibrational motions in movable objects or their surroundings, as we have learned from experiment. One reason is simply that the amplitude of alternating influences increases at resonances, with the result that we can expect a pronounced change in mobility when the energy of the high-frequency vibrations of particles becomes comparable to the height of the potential barriers formed by obstacles to the motion or when these barriers oscillate intensely, so that their effective height decreases significantly.

However, nonlinear mechanisms which result from the particular nature of the energy exchange between slow motions and high-frequency resonances may have a far more important effect on the mobility and may accordingly be

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manifested at significantly smaller jarring amplitudes. The point is that the average force exerted by alternating fields generally becomes an essentially nonpotential influence in a situation with high-Q resonances, and there is a pronounced renormalization of the effective damping for slow motions. As a result, the potential barriers may be surmounted easily, even when the jarring effects are small.

Action of this type by resonances on mobility can be seen even in the lowest-order nonlinear processes, in which the energy of high-frequency oscillation modes c which is circulating in a system, including the energy of the coupling with slow motions x, is small enough and can be approximated by an expression quadratic in c:

$$H_{int}=\omega_{ik}(x)\,c_i^*c_k. \tag{1}$$

On the one hand, interactions (1) generate forces F that act on the slow subsystem; on the other, when motions $x(t)$ cause modulation of the parameters of the resonances. In the case of high-Q resonances, small smooth changes of their parameters cause pronounced changes in the resonance regimes, and furthermore with a delay that also increases with the Q of the resonances. As a result, the forces $\langle F\rangle$ (the angle brackets mean an average of the fast oscillations c over the time) acquire terms which depend strongly and with a delay on $x(t)$. This is the reason why the forces $\langle F\rangle$ are definitely not potential forces. The nonpotential nature of the forces actually stems from the pumping of the resonances and their finite (but large) Q, since these are the reasons why the resonance regimes are extremely sensitive to changes in the parameters ω_{ik} as a function of x.

The elementary analysis which we have just presented is taken from Ref. 6, where it was pointed out that mechanisms of this sort for nonadiabatic interactions with resonances are important in a multitude of effects. Some examples are effects such as stimulated Raman scattering in resonant media, the inverse effects of damping of slow motions by resonant fields, and the principle underlying the operation of parametric amplifiers and oscillators for low-frequency oscillations and waves which use a high-frequency resonant pump, beginning with the Mandel'shtam-Papaleksi motor.[1] The analysis in Ref. 6 dealt with the behavior of ferromagnetic particles in a microwave field exciting high-Q ferromagnetic resonances in particles. It was shown that in a certain interval of the detuning from the ferromagnetic resonance the magnetic retardation of the particles gives way to an antiretardation of significant magnitude, so that at relatively low amplitudes of the microwave field it is possible to effectively surmount the potential barriers formed by the forces of magnetic coupling between particles.

The fact that the amplitudes ω_{ik} in (1) are alternating-sign (and, in general, irregular) functions of x does not change the basic trend in the renormalization of the damping, since the renormalization turns out to be quadratic in $\partial\omega_{ik}/\partial x$. When there are inhomogeneities of some type or other—and inhomogeneities substantially determine (limit) the mobility—a nonlinear relation such as in (1) is quite typical (also typical, of course, is a relation between x and c which is linear in c, but of lower order in the small parameter $|c|^2$, and is unimportant for estimates of the renormalization of the damping when the frequencies of the motions x and c are greatly different. For example, the energy of the dipole interaction of the particles,

$$U_d=\mathbf{M}_i\mathbf{M}_k/r_{ik}^3-3(\mathbf{M}_i\mathbf{r}_{ik})(\mathbf{M}_k\mathbf{r}_{ik})/r_{ik}^5 \tag{2}$$

($\mathbf{r}_{ik}=\mathbf{r}_i-\mathbf{r}_k$, where $\mathbf{r}_i$ is the coordinate of the particle with dipole moment $\mathbf{M}_i$), contributes to the coupling with resonances of the type in (1), and also determines the static magnetic forces that shape the potential relief for various types of relative displacements x of the particles. The closer the particles are packed, the stronger their coupling that limits the mobility. However, there is an equally sharp increase in the parameters $\partial\omega_{ik}/\partial x$, and in the case of high-Q resonances of $\mathbf{M}_i(t)$ the increase of their influence on the mobility of the particles is even faster than quadratic in $\partial\omega_{ik}/\partial x$, and this mobility changes sharply even at relatively small angles of the high-frequency resonant precession of $\mathbf{M}_i(t)$.

It is a straightforward matter to analyze these questions in a simplified model corresponding to the conditions of a single-mode high-frequency resonance, i.e., a one-dimensional subsystem c in (1) (Ref. 6). Typical of the behavior of ferromagnetic particles in microwave fields and of many other problems are conditions such that not one but some set of c resonances are involved in the interactions. The nature of the renormalization of the damping and the effective elasticity in this case depends on many parameters, and the picture of manifestations of the interaction with resonances becomes much more complex. Just what the general features of this picture are and just what fundamentally new features are introduced by the multimode nature of the problem within the framework of interactions (1) are questions with which we are concerned in the present paper.

It is not difficult to see that a frequent case of interactions (1)—that in which ω_{ik} are functions linear in x—is the energy of three-wave interactions, which is widely used in describing wave processes.[7–9] For example, when subsystem x (with $c=0$) is a one-dimensional linear oscillator, while subsystem c is two-dimensional, we have a system of three coupled oscillators (or coupled waves), for which several exact results (if the pumping and damping are ignored) and approximate results are quite well known. In particular, models of this sort describe the stimulated decay of a "strong" wave c of frequency ω into a low-frequency wave x of frequency $\Omega\ll\omega$ and a satellite c of frequency $\omega-\Omega$ and also the inverse processes of wave coalescence. However, such models as well as more general models have not been studied adequately from the standpoint of the present paper. In this sense, the analysis below is not simply a methodological analysis concerning familiar nonlinear processes such as the stimulated decay and coalescence of waves. This analysis also reveals a new category of nonlinear processes, which have apparently not been discussed previously and which we will call "vortical Raman effects."

INITIAL EQUATIONS

We work from interaction model (1). The variables c, which characterize the resonant field, are assumed to be normal variables when the nonlinear couplings with x are ignored, and the energy of the normal modes is incorporated in

(1). In other words, in the absence of sources and sinks, the dynamics of the field obeys the system of equations

$$\dot{c}_k+i\omega_{kn}(x)c_n=0. \qquad (3)$$

The number of field modes may be unbounded. A dummy index implies a summation over the set. We will not discuss here the case of an anharmonic (at $x=$ const) resonant subsystem.

