Conductivity of inhomogeneous media with a low concentration of inclusions

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The effective electrical conductivity of isotropic systems with a low concentration ($c \ll 1$) of inclusions is analyzed to second order in c. General formulas convenient for specific calculations are derived. Their range of applicability is determined. Their relationship with the problem of the conductivity of slightly inhomogeneous media is determined. The conductivity of a two-dimensional system with circular inclusions is calculated explicitly. The analytic properties of the electrical conductivity of this system are discussed in the plane of the complex argument h, the ratio of the conductivities of the components. The point $h = 0$ is a singular point for the effective conductivity. The nature of the singularity is determined. The broadening of the local level in the corresponding LC model due to the "interaction" of inclusions is determined.

1. INTRODUCTION

The effective characteristics of two-component systems can be written as power series in the concentration of one of the components, c (this is a so-called group or virial expansion). The group expansion is essentially the only systematic method for calculating the various characteristics of inhomogeneous media for arbitrary ratios of the properties of the components. Most of the work on the subject has been carried out to first order in the concentration (Ref. 1; see also Ref. 2) or through a general formal analysis of the virial series (Ref. 3, for example). An examination of the correction which is linear in c reveals several important features of two-component systems. There is also considerable interest in studying the next second-order correction in the concentration, to obtain a more detailed picture of the structure of the group expansion and of its range of applicability. We wish to emphasize that going from the linear approximation in c to the second-order approximation (which we will also call the c^2 approximation) does not just provide a quantitative refinement of the results. As we will see below, incorporating the "interaction" of inclusions leads, even in the c^2 approximation, to several qualitatively new effects, which are not found to first order in c and which must be taken into account in a study of the analytic properties of the electrical conductivity.[4] It is also necessary to go beyond first order in the concentration in studying the spectral properties of the so-called LC model.[4]

In the present paper we derive a general expression for the effective electrical conductivity σ_e in the c^2 approximation. In this expression, the quadratic correction is expressed in terms of the polarizability of a pair of inclusions in an external electric field. We relate the problem of the group expansion for σ_e to the problem of the conductivity of a slightly inhomogeneous medium. For slightly inhomogeneous systems, a perturbation theory in $\delta\sigma$—the deviation of the conductivity $\sigma(\mathbf{r})$ from its mean value—leads to an expression[1] for σ_e which is quite common: In the approximation quadratic in $\delta\sigma$, and for arbitrary values of c, the conductivity σ_e depends on neither the shape nor distribution of the inclusions. Analysis of the virial series leads to an explanation of the reason for this universal applicability.

For a specific calculation of the correction quadratic in c, we must find the polarizability of a pair of inclusions. In general, this is an extremely complicated problem. As an example of the application of the general equations, we consider the comparatively simple case of a two-dimensional system with circular inclusions. The problem of determining the polarizability tensor of two circles (or circular cylinders) can be solved exactly, so that we can calculate σ_e explicitly in the c^2 approximation. The expression which we derive for σ_e demonstrates the nature of the group expansion for an arbitrary ratio h of the conductivities of the components, in particular, for systems with insulating or ideally conducting inclusions.

The explicit expression for σ_e makes it possible in this example to study a question of importance to percolation theory[5,6]: the analytic properties of the function f (the dimensionless effective electrical conductivity) in the complex plane of the argument h (Ref. 4). In accordance with the general conclusions of Ref. 4, the function f is analytic over the entire h plane, cut along the negative real semiaxis. Of fundamental importance is a confirmation of the conclusion, reached in Ref. 4, that the point $h = 0$ is a singular point for f for disordered systems. In the model considered here, the function f contains a nonanalytic term of the form $h^3 \ln h$ in the limit $h \to 0$. In the corresponding LC model,[4] only the local frequency is related to an isolated inclusion ("defect"). A pair of defects has a set of impurity levels, which fill the entire energy interval as the inclusions move progressively closer together. Consequently, the gap in the spectrum of the LC model which was discussed in Ref. 4 is not present in this case, and the imaginary part of f is nonzero for all $h < 0$. The explicit expression for σ_e can be used to study the behavior of $\mathrm{Im} f$ on the cut and to study the concentration broadening of the impurity level (corresponding to an isolated defect) which results from the interaction of inclusions. These results make it possible to draw conclusions about the form of the imaginary part of the function f for some other binary systems also.

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2. SECOND-ORDER APPROXIMATION

The general formal scheme for constructing a group expansion for the effective characteristics of binary media is described by Finkel'berg,[3] for example, for the problem of the dielectric constant of a mixture. The approach which we will take below is analogous to that suggested in Ref. 2. This method is fundamentally the same as that in Ref. 3 but more convenient for specific calculations.

We consider a three-dimensional medium consisting of an isotropic matrix of conductivity σ_1 and of identical inclusions of conductivity σ_2. We denote by $\mathbf{E}_0$ the macroscopic electric field in the medium in the absence of inclusions. We place one inclusion at the origin. At large distances ($r\to\infty$) the asymptotic expression for the potential φ then has its usual form[1]:

$$\varphi=-\mathbf{E}_0\mathbf{r}+\mathbf{pr}/r^2+\dots, \tag{1}$$

$$p_\alpha=\Lambda_{\alpha\beta}E_{0\beta},\qquad \hat{\Lambda}=v\hat{\alpha}. \tag{2}$$

Here v is the volume of an inclusion, $\mathbf{p}$ is its dipole moment, and $\hat{\alpha}$ is the polarizability tensor, which depends on the shape of the inclusion and the conductivity ratio $h=\sigma_2/\sigma_1$.