To describe the resonant regime of field oscillations we must introduce sources of a harmonic pump and compensating sinks, i.e., a dissipation. Introducing them in the standard fashion,[8,9] we replace (3) by the equations

$$[d/dt+A(x)]c=he^{-i\omega t}, \qquad (4)$$

where the complex amplitudes $h=\{h_k\}$ characterize the sources of an alternating field of frequency ω and $A(x)$ is the matrix with the elements

$$A_{kn}=i\omega_{kn}+\gamma_{kn},$$

where the matrix γ_{kn} is assumed to be Hermitian, like ω_{kn}. The sum

$$P=\gamma_{kn}\omega_{ns}c_k^*c_s+\text{c.c.}$$

is equal to the dissipated power.[1)] Where necessary in the calculations, we assume that the matrix A is a normal matrix, i.e., that the matrices ω_{kn} and γ_{kn} commute. In this sense we disregard effects of an additional motion mixing caused by slight dissipative forces. We say "slight" here since we are interested in conditions of weak dissipation, i.e., high-Q resonances c, in which the power (P) which is dissipated over a time on the order of the reciprocals of the resonant frequencies is much smaller than the energy in (1).

We characterize the slow motions by real coordinates which we assume are also multidimensional: $x=\{x_\alpha\}$. Arising from interactions (1) are generalized forces F corresponding to the coordinates x:

$$F=-\frac{\partial\omega_{kn}}{\partial x}c_k^*c_n, \qquad (5)$$

where $F=\{F_\alpha\}$ and, respectively, $\partial/\partial x=\{\partial/\partial x_\alpha\}$. These forces combine additively with other forces which are acting on the slow system in the case $c=0$, and determine the dynamics of the slow system. Since the oscillations c depend on the previous behavior of x [in accordance with Eqs. (4)], the forces F generally constitute a complicated retarded functional of $x(t)$. An analysis of the characteristic properties of this functional is the heart of the present paper.

In several places, where we wish to illustrate a point, we will take x to be the mechanical motion of a particle of mass m or the oscillations of an oscillator. In the general analysis, however, we actually do not need to be specific about the dynamics of the slow system. The applicability of this analysis thus extends beyond mobile objects. In this analysis, x could also be understood as any internal motions of systems or low-frequency waves in a medium. Their dynamics does not necessarily have to be approximately Hamiltonian; other possibilities are any forms of the motion of systems which are of a substantially relaxational nature.

Essentially the only limitation is the slowness of subsystem x, which is actually already incorporated in the interaction model. When the frequencies of the motions of subsystems x and c are commensurable, terms c_ic_k, for example, would be just as important as coupling terms with the structure $c_i^*c_k$, which are taken into account in (1), and terms $e^{i\omega t}$ might play a role in Eqs. (4) along with the resonant pump $e^{-i\omega t}$ (at frequencies ω approximately equal to eigenvalues of the matrix ω_{ik}).

Interactions of x with c which are close in terms of effects, and which are taken into account through the functional dependence $\omega_{ik}(x)$, occur when there is a modulation of the amplitudes, $h=h(x)$. An x dependence of h corresponds to a coupling energy in addition to (1),

$$H_{int}^{(h)}=ic_kh_k^*(x)e^{i\omega t}+\text{c.c.},$$

and forces $F^{(h)}=-\partial H_{\text{int}}^{(h)}/\partial x$ appear in addition to (5). For the forces of both types it is necessary tor recognize that the x dependence of h in (4) alters additionally the response of the resonant field to motions $x(t)$. The renormalization of the effective elasticity and of the damping of the slow motions due to these couplings is generally not a small effect. A coupling of this type arises in, for example, an analysis of the behavior of a dipole particle (x would represent its displacements and rotation angles) in a harmonic external field which excites an "internal" resonant dipole subsystem c. This pertains, in particular, to an atom in a laser field which is at resonance with optical transitions in the atom. These couplings also play a significant role in derivations of the mobility of ferromagnetic particles in microwave fields.

The model selected above for interaction with resonances [through forces (5) and Eqs. (4) with $h=$ const], which we will work from in the analysis below, actually encompasses effects of couplings of h from x through an expansion of the number of modes c. The additional set of these modes, $\{c_{k'}\}$, must have suitable interaction parameters, and its eigenfrequencies must be far enough from the frequency ω so that the oscillations $\{c_{k'}\}$ react only weakly to changes in $x(t)$. Obviously, an expansion of the number of modes c also covers the case of "external" resonances associated with the use of resonant systems to amplify a pump field.

ADIABATIC SITUATION AND THE GIVEN-FIELD APPROXIMATION

We begin our study with an analysis of the nature of the interactions of motions x with an ideal resonant system c, described by Hamilton's equations (3). We consider the case in which the matrices ω_{ik} and $\partial\omega_{ik}/\partial x$ commute. In this case we have

$$\frac{\partial\omega_{jk}}{\partial x}\frac{d}{dt}c_j^*c_k=0 \qquad (6)$$

identically. Indeed, from Eqs. (3), we find

$$\sum_{j,k}\frac{\partial\omega_{jk}}{\partial x}\frac{d}{dt}c_j^*c_k=i\frac{\partial\omega_{jk}}{\partial x}\omega_{jn}^*c_n^*c_k-i\frac{\partial\omega_{jk}}{\partial x}\omega_{kn}c_nc_j^*.$$

The second term on the right becomes identical to the first if, in accordance with the commutation condition, we replace $(\partial\omega_{jk}/\partial x)\omega_{kn}$ by $\omega_{jk}(\partial\omega_{kn}/\partial x)$, use $\omega_{jk}=\omega_{jk}^*$, and then permute the indices: $j\to n$, $n\to k$, $k\to j$.

Equation (6) means that the entire effect of subsystem c on the motions x reduces, in accordance with (5) and (6), to the influence of a time-independent potential

$$U_N(x)=N_{jk}\omega_{jk}(x), \tag{7}$$

where N_{jk} are constants equal to the values of the products $c_j^* c_k$ at some time (e.g., the initial time). In this case, multidimensional motions c can be separated.

This point can be easily understood by noting that if the matrices ω_{jk} and $\partial\omega_{jk}/\partial x$ commute during the changes in $x(t)$, the matrices ω_{jk} for different $x(t)$ also commute. There exists thus a basis in the variables c in which the matrices ω_{jk} remain diagonal during motions $x(t)$:

$$\omega_{jk}=\delta_{jk}\omega_k(x(t)).$$

Correspondingly, the new variables $\{c_k\}$ (we will retain the previous notation for them) are normal vibration modes with time-varying eigenfrequencies $\{\omega_k\}$. Each of the quantities $|c_k|^2$ is an invariant (an adiabatic invariant) and does not change during the motions $x(t)$.