To determine the conductivity of this medium at a low inclusion concentration ($c\ll 1$), we average the vector $\mathbf{j}-\sigma_1\mathbf{E}$ over the volume of the entire system V as in Ref. 1. Here $\mathbf{j}=\mathbf{j}(\mathbf{r})$ and $\mathbf{E}=\mathbf{E}(\mathbf{r})$ are the current density and electric field in the medium. Since we have $\langle\mathbf{j}\rangle=\sigma_e\langle\mathbf{E}\rangle$ by definition ($\langle\dots\rangle$ means an average over the entire volume of the sample), where σ_e is the effective conductivity of the system, we find

$$(\sigma_e-\sigma_1)\langle\mathbf{E}\rangle=\frac{1}{V}\sum_a\int_{v_a}(\mathbf{j}-\sigma_1\mathbf{E})\,dV. \tag{3}$$

Expression (3) reflects the fact that the integrand in nonzero only inside inclusions. The summation in (3) runs over all the inclusions (particles), while the integration is over the volume of the ath particle, v_a. As was shown in Ref. 2, the integral on the right side of (3) can be transformed into a surface integral, which can be calculated through the use of the asymptotic expression (1). As a result we find

$$(\sigma_e-\sigma_1)\langle\mathbf{E}\rangle=4\pi\sigma_1\frac{1}{V}\sum_a\mathbf{p}_a, \tag{4}$$

where p_a is the dipole moment of the ath particle. From (4) we find a general expression for σ_e to first order in the concentration of inclusions (see Ref. 2 and the discussion below in the present paper).

To calculate the effective conductivity σ_e with terms $\sim c^2$, we partition the entire set of inclusions into pairs, and we treat the two nearest inclusions as constituting a single "particle." In this case the quantity p_a in (4) is the dipole moment of a pair of inclusions, $p_a^{(2)}$, while the sum in (4) contains $\mathcal{N}/2$ terms, where $\mathcal{N}$ is the total number of inclusions in the sample. Using (2), we find

$$\frac{1}{V}\sum_a\mathbf{p}_a^{(2)}=\frac{1}{V}\sum_a\hat{\Lambda}_a^{(2)}\mathbf{E}_0=\frac{\mathcal{N}}{2V}\overline{\overline{\hat{\Lambda}^{(2)}}}\mathbf{E}_0. \tag{5}$$

The second equality follows from the law of large numbers: As $V\to\infty$ and $\mathcal{N}\to\infty$ we have

$$\sum_a\hat{\Lambda}_a^{(2)}=\mathcal{N}_a\overline{\overline{\hat{\Lambda}^{(2)}}},$$

where $\mathcal{N}_a=\mathcal{N}/2$ is the total number of particles. The double bar over $\hat{\Lambda}(2)$ means an average over the ensemble of particles; the superscript 2 in $\hat{\Lambda}^{(2)}$ means that the tensor $\hat{\Lambda}$ refers to a pair of inclusions.

The tensor $\hat{\Lambda}^{(2)}$ must be averaged over both the orientations of the individual inclusions (over the "internal" coordinates) and the coordinates characterizing the particle as a whole (the pair of inclusions). For simplicity we restrict the analysis to spherical inclusions, in which case there is no averaging over the internal coordinates. We first average over the orientations of a pair for a fixed distance between inclusions:

$$\overline{\overline{\Lambda_{\alpha\beta}^{(2)}}}={}^1/_3\,\delta_{\alpha\beta}\overline{\mathrm{Sp}\hat{\Lambda}^{(2)}}. \tag{6}$$

The single superior bar here means the remaining part of the average, over the distance between inclusions,

$$\overline{\mathrm{Sp}\,\hat{\Lambda}^{(2)}}=\int\mathrm{Sp}\,\hat{\Lambda}^{(2)}(\boldsymbol{\rho})\,w(\boldsymbol{\rho})\,d\boldsymbol{\rho}, \tag{7}$$

where $\omega(\boldsymbol{\rho})=\omega(|\rho|)$ is the probability that the distance between the centers of two inclusions is $\boldsymbol{\rho}$. Here

$$\int w(\boldsymbol{\rho})\,d\boldsymbol{\rho}=1. \tag{8}$$

If ρ increases without bound we have

$$\mathrm{Sp}\,\hat{\Lambda}^{(2)}(\boldsymbol{\rho})\to2\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}, \tag{9}$$

where $\hat{\Lambda}^{(1)}$ is the tensor $\hat{\Lambda}$ corresponding to an isolated inclusion. The expression for $\mathrm{Sp}\hat{\Lambda}^{(2)}$ can be written in the following form, where we are using (7)–(9):

$$\overline{\mathrm{Sp}\,\hat{\Lambda}^{(2)}}=2\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}+\overline{\mathrm{Sp}\,[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]}. \tag{10}$$

In the "normal" situation (as specified below) the second term on the right side of (10) is dominated by $\rho\sim R$, where R is the scale length of an inclusion. For a random distribution of inclusions, the probability for two particles to come this close together is proportional to the particle concentration. We can thus use (5)–(10) to rewrite (4)

$$(\sigma_e-\sigma_1)\langle\mathbf{E}\rangle$$
$$=\sigma_1\frac{4\pi}{3}\left\{N\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}+\frac{1}{2}N^2\,\overline{\mathrm{Sp}\left(\frac{1}{N}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\right)}\right\}\mathbf{E}_0, \tag{11}$$

where $N=\mathcal{N}/V$ is the number density of the inclusions. In accordance with the comment above, we note that the second term on the right side of (11) is quadratic in N.