This adiabatic situation can serve as an illustration of Hertz's concept: "Latent" oscillations c having the energy (1) generate the potential (7). This property evidently holds whenever an isolated subsystem c is a single-mode subsystem. In the case of two modes, $c=(c_1, c_2)$, the property of the separation of the motions c holds if $\omega_{12}=0$ or (if $\omega_{12}\neq 0$) when the expressions ω_{21}/ω_{12} and $(\omega_{11}-\omega_{22})/\omega_{12}$ do not change during the change in $x(t)$. This is the case, for example, for a system of two coupled electric circuits with identical partial frequencies when there are arbitrary changes in their mutual arrangement, which change the inductive and/or capacitive couplings. This property also holds for two magnetic particles represented by dipoles with an energy (2) and with oscillations c_1 and c_2, represented by modes of a uniform precession of $\mathbf{M}_1$ and $\mathbf{M}_2$ for rather arbitrary changes in $\mathbf{r}_{12}(t)$. [The limitation reduces to the requirement that the deviations of $\mathbf{M}_1$ and $\mathbf{M}_2$ in the course of the motions $\mathbf{r}_{12}(t)$ from their equilibrium configuration $\mathbf{M}_1\|\mathbf{M}_2\|\mathbf{r}_{12}(0)$ must be small. If $M_1\neq M_2$, then we have $\omega_{11}\neq\omega_{22}$, but the ratio $(\omega_{11}-\omega_{22})\omega_{12}$ remains unchanged.]

The result in (7) remains valid if we take the N_{jk} to be not fixed values of the products $c_j^* c_k$ but functions of the time $N_{jk}(t)=c_j^*(t)C_k(t)$, where $\{C_j(t)\}$ are oscillations of $\{c_j\}$ which occur in accordance with laws (3) for a value of x which is constant over time but otherwise arbitrary. In other words, the given-field approximation, i.e., the approximation in which we ignore the response of the field to changes $x(t)$, is an exact result in this case.

In the general case of an arbitrary functional dependence $\omega_{ik}(x)$ and for a dynamics of c in accordance with Eqs. (4), the average force exerted by the field also reduces to a potential force if we ignore its response to changes in $x(t)$. For given oscillations $c(t)\propto e^{-i\omega t}$, the forces in (5) do not have high-frequency components and can be expressed in terms of a potential which is of the same form as (7).

Not only for interactions (1) but also for $H_{\rm int}(x, c, c^*)$ of arbitrary structure, the average force $F=-\partial H_{\rm int}/\partial x$ is very accurately a potential force in the given-field approximation [with short-priod oscillations of $c(t)$], if we are dealing with sufficiently inertial forms of the motions x (with large inertial coefficients). In this case, the high-frequency components $F_\sim(x,t)=F-\langle F\rangle$ add to the potential a contribution equal to the kinetic energy of the jarring, $x_\sim=x-\langle x\rangle$ (Ref. 3). The condition under which the oscillations c are fast is that there be a small parameter $|x_\sim|/l\ll 1$, where l is the scale length of the inhomogeneity along x of the forces exerted on the system. Characteristically, if the forces $F_\sim$ do not exceed $\langle F\rangle$ in amplitude, we have

$$|\langle F_\sim\rangle|\approx\left|\left\langle x_\sim\frac{\partial F_\sim}{\partial x}\right\rangle\right|\leqslant\frac{|x_\sim F_\sim|}{l}\leqslant\frac{|x_\sim|}{l}|\langle F\rangle|\ll|\langle F\rangle|;$$

i.e., to incorporate the forces $F_\sim$ is to exceed the accuracy of the treatment in terms of the small parameter.

The given-field approximation is frequently used in estimates of the interaction of forms of motions with greatly different frequencies. All-in-all, it yields a correct estimate of the forces $\langle F\rangle$ *in order of magnitude*. Regarding the *functional structure* of the force $\langle F\rangle$, however, it may give incorrect results, making it impossible to draw conclusions about the renormalization of the damping of the slow motions and even about the effective elasticity. This is true in particular of regimes with a resonant excitation of oscillations c. In this case, it is very important to take into account the resonant pumping of the modes c and the offsetting dissipation of these modes.

QUASISTATIC APPROXIMATION

In this section we consider extremely slow motions x, which are such that the changes $\omega_{kk}(x(t))$ set in more smoothly than the establishment of resonant regime of oscillations c according to (4) with $x=\text{const}$. This case evidently corresponds to motions x which are slower than in the adiabatic situation, in which the pumping and dissipation of the oscillations c are ignored.

From Eqs. (4) we find

$$c(t)=e^{-i\omega t}[d/dt+D(x(t))]^{-1}h=e^{-i\omega t}\int_{-\infty}^{t}\exp_t\left\{-\int_{\tau}^{t}D(x(t'))\right.$$

$$\left.\times dt'\right\}h\,d\tau=e^{-i\omega t}\left[1-D^{-1}\frac{d}{dt}+D^{-1}\frac{d}{dt}D^{-1}\frac{d}{dt}-\ldots\right]D^{-1}h, \tag{8}$$

where

$$D(x)=A(x)-i\omega I, \tag{9}$$

I is the unit matrix, $\exp_t$ is the chronologically ordered exponential function, and D^{-1} is the inverse of matrix (9).

We note that we have $dD^{-1}/dt=-D^{-1}AD^{-1}$, and the series on the right in (8) is, for slow changes in ω_{ik}, a power series in the small parameter

$$\varepsilon\sim|D^{-2}||d\omega_{ik}/dt|\sim\Omega_*/\Delta, \tag{10}$$

where $1/\Omega_*$ is the scale time for the crossing of the resonances upon changes in ω_{ik}, and Δ is the scale of the eigenvalues of matrix $|D|$, i. e., of the quantities $[\gamma_k^2+(\omega_k-\omega)^2]^{1/2}$.

The expression found for the force F as a functional of x by substituting the series (8) into (5) is complicated. To analyze it, we ignore terms $O(\varepsilon^2)$. In this approximation we have

$$F=F^0(x)-\Gamma(x)\dot{x}, \qquad (11)$$

where $F^0(x)$ are the forces in the quasistatic limit, $\varepsilon = 0$, and their components are

$$F_\alpha{}^0=-a^* \frac{\partial\hat{\omega}}{\partial x_\alpha} a, \qquad (12)$$

$\hat{\omega}$ is the matrix ω_{ik}, and $a = \{a_k\} = D^{-1}h$ are stationary [for a given $x = \text{const}$ in (11)] amplitudes of the oscillations c. The matrix $\Gamma(x)$ in (11) has the structure

$$\Gamma_{\alpha\beta}=ia^* \frac{\partial\hat{\omega}}{\partial x_\alpha} D^{-2} \frac{\partial\hat{\omega}}{\partial x_\beta} a+\text{c.c.} \qquad (13)$$

The terms in (11) generally do not correspond to a separation into potential forces and a damping, since in general we have

$$\partial F_\alpha{}^0/\partial x_\beta \neq \partial F_\beta{}^0/\partial x_\alpha, \quad \Gamma_{\alpha\beta}\neq\Gamma_{\beta\alpha}.$$

These inequalities become equalities if the matrices $\partial\omega/\partial x$ and D commute, i.e., if the motions c separate. In such a case, the term F^0 is a potential term, while $-\Gamma\dot{x}$ represents forces of a damping which is linear in the velocity. We first discuss the properties of the potential and the damping for this case.

Separable fast motions. We use a basis in the variables c in which the matrix $D(x)$ remains diagonal in the course of motion $x(t)$, i.e., is of the form

$$D_{ik}=\delta_{ik}D_k(x), \quad D_k=\gamma_k+i(\omega_k(x)-\omega). \qquad (14)$$

In this basis, we find the following expression for the potential of the forces $F^0(x)$:

$$U_0(x)=\sum_k \frac{|h_k|^2}{\gamma_k} \operatorname{arctg} \frac{\omega_k(x)-\omega}{\gamma_k}. \qquad (15)$$

The relation between the parameters $|h_k|^2/\gamma_k$ and the power $P_k = 2\gamma_k\omega_k|a_k|^2$ expended in sustaining the oscillations of mode k for a given $x = \text{const}$ is

$$\frac{|h_k|^2}{\gamma_k}=\frac{P_k}{2\omega_k}\left[1+\left(\frac{\omega_k-\omega}{\gamma_k}\right)^2\right].$$

The potential (15) differs substantially from the potential (7), which corresponds to relatively fast (adiabatic) changes in x.

In this basis we have

$$U_N(x)=\sum_k N_k\omega_k(x).$$

The contribution of each of the resonances k to (15) is of such a nature that there is a tendency for a particle to be displaced into the region where $\omega_k(x)$ is a minimum. The tendency is the same in the case of the potential $U_N(x)$, but the corresponding dependence in $U_0(x)$ becomes much steeper in the resonant region, $\omega_k \approx \omega$, and it acquires a substantially smooth form far from the resonances. For this rason, $U_0(x)$ may, for example, form quite steep wells and hills where $U_N(x)$ behaves smoothly and monotonically.

To illustrate these arguments we consider a one-dimensional motion x in the resonant field of two modes with normal frequencies

$$\omega_k(x)=\omega_0+(x-x_0)v_k,$$

where $k = 1, 2$, and the coefficients v_1 and v_2 differ in sign. In this case the potential $U_N(x)$ has a monotonic behavior: linear in x. The potential $U_0(x)$, on the other hand, forms a well near the point $x = x_0$ under the condition $\omega > \omega_0$, as shown in Fig. 1. Under the condition $\omega < \omega_0$, a hill is formed. The width of the well or hill depends on the frequency difference $\omega-\omega_0$; the minimum width is determined by the width of the edges, $\sim\gamma_k/|v_k|$; and the depth of the well is $\sim P/\omega_0$, where $P = P_1 + P_2$ is the power dissipated at the resonances. We note, however, that a free particle will not become localized in such a well because of the negative damping introduced by forces $-\Gamma\dot{x}$.

The damping coefficient $\Gamma(x)$ approximately duplicates the behavior of the second derivative of $-U_0(x)$; its profile is also shown in Fig. 1. In the potential well, Γ is negative; the particle is driven, and it begins to jump out of the well. Around the edges of the well, the damping Γ is positive, and the particle is slowed down, but the net effect is (as an analysis shows) that the amplitude of the oscillations of the particle continues to grow monotonically. For the simplified case with which we are dealing here, this growth is unbounded [in particular, as long as approximation (11), which requires that $\dot{x}$ be small, holds]. At $\omega < \omega_0$, when the well is replaced by a hill, the $\Gamma(x)$ profile is inverted. Now the damping Γ is negative around the potential barrier $U_0(x)$, and a particle whose velocity is too low to surmount the barrier will be reflected and will acquire some additional energy in the process. Untrapped particles also acquire energy on the average and surmount the barrier.

In general,the structure of the matrix Γ in (11) is as follows (in the basis in which D is diagonal):

$$\Gamma_{\alpha\beta}=\sum_k \frac{4(\omega_k-\omega)\gamma_k|h_k|^2}{[(\omega_k-\omega)^2+\gamma_k^2]^3} \frac{\partial\omega_k}{\partial x_\alpha} \frac{\partial\omega_k}{\partial x_\beta}. \qquad (16)$$

Since the sum over α,β for a given k obeys for arbitrary $x(t)$

$$\frac{\partial\omega_k}{\partial x_\alpha} \frac{\partial\omega_k}{\partial x_\beta} \dot{x}_\alpha\dot{x}_\beta=\left(\sum_\alpha \frac{\partial\omega_k}{\partial x_\alpha}\dot{x}_\alpha\right)^2 \geqslant 0,$$

we conclude from (16) that the power of the friction forces (this power is $\dot{x}\Gamma\dot{x}$) receives a positive contribution from

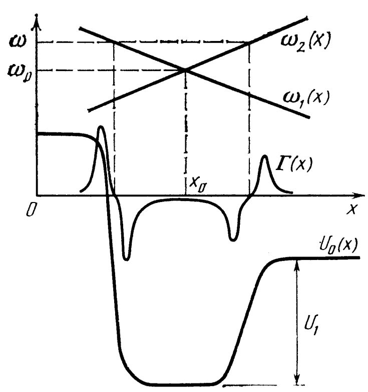

FIG. 1. Profiles of the potential $U_0(x)$ and of the damping $\Gamma(x)$ for a slow crossing of two resonances with frequencies $\omega_1(x)$ and $\omega_2(x)$. It is assumed here that we have $\gamma_1 = \gamma_2$ and $|h_1| < |h_2|$. $U_1 = \pi|h_1|^2/\gamma_1$.

interactions with normal modes of frequencies $\omega_k > \omega$ and a negative contribution from modes of frequencies $w_k < \omega$. This is the case for arbitrary forms of the motions $x(t)$, for aribtrary functions $\omega_k(x)$, and for an arbitrary nature of the excitation of resonances. It follows in particular that if $\omega_k > \omega$ for all of the modes involved in the interactions there should be a damping, while in the case in which we have $\omega_k < \omega$ for all these modes we should see an antidamping—a buildup of the motions x.

Nonseparable fast motions. In this case, only the symmetric part of the matrix Γ in (11), i.e., the matrix with the elements

$$\Gamma_{\alpha\beta}{}^{+}={}^1/_2(\Gamma_{\alpha\beta}+\Gamma_{\beta\alpha}),$$

represents the forces of a damping which is linear in the velocity. The skew-symmetric part of Γ, which is equal to $\Gamma^- = \Gamma - \Gamma^+$, represents gyroscopic forces. The power which they generate, $\dot{x}\Gamma^-\dot{x}$, is identically zero by virtue of the relation $\Gamma^-_{\alpha\beta} + \Gamma^-_{\beta\alpha} = 0$.