To calculate $\langle\mathbf{E}\rangle$ it is sufficient to work only to first order in the concentration. Taking an average of $\mathbf{E}(\mathbf{r})-\mathbf{E}_0$ over the entire volume V, we find

$$\langle\mathbf{E}\rangle-\mathbf{E}_0=\frac{1}{V}\sum_a\int_{V_a}(\mathbf{E}-\mathbf{E}_0)\,dV, \tag{12}$$

where the summation runs over the individual inclusions. The integration on the right side of (12) is extended to the volume V_a of a sphere surrounding the ath inclusion. To first order in N, the radius of this sphere can be assumed arbitrarily large. The integral on the right side of (12) transforms into a surface integral, which we can evaluate with the help of expression (1). As a result we find

$$\int_{V_a} (\mathbf{E}-\mathbf{E}_0)\,dV=-\frac{4\pi}{3}\mathbf{p}_a^{(1)}, \tag{13}$$

where $\mathbf{p}_a^{(1)}$ is the dipole moment of an individual inclusion. Substituting (13) into (12), using (2), and taking an average over the angles, we find an expression for $\langle\mathbf{E}\rangle$ in the approximation linear in N:

$$\langle\mathbf{E}\rangle=\{1-{}^4/_9\pi N\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}\}\mathbf{E}_0. \tag{14}$$

Substitution of (14) into (11) gives us our final expression for the effective conductivity σ_e to second order in the concentration:

$$\sigma_e=\sigma_1\Big\{1+\frac{4\pi}{3}N\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}$$

$$+\frac{4\pi}{3}N^2\Big[\frac{1}{2}\overline{\mathrm{Sp}\Big(\frac{1}{N}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\Big)}+\frac{4\pi}{9}(\mathrm{Sp}\,\hat{\Lambda}^{(1)})^2\Big]\Big\}. \tag{15}$$

Introducing the polarizabilities $\hat{\Lambda}^{(1)}=v\hat{a}^{(1)}$ and $\hat{\Lambda}^{(2)}=2v\hat{a}^{(2)}$, we can rewrite (15) as

$$\sigma_e=\sigma_1\Big\{1+\frac{4\pi}{3}c\,\mathrm{Sp}\,\hat{\alpha}^{(1)}$$

$$+\frac{4\pi}{3}c^2\Big[\overline{\mathrm{Sp}\Big(\frac{1}{c}[\hat{\alpha}^{(2)}-\hat{\alpha}^{(1)}]\Big)}+\frac{4\pi}{9}(\mathrm{Sp}\,\hat{\alpha}^{(1)})^2\Big]\Big\}, \tag{16}$$

where $c=vN$ is the dimensionless concentration of inclusions (the fraction of the volume which they occupy). To first order in the concentration, expressions (15) and (16) are the same as the corresponding expressions of Ref. 2.

To calculate the second-order correction explicitly, we must specify the distribution function $w(\boldsymbol{\rho})$ [see (7)]. Let us assume that the centers of the inclusions are distributed at random (a Poisson distribution). In this case we have[7]

$$w(\boldsymbol{\rho})=N\exp\{-NV(\boldsymbol{\rho})\},\qquad V(\boldsymbol{\rho})={}^4/_3\pi\rho^3. \tag{17}$$

For "rigid" inclusions, which cannot overlap, the function $w(\boldsymbol{\rho})$ is generally different from a Poisson distribution). On the other hand, if $c=nV\ll1$, the deviation is only slight[7] and should be considered in approximations of higher order in c. We can therefore use expression (17) as $w(\boldsymbol{\rho})$ in (7) in the c^2 approximation. The integral in (7) is evaluated under the condition $\rho\geqslant\rho$, where ρ_0 is the minimum distance to which two inclusions can approach each other (for example, we would have $\rho_0=2R$ for spherical inclusions of radius R). We also note that in the normal situation (again, as specified below) the quantity $\mathrm{Sp}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]$ falls off quite rapidly with increasing ρ, so that, to the accuracy of this treatment, the exponential function in (17) can be replaced by unity. We can then write

$$\overline{\mathrm{Sp}\Big(\frac{1}{N}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\Big)}=\int_{\rho\geqslant\rho_0}\mathrm{Sp}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\,d\boldsymbol{\rho}. \tag{18}$$

The condition for the applicability of the c^2 approximation is that the values of ρ which are important in the integral in (18) be small in comparison with the average distance between inclusions, $\sim N^{-1/3}$. In this case, we can ignore the effects of other inclusions on the pair under consideration. We describe as the "normal situation" that in which the integral in (18) converges for $\rho\sim R$, so that the inequality $\rho\ll N^{-1/3}$ holds for $c\ll1$. Thus in the normal situation, the condition for the applicability of the quadratic approximation (or the linear approximation) is that the concentration be low: $c\ll1$. As we will see in Section 4, there can also be an "anomalous situation," in which distances $\rho\gg R$ are important in integral (18). In this case the condition for the applicability of a group expansion is slightly different [see (43), for example].

At small values of the difference $\sigma_1-\sigma_2$, expressions (15) and (16) are the same as the expression for the conductivity of a slightly inhomogeneous medium,[1] to an accuracy suitable for the present purposes. According to the results of Appendix 1, including terms quadratic in $\sigma_1-\sigma_2$ for an object of arbitrary shape (and of conductivity σ_2 and v) we can write

$$\mathrm{Sp}\,\hat{\Lambda}=-\frac{v}{4\pi}\Big\{3\frac{\sigma_1-\sigma_2}{\sigma_1}+\Big(\frac{\sigma_1-\sigma_2}{\sigma_1}\Big)^2\Big\}. \tag{19}$$

Expression (19) also applies to a multiply connected "object," i.e., to an arbitrary number of inclusions. In this approximation in the difference $\sigma_1-\sigma_2$, the quantity $\mathrm{Sp}\hat{\Lambda}^{(2)}$ does not depend on the distance between inclusions, so that we can write $\mathrm{Sp}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\equiv0$. For the same reason, there are no corrections to (15) and (16) in the higher-order approximations in the concentration ($\sim c^3$, etc.), so that in the approximation quadratic in $\sigma_1-\sigma_2$ expressions (15) and (16) are applicable at arbitrary values of c. Substitution of (19) into (15) leads to an expression for σ_e which holds to within terms $\sim(\sigma_1-\sigma_2)^2$

$$\sigma_e=\sigma_1\Big\{1-c\frac{\sigma_1-\sigma_2}{\sigma_1}-\frac{1}{3}c(1-c)\Big(\frac{\sigma_1-\sigma_2}{\sigma_1}\Big)^2\Big\}. \tag{20}$$

It is not difficult to see that expression (20) is an expanded form of the standard expression for the conductivity of a slightly inhomogeneous medium[1]:

$$\sigma_e=\langle\sigma\rangle\Big\{1-\frac{1}{D}\frac{\langle(\sigma-\langle\sigma\rangle)^2\rangle}{\langle\sigma\rangle^2}\Big\}, \tag{21}$$

where D is the dimensionality of the space. For a binary system we would have

$$\langle\sigma\rangle=p\sigma_1+(1-p)\sigma_2,\qquad\langle(\sigma-\langle\sigma\rangle)^2\rangle=p(1-p)(\sigma_1-\sigma_2)^2$$

($p=1-c$ is the concentration of the first component), so that we find (20) from (21) for the case $D=3$. The reason for the universal applicability of (20) and (21), which we mentioned in the Introduction, is that $\mathrm{Sp}\hat{\Lambda}^{(n)}$ does not depend on the relative positions of the inclusions to second order in $\sigma_1-\sigma_2$.