The forces F^0 can also be broken up into two parts: a conservative part described by a potential, and a vortical part. The potential part of F^0 is characterized by the matrix of elasticity coefficients

$$K_{\alpha\beta}{}^{+}=-{}^1/_2(\partial F_\alpha{}^0/\partial x_\beta+\partial F_\beta{}^0/\partial x_\alpha).$$

The vortical part of F^0, like the forces $-\Gamma^+\dot{x}$, contributes to an irreversible exchange of energy with resonances, but the work performed by the vortical forces over the cycle of the $x(t)$ change does not depend on the velocity. Such forces are similar to rotational forces (e.g., those which act on a Stokes particle in a rotating liquid when the velocity in the flow is much higher than that of the particle). The intensity of the vortices of the field $F^0(x)$ is characterized by the skew-symmetric matrix with elements

$$K_{\alpha\beta}{}^{-}=-{}^1/_2(\partial F_\alpha{}^0/\partial x_\beta-\partial F_\beta{}^0/\partial x_\alpha), \tag{17}$$

whose meaning is clear from the circumstance that the work performed by the forces F^0 in the motions $x(t)$ in a small neighborhood of the point x, along a contour of area s in the x_α, x_β plane, is

$$\oint F^0\,dx=2sK_{\alpha\beta}{}^{-}(x).$$

Let us examine the distinctive features of the damping $\Gamma^+(x)$. The vortical forces are discussed in a separate subsection. For convenience in comparison with the case of separable motions, we use a representation in which, for a *given* x in $\Gamma^+(x)$, the matrix $D(x)$ is diagonal. This approach does not limit the generality of the discussion. The elements of $D(x)$ are evidently of the form in (14), but $\dot{D}(x)$ is no longer the same as the derivative of the diagonal matrix with the elements in (14). In this basis we have

$$\Gamma^+=\sum_k\frac{4(\omega_k-\omega)\gamma_k}{[(\omega_k-\omega)^2+\gamma_k{}^2]^2}g(k), \tag{18}$$

where $g(k)$ are symmetric matrices with the elements

$$g_{\alpha\beta}(k)=\mathrm{Re}\left\{\frac{\partial\omega_{ki}{}^*}{\partial x_\alpha}\frac{\partial\omega_{kn}}{\partial x_\beta}a_i{}^*a_n\right\}. \tag{19}$$

Expression (18) differs from (16) only in the form of the matrices $g(k)$, which, for the case of separable motions, become

$$g_{\alpha\beta}(k)=\frac{\partial\omega_k}{\partial x_\alpha}\frac{\partial\omega_k}{\partial x_\beta}|a_k|^2.$$

In the particular case in which $h_k = \delta_{0k}h_0$ corresponds to the given x, i.e., in which only a single mode c (that with the index 0) is excited, matrix (18) is again a sum over k; only the form of the matrices $g(k)$ simplifies:

$$2g_{\alpha\beta}(k)=\frac{\partial\omega_{k0}{}^*}{\partial x_\alpha}\frac{\partial\omega_{k0}}{\partial x_\beta}|a_0|^2+\text{c.c.} \tag{20}$$

This result shows clearly that a modulation of the coefficients of the couplings between the resonances add to the damping a contribution comparable to that of the modulation of the eigenfrequencies.

Each of the matrices in (19) has nonnegative eigenvalues, since for arbitrary $x(t)$ the quadratic form $\dot{x}g(k)\dot{x}$ takes on only nonnegative values:

$$\dot{x}_\alpha g_{\alpha\beta}(k)\dot{x}_\beta=\left|\sum_{i,\alpha}\frac{\partial\omega_{ki}}{\partial x_\alpha}a_i\dot{x}_\alpha\right|^2\geqslant 0. \tag{21}$$

It thus follows from (18) that the sign of the damping Γ^+, i.e., the sign of the eigenvalues of the matrix Γ^+, is determined by the position of the spectrum $\{\omega_k(x)\}$ with respect to the frequency ω, and it is of the same nature as in the case of separable motions c, discussed above.

Vortical forces. The matrix in (17) characterizes the intensity of the vortical forces. For it we find from (12)

$$K^-=\sum_k\frac{2\gamma_k}{(\omega_k-\omega)^2+\gamma_k{}^2}\varkappa(k), \tag{22}$$

where $\varkappa(k)$ are skew-symmetric matrices with the elements

$$\varkappa_{\alpha\beta}(k)=\mathrm{Im}\left\{\frac{\partial\omega_{ki}{}^*}{\partial x_\alpha}\frac{\partial\omega_{kn}}{\partial x_\beta}a_i{}^*a_n\right\}. \tag{23}$$

We note that the behavior of the vortical forces as a function of the frequency differences $\omega_k - \omega$ is quite different from that of the damping forces: The frequency-dependent coefficients of $\varkappa(k)$ in (22) are at a maximum at $\omega = \omega_k$, while the corresponding factors in (18) vanish.

Let us compare the vortical action with the dissipative action. Specifically, we compare the average power levels of the forces of the two types in the case of small periodic motions with respect to some fixed value of x, which is taken as the origin of the $x(t)$ scale:

$$\begin{aligned}\langle\dot{x}\Gamma^+\dot{x}\rangle&=\sum\frac{4(\omega_k-\omega)\gamma_k}{[(\omega_k-\omega)^2+\gamma_k{}^2]^2}g_{\alpha\beta}(k)\langle\dot{x}_\alpha\dot{x}_\beta\rangle,\\ \langle\dot{x}K^-x(t)\rangle&=\sum\frac{2\gamma_k}{(\omega_k-\omega)^2+\gamma_k{}^2}\varkappa_{\alpha\beta}(k)\langle\dot{x}_\alpha x_\beta(t)\rangle.\end{aligned} \tag{24}$$

Here the angle brackets mean averaging over the period of $x(t)$. We consider the expressions

$$I_g(k)=\langle\dot{x}g(k)\dot{x}\rangle,\quad I_\varkappa(k)=\Omega\langle\dot{x}\varkappa(k)x(t)\rangle \tag{25}$$

in the case of harmonic changes in the frequency Ω:

$$x(t)=\{x_\alpha(t)\}=x_\alpha\cos(\Omega t+\varphi_\alpha).$$

Using (19) and (24) we find (we are not writing the index k explicitly)

$$\begin{aligned}I_g&=\Omega^2x_\alpha x_\beta|z_\alpha z_\beta|\cos(\varphi_\alpha-\varphi_\beta)\cos(\psi_\alpha-\psi_\beta),\\ I_\varkappa&=\Omega^2x_\alpha x_\beta|z_\alpha z_\beta|\sin(\varphi_\alpha-\varphi_\beta)\sin(\psi_\alpha-\psi_\beta),\end{aligned} \tag{25'}$$

where

$$z_\alpha=|z_\alpha|e^{i\psi_\alpha}=\frac{\partial\omega_{kn}}{\partial x_\alpha}a_n.$$

Hence

$$I_g+I_\varkappa=\Omega^2x_\alpha x_\beta|z_\alpha z_\beta|\cos[(\varphi_\alpha-\psi_\alpha)-(\varphi_\beta-\psi_\beta)]$$
$$=\Omega^2\Big[\sum_\alpha x_\alpha|z_\alpha|\cos(\varphi_\alpha-\psi_\alpha)\Big]^2$$
$$+\Omega^2\Big[\sum_\alpha x_\alpha|z_\alpha|\sin(\varphi_\alpha-\psi_\alpha)\Big]^2\geqslant 0.$$

Since $I_g\geqslant 0$ [see (21)], we find

$$I_g(k)\geqslant|I_\varkappa(k)|. \tag{26}$$

A stronger opposite inequality holds for the corresponding frequency factors for I_g and $I_\varkappa$ in (24). The ratio of the k th terms in the sums in (24) is

$$\left|\left(\frac{\langle\dot{x}K^-x(t)\rangle}{\langle\dot{x}\Gamma^+\dot{x}\rangle}\right)_k\right|=\frac{|I_\varkappa(k)|}{I_g(k)}\Phi(k)\gg\frac{|I_\varkappa(k)|}{I_g(k)}. \tag{27}$$

Here the quantity

$$\Phi(k)=\frac{\gamma_k^2+(\omega_k-\omega)^2}{|\Omega(\omega_k-\omega)|}\gtrsim O\left(\frac{1}{\varepsilon}\right)$$

is very large, since the analysis is being carried out in the quasistatic approximation, in which the parameter in (10) is small.