For the two-dimensional case, an expansion in the concentration can be constructed in an analogous way. We write the asymptotic expression for the potential in the form

$$\varphi=-\mathbf{E}_0\mathbf{r}+\mathbf{pr}/r^2+\ldots, \tag{22}$$

where the relationship between $\mathbf{p}$ and the field $\mathbf{E}_0$ is given by (2). To second order in the inclusion concentration, the expression for the effective conductivity σ_e in the two-dimensional case is

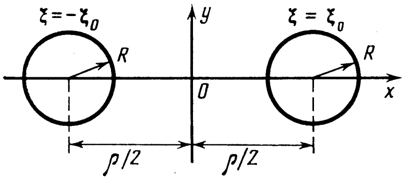

FIG. 1.

$$\sigma_e=\sigma_1\left\{1+\pi N\,\mathrm{Sp}\,\hat{\Lambda}^{(1)}+\pi N^2\left[\frac{1}{2}\overline{\mathrm{Sp}\left(\frac{1}{N}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]\right)}+\frac{\pi}{2}(\mathrm{Sp}\,\hat{\Lambda}^{(1)})^2\right]\right\}. \quad (23)$$

For rigid (nonoverlapping) inclusions, in evaluating the average in (23) we should use expression (18), where the integration in the two-dimensional case is carried out over the area. The condition for the applicability of (23) is that the corresponding integral for $\rho \ll N^{-1/2}$ must converge; i.e., the condition $c \ll 1$ must hold in the normal situation, while an inequality such as (43) below must hold in the anomalous situation. For slightly inhomogeneous media, expression (23) with (A1.9) is the same as (21) with $D = 2$.

3. CONDUCTIVITY OF A TWO-DIMENSIONAL MODEL

Let us consider a two-dimensional system with a low concentration of circular inclusions (of radius R). According to the results of the preceding section, we must find the polarizability of a pair of circles (or circular cylinders) in order to calculate the effective electrical conductivity of such a medium in the c^2 approximation. This problem can be solved exactly in bipolar coordinates.[8]

We choose the coordinate system x, y shown in Fig. 1. The bipolar coordinates θ, ξ are introduced by means of the relations[8]

$$x=\frac{a\,\mathrm{sh}\,\xi}{\mathrm{ch}\,\xi+\cos\theta},\quad y=\frac{a\sin\theta}{\mathrm{ch}\,\xi+\cos\theta}. \quad (24)$$

The curve $\xi = \mathrm{const}$ is a circle of radius $a/\mathrm{sh}\xi$ centered at the point $x = a\mathrm{cth}\xi$, $y = 0$. The boundaries of the inclusions thus correspond to $\xi = \xi_0$ (for the circle on the right) and $\xi = -\xi_0$ (for the circle on the left), where

$$\xi_0=\ln\{[\rho+(\rho^2-4R^2)^{1/2}]/2R\},\quad a={}^1/_2(\rho^2-4R^2)^{1/2}. \quad (25)$$

Solving the problem of finding the potential in the case in which a uniform field $\mathbf{E}_0$ is specified at infinity (see Appendix 2), we determine the principal values of the tensor $\hat{\Lambda}^{(2)}$:

$$\Lambda_{xx}^{(2)}=-8a^2\delta_0\sum_{n=1}^{\infty}\frac{ne^{-2n\xi_0}}{1+\delta_0e^{-2n\xi_0}},$$

$$\Lambda_{yy}^{(2)}=-8a^2\delta_0\sum_{n=1}^{\infty}\frac{ne^{-2n\xi_0}}{1-\delta_0e^{-2n\xi_0}}, \quad (26)$$

$$\delta_0=(1-h)/(1+h),\quad h=\sigma_2/\sigma_1,$$

where ξ_0 and a are given by expressions (25). In the limit $\rho\to\infty$ we have $\xi_0\to\ln(\rho/R)$, $a\to\rho/2$, so that

$$\Lambda_{xx}^{(2)}(\infty)=\Lambda_{yy}^{(2)}(\infty)=2\Lambda^{(1)}=-2R^2\delta_0, \quad (27)$$

where $\Lambda^{(1)}/\pi R^2$ is the polarizability of an isolated (individual) inclusion. Using the equality

$$\sum_{n=1}^{\infty}n\exp(-2n\xi_0)=(R/2a)^2,$$

which is easily verified, we find

$$\mathrm{Sp}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}]=-16a^2\delta_0^3\sum_{n=1}^{\infty}\frac{ne^{-6n\xi_0}}{1-\delta_0^2e^{-4n\xi_0}}. \quad (28)$$

We average (28) in accordance with (18). Here we have $d\boldsymbol{\rho} = 2\pi\rho d\rho$, $\rho_0 = 2R$. Replacing ρ by the variable

$$x=\exp\{-2\xi_0(\rho)\}=\{[\rho-(\rho^2-4R^2)^{1/2}]/2R\}^2,$$

we find from (18) and (28)

$$\overline{\mathrm{Sp}(N^{-1}[\hat{\Lambda}^{(2)}-2\hat{\Lambda}^{(1)}])}=-4\pi R^4\delta_0^3F(\delta_0), \quad (29)$$

$$F(\delta_0)=\sum_{n=1}^{\infty}n\int_0^1\frac{x^{3n-3}(1-x)^2(1-x^2)}{1-\delta_0^2x^{2n}}dx. \quad (30)$$

Using (29) and also expression (27) for $\hat{\Lambda}^{(1)}$ ($\mathrm{Sp}\hat{\Lambda}^{(1)} = -2R^2\delta_0$), we finally find from (23)

$$\sigma_e=\sigma_1\{1-2c\delta_0+2c^2\delta_0^2[1-\delta_0F(\delta_0)]\};$$
$$\delta_0=\frac{1-h}{1+h},\quad h=\sigma_2|\sigma_1. \quad (31)$$

Here $c = N\pi R^2$ is the dimensionless concentration, and the function $F(\delta_0)$ is defined in (30). Expression (31) is the expression which we have been seeking for the effective electrical conductivity of a two-dimensional system with circular rigid inclusions to second order in the concentration.

We write the effective conductivity of an isotropic binary medium as

$$\sigma_e=\sigma_e(p;\ \sigma_1,\ \sigma_2)=\sigma_1 f(p,\ h);\quad h=\sigma_2/\sigma_1, \quad (32)$$

where p is the concentration of the first component ($p = 1 - c$). In the two-dimensional case, the function f obeys the so-called reciprocity relation (Ref. 9, for example)

$$f(p,\ h)f(p,\ 1/h)=1. \quad (33)$$

It is not difficult to see that to second order in $c = 1 - p$ the function f defined by (31), (32) satisfies relation (33).

For real, positive values of σ_1 and σ_2, the parameter h lies in the interval $0\leqslant h\leqslant\infty$, which corresponds to $-1\leqslant\delta_0\leqslant 1$, i.e., $\delta_0^2\leqslant 1$. Under these conditions, the function $F(\delta_0)$ given by (30) is well-defined [expression (30) converges] and is of even parity: $F(-\delta_0) = F(\delta_0)$. As δ_0 is varied from zero to 1, the function $F(\delta_0)$ increases monotonically, and we have $F(0) = 1/3 \approx 0.33$ and $F(1) \approx 0.37$. In accordance with the results of Section 2, the condition for the applicability of expression (31) for real positive conductivities of the components is that the concentration be low, $c \ll 1$.

For insulating or dielectric (d) inclusions ($\sigma_2 = 0$), we find from (31)

$$\sigma_e^{(d)}=\sigma_1 f_d,\quad f_d=1-2c+2c^2[1-F(1)]. \quad (34)$$

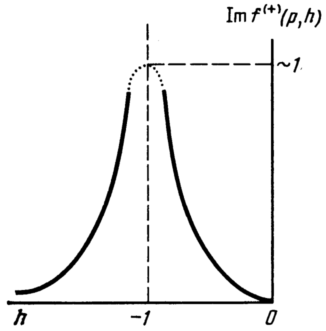

FIG. 2.

In the case of insulating inclusions, the second-order correction in the concentration is positive (in contrast with the linear correction). For ideally conducting (s) inclusions ($\sigma_2 \to \infty$), we find from (31)

$$\sigma_e^{(s)} = \sigma_1 f_s, \quad f_s = 1 + 2c + 2c^2[1 + F(1)]. \tag{35}$$

Expressions (34) and (35) hold for $c \ll 1$.

Expression (34) is the zeroth term in an expansion of the function $f(p, h)$ in powers of h at $h = 0$. By evaluating the derivatives of $F(\delta_0)$ with respect to δ_0 at $\delta_0 = 1$ ($h = 0$), we can also find the terms $\sim h$ and $\sim h^2$ of this expansion. The third derivative of F with respect to δ_0 diverges in the limit $\delta_0 \to 1$ ($h \to 0$), however, because of the presence in F of a term

$$\Delta F = 8\zeta(3) h^3 \ln h, \quad h \to 0, \tag{36}$$

where $\zeta(3) = 1.202$ ... is the Riemann ζ function. Consequently, and in agreement with Ref. 4, the function f given by (31) cannot be expanded in a converging series near the point $h = 0$.

Taking a similar approach, we can determine the effective conductivity tensor of this two-dimensional system in a transverse magnetic field. We will not reproduce the rather lengthy results here, but we do note that these results can be derived from the general expressions for the effective galvanomagnetic characteristics of two-dimensional binary media (isomorphism relations)[10] through the use of the function f in (31).

4. ANALYTIC PROPERTIES OF THE CONDUCTIVITY

Let us examine the analytic properties of the function $f(p, h)$, defined in accordance with (31) and (32), in the plane of the complex argument h (we use z to denote complex h). To first order in c we can write

$$f(p, z) = 1 - 2c(1 - z)/(1 + z), \tag{37}$$

and the only singularity of f is a pole at $z = -1$ (its meaning will be discussed below). In the c^2 approximation the analytic properties of $f(p, z)$ are determined by the properties of the function F. It follows from (30) that the function F is defined for all z except real negative values ($\mathrm{Im} z = 0$, $\mathrm{Re} z \leqslant 0$, corresponding to $\delta_0^2 \geqslant 1$), for which expression (30) becomes meaningless. In the limit $z \to 0$ there is, according to (36), a nonanalytic term in the expansion of f,

$$\Delta f = -16\zeta(3) c^2 z^3 \ln z, \tag{38}$$

so that $z = 0$ is a logarithmic branch point for the function $f(p, z)$. In the limit $z \to \infty$ we have $f = f_s + \ldots$ with f_s from (35); the terms which we have not written explicitly are of order z^{-1} or higher. Furthermore, the expansion of f in the limit $z \to \infty$ contains a nonanalytic term which is found from (38) by changing the sign and making the substitution $z \to z^{-1}$. An infinitely remote point is therefore also a logarithmic branch point for $f(p, z)$. In accordance with Ref. 4, the function $f(p, z)$ in (31) is therefore analytic in the entire z plane, cut along the negative real semiaxis. The point $z = 0$ (like $z = \infty$) is singular for $f(p, z)$, and the singularity in f at $z = 0$ is "weak."