According to (27), the irreversible energy exchange generated by vortical forces may significantly exceed the work performed by friction forces, since the factors I_g and $I_\varkappa$ are generally comparable in magnitude, as can be seen from their stucture in (25′). A necessary condition here, in additon to the condition that there is a subsantial mixing of the multidimensional motions c upon changes in $x(t)$, is that the corresponding forms of the motion x must have approximately equal frequencies.

When a motion x is represented by only a single degree of freedom, the concepts of vortical and gyroscopic forces evidently lose their meaning; the forces $F^0(x)$ in (11) are potential forces, and Γ_x is the same as the damping Γ^+, with all its properties as discussed in the preceding section. The distinctive features in the case of a two-dimensional motion $x=(x_1,x_2)$ can be seen from the example described by the equations

$$m_1\ddot{x}_1+k_1x_1=\lambda x_2,\quad m_2\ddot{x}_2+k_2x_2=-\lambda x_1,$$

where vortical forces which are linear in x appear on the right sides. The solution of these equations is a superposition of rotational motions with characteristic frequencies Ω_+ and Ω_- given by

$$\Omega_\pm^2={}^1\!/_2(\Omega_1^2+\Omega_2^2)\pm[{}^1\!/_4(\Omega_1^2-\Omega_2^2)^2-\lambda^2/m_1m_2]^{1/2},$$

where $\Omega_i^2=k_i/m_i$. We see that the vortical forces cause rotational motions to grow only if the frequencies Ω_i are quite close together. If the frequencies are separated substantially, and the condition $(\Omega_1^2-\Omega_2^2)^2\gg 4\lambda^2/m_1m_2$ holds, the effect of these forces is slight.

We have yet another comment. Vortical (and gyroscopic) forces arise when the interactions with the resonant field are of such a nature that the matrices ω_{ik} and $\partial\omega_{ik}/\partial x$ do not commute; i.e., there is a mixing of the motions c during a change in $x(t)$. This condition, however, is a necessary but not sufficient condition, as we have already seen clearly in the example of a one-dimensional motion x. Another example is that in which, during multidimensional motions $x(t)$, only a single pair of elements $(\omega_{ik},\omega_{ki})$ changes in $\{\omega_{ik}\}$, and their ratio ω_{ik}/ω_{ki} does not depend on $x(t)$. In this case we have $K^-=0$ and $\Gamma^-=0$. The vortical and gyroscopic forces are also absent if ω_{ik} and $\partial\omega_{ik}/\partial x$ do not commute, but for a given x only a single mode c is excited [in this case, the matrix $\Gamma(x)$ is as given by expression (18), with matrices $g(k)$ as in (20)]. Furthermore, vortical forces do not arise in the case of the simultaneous excitaiton of a set of modes c if the phases of $\{h_k\}$ are selected in a certain way.

If, for example, subsystem c is nongyroscopic (in this case, we can assume, without any loss of generality, that all the elements ω_{ik} and all the corresponding $\partial\omega_{ik}/\partial x$ are real), then it is easy to see that we have $k^-=0$ and $\Gamma^-=0$ when the phases of all the $\{a_k\}$ are identical or differ by π. The presence of a phase shift which is not a multiple of π, i.e., the presence of rotating or traveling components in the excited resonant field, does not by itself lead to rotational forces, since there is the further necessary condition that ω_k and $\partial\omega_{ik}/\partial x$ do not commute.

The action of these vortical forces is thus a rather unique effect of the reaction forces (recoil forces) of resonant fields. Nevertheless, it may be predominant because of strong inequality (27). As the frequency Ω is increased, and the quasistatic approximation must be abandoned, the parameter $\Phi(k)$ in (27) becomes modified, but it does not remain greater than unity, as before (more on this below).

RENORMALIZATION OF THE MOBILITY PARAMETERS AT VALUES OF ε WHICH ARE NOT SMALL

In this section of the paper we examine the quantity $\delta F(t)/\delta x(\tau)$. It determines a renormalization of the mobility parameters of system x and is a convenient and rather comprehensive (along with the value of F in the case $\delta x=0$) characteristic of the interactions if we are concerned with the stability of dynamics of oscillations $x(t)$ whose amplitude is small in comparison with the scale l. In the limit $\varepsilon\ll 1$, the quantity $\delta F(t)/\delta x(\tau)$ is represented by the matrices of coefficients $K^\pm$, $\Gamma^\pm$, discussed above. Let us examine the particular features of these matrices and the variance which arise as the rate of change $\delta x(t)$ increases with distance from the quasistatic limit.

To find $\delta F(t)/\delta x(\tau)$ we need to calculate the response of a resonant regime of oscillations c to changes $\delta x=x(t)-x$ in the limit $\delta x\to 0$. From (4) we find the following expression for the response δc:

$$\delta c(t)=-\int_{-\infty}^{t}e^{-(t-\tau)A}\frac{\partial A}{\partial x}\delta x(\tau)c(\tau)d\tau, \tag{28}$$

where A and $\partial A/\partial x$ in the integral are functions of $x=\text{const}$, and $c(\tau)=e^{-i\omega\tau}D^{-1}h$ is a stationary regime of oscillations c with $\delta x=0$. As before, we are using a representation in which $A|_{\delta x=0}$ is diagonal. Using the spectral representation

$$\delta x(t)=\int_{-\infty}^{\infty}\delta x(\Omega)e^{-i\Omega t}d\Omega$$

and calculating the response δF from (5) and (28), we put it in the form

$$\delta F(t)=F-F|_{\delta x=0}=-\int_{-\infty}^{\infty}[K(x,\Omega)-i\Omega\Gamma(x,\Omega)]\delta x(\Omega)e^{-i\Omega t}d\Omega, \tag{29}$$

where K and Γ are matrices with real elements which are even in the variable Ω. This approach gives the symmetric and skew-symmetric parts of $K=K^{+}+K^{-}$ the meaning of elasticity coefficients and vortical forces which are generated by interactions with resonances during harmonic changes $\delta x(t)$ of frequncy Ω. Analogously, the symmetric (Γ^{+}) and skew-symmetric (Γ^{-}) parts of Γ are the coefficients of a linear damping and gyroscopic forces.

The matrices $K^{\pm}$ are of the form

$$K^{-}(x,\Omega)=-\sum_k\frac{\varkappa(k)}{\gamma_k}\left[\frac{1}{1+\xi_+^2(k)}+\frac{1}{1+\xi_-^2(k)}\right],$$
$$K^{+}(x,\Omega)=k(x)-\sum_k\frac{g(k)}{\gamma_k}\left[\frac{\xi_+(k)}{1+\xi_+^2(k)}+\frac{\xi_-(k)}{1+\xi_-^2(k)}\right], \tag{30}$$

where

$$\xi_\pm(k)=[\omega_k(x)-\omega\pm\Omega]/\gamma_k,$$

and $g(k)$ and $\varkappa(k)$ are the matrices introduced in (19) and (24). the matrix $k(x)$ in K^{+} is symmetric and independent of Ω; it represents the elasticity of the forces F in the approximation of a given field and is expressed in terms of a potential:

$$k_{\alpha\beta}(x)=\frac{\partial^2}{\partial x_\alpha\,\partial x_\beta}U_{\langle N\rangle}(x),\qquad U_{\langle N\rangle}(x)=\langle N_{ik}\rangle\omega_{ik}(x),$$

where the $\langle N_{ik}\rangle$ are the amplitudes $a_i^*a_k$, which are stationare in the case $\delta x=0$.

The matrices $\Gamma^{\pm}$ are given by

$$\Gamma^{+}(x,\Omega)=\sum_k\frac{g(k)}{\gamma_k^2}\left[-\frac{\gamma_k/\Omega}{1+\xi_+^2(k)}+\frac{\gamma_k/\Omega}{1+\xi_-^2(k)}\right],$$
$$\Gamma^{-}(x,\Omega)=\sum_k\frac{\varkappa(k)}{\gamma_k^2}\left[\frac{\gamma_k/\Omega}{1+\xi_+^2(k)}-\frac{\gamma_k/\Omega}{1+\xi_-^2(k)}\right]. \tag{31}$$

In the limit $\Omega\to 0$, expressions (31) remain finite and go to their quasistatic limits discussed in the preceeding section. The same is true of matrices (30). The terms in the sums in (30) and (31) are structured matrices which do not depend on Ω, with weight factors [in the square brackets] which increase sharply in the region of Raman resonances, $\omega=\omega_k\pm\Omega$. These factors are dimensionless and do not exceed unity in order of magnitude. In K^{-} and Γ^{-}, these factors are characteristically even in $\omega-\omega_k$, while in Γ^{+} and K^{+} they are odd and change sign when the sign of $\omega-\omega^k$ is reversed. It follows, for example, that when we also recognize that all the matrices $g(k)$ are not negative-definite [property (21)], the modes of the frequencies $\omega_k>\omega(\omega_k<\omega)$ make a positive (negative) contribution to the damping Γ^{+}.

If, in the estimates of Γ^{+} and K^{-}, we take into account only interactions with Raman resonances $\omega=\omega_k+\Omega$ (or only those with $\omega=\omega_k-\Omega$), i.e., if we retain in (30) and (31) only those terms which depend on $\{\xi_+(k)\}$ [or only on $\{\xi_-(k)\}$] in the sums over all k, we find the following result, where we are making use of property (26):

$$|\langle\dot{x}\Gamma^{+}\dot{x}\rangle|\geqslant|\langle\dot{x}K^{-}x(t)\rangle|.$$

This result means that for any forms of motions $x(t)$, for any structure of the interactions $\omega_{ik}(x)$, and for any level of excitation of the resonances, the direction of the resultant energy flux between c and x is determined by the nature of the damping Γ^{+}. We have either quenching [if only the terms which depend on $\{\xi_-(k)\}$ are taken into account] or amplification [only the terms which depend on $\{\xi_+(k)\}$] of the oscillations $x(t)$. If such conditions are to be met, the frequencies of the motions Ω must obviously be large in comparison with the frequencies of the damping γ_k, and the spectrum $\{\omega_k\}$ of the modes which are involved in the interactions must be relatively narrow.

When the spectrum $\{\omega_k\}$ contains widely spaced frequencies, there may be conditions, depending on the position of the frequency ω, even in the case $\Omega\gg\gamma_k$, such that resonances $\omega=\omega_k+\Omega$ are effective for certain modes, while resonances $\omega=\omega_k-\Omega$ are effective for others. The result may be a substantial lowering of the value (eigenvalues) of Γ^{+} but without affecting K^{-}, since K^{-} is even in $\omega_k-\omega$. Under these conditions we would expect a significant predominance of the vortical action of the resonant fields.

Comparing the irreversible energy fluxes generated by the partial contributions from the k th terms to K^{-} and Γ^{+} for arbitrary Ω, we find relations which generalize (27) in that $\Phi(k)$ should be replaced by

$$\Phi(k)=\frac{\gamma_k^2+(\omega_k-\omega)^2+\Omega^2}{|\Omega(\omega_k-\omega)|}.$$

Correspondingly, the symbol for "much greater than" in (27) would be replaced in the general case by a "greater than" symbol, and $O(1/\varepsilon)$ would be replaced by $O[(1+\varepsilon^2)/\varepsilon]$.

When we were discussing the quasistatic limit $\varepsilon\ll 1$, we mentioned the property that the damping Γ^{+} is of one sign for all forms of the motions $x(t)$ for which the spectrum $\{\omega_k\}$ of the modes c that are effectively involved in the interactions lie on one side of the frequency ω. We see that this property holds for arbitrary values of ε. This property may be interpreted from the standpoint of processes of a stimulated decay, $\omega\to\omega_k+\Omega$, or coalescence, $\omega+\Omega\to\omega_k$, of elementary excitations, by associating with the processes the first and second terms in square brackets in (31), which differ in sign. In the decay, energy is pumped into the oscillations at the frequencies ω_k and Ω; the oscillations $x(t)$ are amplified; and the contribution to the damping Γ^{+} is correspondingly negative. In the case of a coalescence, there is an energy tansfer from the excitations of frequency Ω; the motions $x(t)$ are damped; and the contribution to Γ^{+} is positive.

If we were to pursue this approach systematically, we would associate the pumping of vortical energy from c to x with decay and coalescence processes. Actually, we are talking here about the same approximation, and the same Ra-

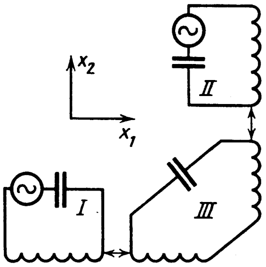

FIG. 2. The "parametric motor." Movable circuit III becomes entrained in translational motions along circular trajectories in the x_1, x_2 plane because of a resonant excitation of circuits I and II, which form the stator.

man resonances are acting in both Γ^+ and K^- in (30) and (31). The vortical forces do not necessarily arise because of the presence of tails of the resonant functions, as could be shown simply for the situation $\gamma_k \gg \Omega$, $|\omega_k - \omega|$. It follows from the discussion above that these forces can also be predominant in the case $\gamma_k \ll \Omega$, $|\omega_k - \omega|$. In this case, however, how would we interpret the circumstance that the pumping of vortical energy from c to x does not change sign and occurs identically efficiently when high-frequency energy is pumped either up or down the spectrum? Apparently, to speak in terms of frequencies and energy levels, as is customary in discussions of stimulated Raman scattering, is not appropriate here, or at any rate it is not sufficient for an interpretation of the vortical Raman phenomena involved here.