As was shown in Ref. 4, the function $f(p, z)$ can be defined at any point in the z plane by means of a dispersion relation if the imaginary part of f on the cut is known. It is thus natural to focus on an analysis of Imf. To first order in c, with $z = -t + i\delta$ ($t > 0, \delta \to +0$), we find from (37)

$$\mathrm{Im}\, f = 4\pi c \delta(t - 1). \tag{39}$$

Switching to the c^2 approximation causes "blurring" of the δ-function in (39) because of the interaction of inclusions. Making the replacement $h \to z$ in (30) and (31), and setting $z = -t + i\delta$, we find ($t \neq 1$)

$$\mathrm{Im}\, f^{(+)}(p, -t) = \pi c^2 \sum_{n=1}^{\infty} (1 + \varepsilon^{1/n})(1 - \varepsilon^{1/n})^3 \varepsilon^{-2/n}; \quad \varepsilon = \left| \frac{1-t}{1+t} \right|. \tag{40}$$

The superscript plus sign on f means that the imaginary part of f is taken on the upper edge of the cut. It follows from (40) that Im$f^{(+)}$ is nonzero for all $0 < t < \infty$, i.e., on the entire negative real semiaxis in the z plane.

In the limit $t \to 0$ we have $\varepsilon \approx 1 - 2t \to 1$, so that $1 - \varepsilon^{1/n} \approx 2t/n$. As a result, for $t \ll 1$ we find from (40)

$$\mathrm{Im}\, f^{(+)}(p, -t) = 16\pi\zeta(3) c^2 t^3, \quad t \ll 1. \tag{41}$$

Expression (41) also follows from (38) if we note that we have $z = t\exp(i\pi)$ in this case. Similarly, for $t \gg 1$ we find an expression for Im$f^{(+)}$ which differs from (41) by the replacement $t \to 1/t$.

According to (40), the imaginary part of f increases sharply in the limit $t \to 1$ ($\varepsilon \to 0$):

$$\mathrm{Im}\, f^{(+)}(p, -t) \approx \pi c^2/\varepsilon^2, \quad c \ll \varepsilon \ll 1, \tag{42}$$

where ε is the same as in (40). The origin of the inequality $\varepsilon \gg c$ in (42) is as follows: The case $\varepsilon \to 0$ ($t \to 1$) is anomalous (Section 2) in the sense that large distances, $\rho \gg R$, are important in the integral in (18). As follows from (26), this is because the asymptotic behavior in (9) or (27) is reached (for $n = 1$) at $\rho \gg R/\varepsilon^{1/2}$. This inequality does not contradict the restriction $\rho \ll N^{-1/2}$ (the c^2 approximation is valid under this restriction) if $\varepsilon \gg c$, and the latter is the condition for the applicability of (31) and thus of (40) and (42). The point $t = 1$ ($z = -1$) is therefore the only "dangerous" point in this model. Consequently, for this system, with complex z, the c^2 approximation is valid if

$$c \ll |(1+z)/(1-z)|. \quad (43)$$

This inequality is also the applicability condition for the linear approximation (37) in the concentration.

At the boundary of the range of applicability of (42) ($\varepsilon \sim c$), the quantity $\mathrm{Im} f^{(+)}$ becomes on the order of unity. This is apparently its maximum value; at any rate, it can be asserted that at $t=1$ the imaginary part of $f^{(+)}$ is less than unity the reciprocity relation (33), which also holds at complex h, takes the following form on the cut ($h=-t+i\delta$):

$$f^{(+)}(p,-t)f^{(-)}(p,-1/t)=1.$$

(The superscript minus sign means that the function f is taken on the lower edge of the cut.) Accordingly, using[4]

$$\mathrm{Re}\, f^{(-)}=\mathrm{Re}\, f^{(+)}, \qquad \mathrm{Im}\, f^{(-)}=-\mathrm{Im}\, f^{(+)} \quad (44)$$

at $t=1$, we have

$$[\mathrm{Re}\, f^{(+)}(p,-1)]^2+[\mathrm{Im}\, f^{(+)}(p,-1)]^2=1,$$

and thus

$$\mathrm{Im}\, f^{(+)}(p,-1)\leqslant 1.$$

An analogous inequality holds for the quantity $|\mathrm{Re} F^{(+)}(p,-1)|$.

To analyze the form of $\mathrm{Im} f^{(+)}$ for $\varepsilon < c$ ($|t-1|<c$) would be to go beyond the scope of the c^2 approximation. Putting aside this narrow interval of t, we see that at $t\sim 1$ the imaginary part of $f^{(+)}$ has a sharp peak of height ~ 1 and width $\sim c$. Figure 2 is a schematic plot of $\mathrm{Im} f^{(+)}$ as a function of h for $c \ll 1$.

For a physical interpretation of these results, we consider the corresponding LC model,[4] whose properties are directly related to the behavior of the function $f(p,z)$ on the cut. In a discrete problem, the LC model is a two-component lattice made up of inductive reactances (with an impedance $Z_L=-i\omega L/c^2$) and capacitive reactances ($Z_c=i/\omega C$); here ω is the frequency of the (quasisteady[1]) alternating electric field, c is the velocity of light, L is the inductance, and C is the capacitance. To study the LC model in a continuous problem, we assign the first component a conductivity $\sigma_1=Z_L^{-1}$ and the second a conductivity $\sigma_2=Z_c^{-1}$, where Z_L and Z_c are the same as above. In this case we have

$$z=h(\omega)=-\omega^2/\Omega^2, \quad \Omega=c/(LC)^{1/2}.$$

Here Ω is the Thomson frequency (the resonant frequency of the LC circuit), and ω is to be understood as the quantity $\omega+i\delta$, $\delta\to+0$. From (32) we find the effective impedance ($Z_e=\sigma_e^{-1}$) of the LC model (Ref. 4):

$$Z_e^{-1}(\omega)=i\frac{c^2}{\omega L}\mathrm{Im}\, f^{(-)}\left(p,-\frac{\omega^2}{\Omega^2}\right), \quad \omega>0. \quad (45)$$

Using (44), we find from (45)

$$\mathrm{Re}\, Z_e^{-1}(\omega)=\frac{c^2}{\omega L}\mathrm{Im}\, f^{(+)}\left(p,-\frac{\omega^2}{\Omega^2}\right), \quad \omega>0. \quad (46)$$

According to Ref. 1, the real part of the impedance, which is responsible for the energy dissipation, is nonnegative, so that it follows from (46) that

$$\mathrm{Im}\, f^{(+)}(p,-t)\geqslant 0.$$

Expression (40) obviously satisfies this condition.