Questions of the existence of vortical forces and mutual transfer of the energy of vortical motions, or of the excitation of rotational oscillations and waves by this mechanism, have apparently not previously been raised in the literature. Although the mechanism for interactions with high-frequency resonances which we have discussed here is quite general, in analyzing the stimulated scattering or damping of excitations by a strong wave of frequency ω it is customary to retain, from the entire reservoir of waves which interact in a nonlinear way, the smallest number necessary for some low-order process or another to occur. When this approach is taken, the vortical effects which we have been discussing here drop out of the picture, since their incorporation requires resorting to a large number of normal vibrations, of both high and low frequencies. This assertion by no means implies that the vortical processes are atypical, since they are manifested in the same low order in the interaction nonlinearities as the decay and coalescence processes we have mentioned.

We conclude with a discussion of an example which demonstrates the vortical action of multimode resonant fields.

THREE-MODE VORTICAL-ACTION "PARAMETRIC MOTOR"

Let us examine the system shown in Fig. 2. It consists of three high-Q RLC circuits with $R \ll (L/C)^{1/2}$. Circuit III is movable; it is coupled inductively with fixed circuits I and II, which are supplied power from an external oscillator of frequency ω. As circuit III moves along the x_1 direction, the coupling between circuits I and III changes; as it moves along the x_2 direction, the coupling between II and III changes.

We choose the variables $c = (c_1, c_2, c_3)$ to be normal when the circuits are separated from each other and the coupling between circuits can be ignored. In this case,the elements ω_{ik}, which depend on $x = (x_1, x_2)$, are

$$\omega_{13}(x_1)=\omega_{31}(x_1), \quad \omega_{23}(x_2)=\omega_{32}(x_2).$$

Let us determine the matrices $\varkappa(k)$ and $g(k)$ in (30) and (31). These are 2×2 matrices with $k = 1, 2, 3$. They depend on the amplitudes a_1, a_2, a_3, which are stationary for a given x, and on the parameters

$$v_1=\partial\omega_{13}/\partial x_1, \quad v_2=\partial\omega_{23}/\partial x_2.$$

For the matrices $\varkappa(k)$ we find

$$\varkappa(1)=0, \quad \varkappa(2)=0,$$

$$\varkappa(3)=\begin{vmatrix} 0 & v_1v_2\,\mathrm{Im}\,(a_1^*a_2) \\ -v_1v_2\,\mathrm{Im}\,(a_1^*a_2) & 0 \end{vmatrix}.$$

For the matrices $g(k)$ we find

$$g(1)=\begin{vmatrix} v_1^2|a_3|^2 & 0 \\ 0 & 0 \end{vmatrix}, \qquad g(2)=\begin{vmatrix} 0 & 0 \\ 0 & v_2^2|a_3|^2 \end{vmatrix},$$

$$g(3)=\begin{vmatrix} v_1^2|a_1|^2 & v_1v_2\,\mathrm{Re}\,(a_1^*a_2) \\ v_1v_2\,\mathrm{Re}\,(a_1^*a_2) & v_2^2|a_2|^2 \end{vmatrix}$$

We consider conditions such that the eigenfrequencies $\omega_1(x)$ and $\omega_2(x)$ for given geometry x are identical and are at exact resonance: $\omega_1 = \omega_2 = \omega$. In this case, only the terms with $k = 3$ are nonzero in the matrices K^- and Γ^+ given by expressions (30) and (31); in other words, only $\varkappa(3)$ and $g(3)$ are important. They do not depend on the amplitude a_3. Let us assume that circuits I and II are excited in such a way that the amplitudes a_1 and a_2, which are stationary for a given $x =$ const, are $\pi/2$ out of phase. For simplicity we also assume $v_1|a_1| = v_2|a_2|$. For harmonic variations $x_{1,2}(t)$ of frequency Ω, the ratio of the power levels of the vortical forces and of the friction forces is

$$\left|\frac{\langle \dot{x}K^-x(t)\rangle}{\langle \dot{x}\Gamma^+\dot{x}\rangle}\right| = \frac{|\langle \dot{x}_1x_2(t)-x_1(t)\dot{x}_2\rangle|}{\langle \dot{x}_1^2\rangle+\langle \dot{x}_2^2\rangle}\,\frac{\gamma_3^2+(\omega_3-\omega)^2+\Omega^2}{|\Omega(\omega_3-\omega)|}. \tag{32}$$

The first fraction on the right depends strongly on the polarization of the oscillations $x(t)$. This fraction vanishes in the case of a linear polarization (an arbitrary polarization in the x_1, x_2 plane) and becomes equal to unity for rotational motions $x(t)$. Correspondingly, in the first case we have $\langle \dot{x}K^-x(t)\rangle = 0$, while in the second the ratio in (32) becomes greater than unity at arbitrary values of the frequency difference $\omega_3 - \omega$ and of the frequency Ω. We are talking here about rotational motions of the center of mass of system III, not about an angular rotation of this system around some axis. Reversing the direction of the rotation of the resonant field (replacing the phase shift $\pi/2$ between a_1 and a_2 by $-\pi/2$) reverses the direction of the vortical action of the forces F. This does not occur when the sign of the frequency difference $\omega_3 - \omega$ is changed.

Consequently, not only in the case $\omega_3 < \omega$, in which the damping Γ^+ is negative, but also in the case $\omega_3 > \omega$, when Γ^+ is positive, and the oscillations $x(t)$ or arbitrary linear polarization are suppressed, rotational motions can grow, since the power of the vortical forces is predominant. At $\Omega \gg \gamma_k$, the rotational action is greatest near the Raman resonances $\omega = \omega_3 - \Omega$ and $\omega = \omega_3 + \Omega$.

This example obviously does not come close to exhausting the variety of systems and manifestations of the vortical Raman effects which we have been discussing here. Study of these effects might apparently be both of applied interest, e.g., for the development of parametric devices of a new type, and of interest for research into new physical phenomena in the motion of particles and the excitationof oscillations and waves in resonant fields (it might be suggested, e.g., that there are gas rotation effects in resonant light fields).

[1)]An arbitrary complex matrix A can always be written unambiguously in the form $A = A' + iA''$, where the complex matrices A' and A'' are Hermitian.[10] If we introduce a normalization such that the expression $c^*A''c$ [i.e., the sum in (1)] is the energy of the oscillations, then the rate of change of this energy, prescribed by (4) with $h = 0$ and $x = \text{const}$, is $c^*(A'A'' + A''A')c$, in agreement with the expression given here for P.

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