As was pointed out in Refs. 4 and 11, the reason for the nonvanishing real part of Z_e (i.e., for the real energy absorption) is the existence of impurity levels (local oscillations) in the LC model; the resonant excitation of these levels leads to the energy dissipation. According to a suggestion in Ref. 4, the cut in the z plane corresponds to an impurity band in the LC model. This suggestion turns out to be correct in the two-dimensional system under consideration here.

The spectrum of local oscillations associated with a pair of circular inclusions (the conductivity of the host is $\sigma_1=Z_L^{-1}$, and that of the inclusions is $\sigma_2=Z_c^{-1}$) can also be obtained in bipolar coordinates (Appendix 2). As a result it is found that there are two sets of discrete frequencies which correspond to a pair of circular inclusions[4]:

$$\omega_{1n}^2=\Omega^2\,\mathrm{th}\, n\xi_0, \quad \omega_{2n}^2=\Omega^2\,\mathrm{cth}\, n\xi_0, \quad n=1,2,\ldots, \quad (47)$$

where ξ_0 is the same as in (25). For a fixed distance between inclusions, the frequencies in (47) form a "band" of finite size which consists of discrete levels with an accumulation point $\omega=\Omega$. Averaging over different pairs, i.e., over the distance between inclusions, causes both the individual discrete levels and the boundaries of this band to blur. In particular, if the inclusions move closer together without bound ($\rho\to 2R$), we have $\xi_0\to 0$, and the frequencies in (47) form a quasicontinuum which stretches from zero to infinity. Accordingly, for this model the impurity band includes the entire frequency range even in the c^2 approximation, so that this band corresponds to the entire real negative semiaxis in the z plane.

Using (47), we find that expressions (26) with $h=-\omega^2/\Omega^2$ take the form of spectral decompositions:

$$\Lambda_{xx}^{(2)}=-4a^2\left(1+\frac{\omega^2}{\Omega^2}\right)\sum_{n=1}^{\infty} n\frac{\omega_{2n}^2-\Omega^2}{-\omega^2+\omega_{2n}^2},$$

$$\Lambda_{yy}^{(2)}=-4a^2\left(1+\frac{\omega^2}{\Omega^2}\right)\sum_{n=1}^{\infty} n\frac{\Omega^2-\omega_{1n}^2}{-\omega^2+\omega_{1n}^2}; \quad (48)$$

i.e., the frequencies in (47) coincide with the poles of the corresponding polarizability tensor. This agreement is not fortuitous, since the local oscillations associated with a pair of circular inclusions can also be determined in a polarizability problem if we let the amplitude E_0 of the external field tend toward zero. Analogously, the frequencies of uniform local oscillations (excited by a uniform external electric field) which are coupled with an arbitrary (possibly multiple connected) inclusion can be found as the poles of its polarizability.

In the limit $\rho\to\infty$ we have $\xi_0\to\infty$, and the frequencies in (47) approach a limit ω_0, where

$$\omega_0=\Omega \quad (49)$$

is the frequency of a local oscillation associated with an isolated circular inclusion. The pole in expression (37) thus corresponds to an impurity level with a frequency (49). The width of this level is zero, and its contribution to $\mathrm{Im} f$ is a δ-function in the linear approximation [see (39)]. When the interaction of two inclusions is taken into account, the level (49) splits, forming a set of frequencies (47); the result in the c^2 approximation is a concentration broadening of the original local level [see (40)]. This broadening may also be thought of as a "spreading" of local level (49) into an impurity band. There is an obvious similarity between this problem and that of the spectrum of a particle in a random potential.[12] The methods of Ref. 12 can also be used to study the spectral properties of the LC model.

The two-dimensional system under consideration here is interesting since it is an exactly solvable (in the c^2 approximation) model. A study of its properties constitutes a test of the basic conclusions of Ref. 4 and yields a general picture of the imaginary part of the function $f(p, z)$ on the cut in the case $c \ll 1$. Let us briefly examine the changes which we can expect in the results when we take up other models.

The concentration broadening of an impurity level in the region $c \ll \varepsilon \ll 1$ [see (42)] is determined by distances which are large in comparison with the typical sizes of the inclusions, so that an expression like (42) apparently holds for all two-dimensional models, including lattice models. On the other hand, the positions of the impurity levels corresponding to isolated defects and also the number of these levels depend on the shape of the inclusions. If, for example, the inclusion is elliptical with semiaxes a and b, then two local frequencies, $\omega_1 = \Omega(a/b)^{1/2}$ and $\omega_2 = \Omega(b/a)^{1/2}$, will be associated with it. In this case, at $c \ll 1$, there will be two peaks in the imaginary part of f.

The conclusion that $\operatorname{Im} f(p, -t) \neq 0$, holds for all $0 < t < \infty$ in the c^2 approximation appears to be valid for all disordered continuous models. The particular form of the singularity in the function $f(p, z)$ near $z = 0$, however, depends on the shape of the inclusions in the "contact" region; i.e., this singularity is not universal. Behavior of the type (38) (at $c \ll 1$) should be expected for disordered two-dimensional systems with convex smooth inclusions. To determine the nature of the singular part of the function $f(p, z)$ at concentrations which are not low (in particular, near the percolation threshold[5]) we need to determine the typical form of the contact in this case.

For lattice models, the spectrum of a pair of defects has both upper and lower limits. In the approximations of higher order in c the interval of "allowed" impurity frequencies broadens. Very low frequencies appear in the spectrum in the higher-order approximations in the concentration (the lowest frequencies are those of "dislocations," defect bonds arranged in a straight line[4]), so that at $c \ll 1$ and in the limit $\omega \to 0$ the quantity $\operatorname{Im} f$ is apparently exponentially small (cf. Ref. 12).

It is possible that the basic features of two-dimensional models will be retained in a qualitative way in three-dimensional systems.

APPENDIX 1

We assume an inclusion of conductivity σ_2, of arbitrary shape, in an isotropic host of conductivity σ_1. We wish to find the polarizability of this inclusion under the condition $|\sigma_1 - \sigma_2| \ll \sigma_1$ with an accuracy to terms $\sim (\sigma_1 - \sigma_2)^2$ inclusively. We begin with an equation for the potential:

$$\sigma_1 \nabla^2 \varphi(\mathbf{r}) = \frac{\partial}{\partial x_\alpha}\left\{\delta\sigma(\mathbf{r}) \frac{\partial \varphi(\mathbf{r})}{\partial x_\alpha}\right\}, \quad \delta\sigma(\mathbf{r}) = \sigma_1 - \sigma(\mathbf{r}). \quad (A1.1)$$

The inhomogeneity due to the presence of the inclusion is on the right side of (A1.1). Using the well-known identity

$$\nabla^2 |\mathbf{r}-\mathbf{r}'|^{-1} = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \quad (A1.2)$$

we can write Eq. (A1.1) in integral form:

$$\varphi(\mathbf{r}) = -\mathbf{E}_0\mathbf{r} - \frac{1}{4\pi\sigma_1}\int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial x_\alpha'}\left\{\delta\sigma(\mathbf{r}') \frac{\partial \varphi(\mathbf{r}')}{\partial x_\alpha'}\right\}. \quad (A1.3)$$

The first term on the right here is the potential of a uniform external field $\mathbf{E}_0$; the second describes the distortion of the potential by the inclusion. Integrating by parts, we can put (A1.3) in the form

$$\varphi(\mathbf{r}) = -\mathbf{E}_0\mathbf{r} + \frac{1}{4\pi\sigma_1}\int d\mathbf{r}'\delta\sigma(\mathbf{r}') \frac{\partial \varphi(\mathbf{r}')}{\partial x_\alpha'}\frac{\partial}{\partial x_\alpha'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \quad (A1.4)$$

We solve Eq. (A1.4) by an iterative procedure, expanding in powers of the quantity $\delta\sigma$:

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \dots . \quad (A1.5)$$

Substituting (A1.5) into (A1.4), we find

$$\varphi^{(0)}(\mathbf{r}) = -\mathbf{E}_0\mathbf{r}, \quad \varphi^{(1)}(\mathbf{r}) = -\frac{E_{0\alpha}}{4\pi\sigma_1}\int d\mathbf{r}'\delta\sigma(\mathbf{r}')\frac{\partial}{\partial x_\alpha'}\frac{1}{|\mathbf{r}-\mathbf{r}'|},$$
$$\varphi^{(2)}(\mathbf{r}) = \frac{E_{0\beta}}{(4\pi\sigma_1)^2}\iint d\mathbf{r}'\, d\mathbf{r}''\left\{\delta\sigma(\mathbf{r}')\delta\sigma(\mathbf{r}'')\frac{\partial^2}{\partial x_\alpha'\,\partial x_\beta'}\frac{1}{|\mathbf{r}'-\mathbf{r}''|}\right\}\frac{\partial}{\partial x_\alpha'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \quad (A1.6)$$

Finding an asymptotic expression for the potential in (A1.5), (A1.6) by means of the equality ($r \to \infty$)

$$1/|\mathbf{r}-\mathbf{r}'| = 1/r + \mathbf{r}\mathbf{r}'/r^3 + \dots,$$

and comparing the result with (1) and (2), we can find the tensor Λ to second order in $\delta\sigma$:

$$\Lambda_{\alpha\beta} = -\frac{\delta_{\alpha\beta}}{4\pi\sigma_1}\int d\mathbf{r}'\delta\sigma(\mathbf{r}') + \frac{1}{(4\pi\sigma_1)^2}\iint d\mathbf{r}'\, d\mathbf{r}''\,\delta\sigma(\mathbf{r}')\delta\sigma(\mathbf{r}'')\frac{\partial^2}{\partial x_\alpha'\partial x_\beta'}\frac{1}{|\mathbf{r}'-\mathbf{r}''|}. \quad (A1.7)$$

Using identity (A1.2), we thus find

$$\operatorname{Sp}\hat{\Lambda} = -\frac{1}{4\pi\sigma_1}\left\{3\int\delta\sigma(\mathbf{r})\,d\mathbf{r} + \frac{1}{\sigma_1}\int[\delta\sigma(\mathbf{r})]^2 d\mathbf{r}\right\}. \quad (A1.8)$$

For an inclusion of conductivity σ_2 we find expression (19) from (A1.8).

In an analogous way, we find the following result in the two-dimensional case:

$$\operatorname{Sp}\hat{\Lambda} = -\frac{1}{2\pi\sigma_1}\left\{2\int\delta\sigma(\mathbf{r})\,d\mathbf{r} + \frac{1}{\sigma_1}\int[\delta\sigma(\mathbf{r})]^2 d\mathbf{r}\right\}. \quad (A1.9)$$

APPENDIX 2

It is convenient to introduce the bipolar coordinates ξ, θ by means of the relation[8]

$$\xi+i\theta=\ln[(a+z)/(a-z)], \quad z=x+iy, \tag{A2.1}$$

from which we find expressions (24) for x and y. The parameter a is given in (25) for the case in Fig. 1. The half-plane $x>0$ corresponds to $\xi>0$, while $x<0$ corresponds to $\xi<0$. For x and y we will therefore use some expansions[8] which follow from (A2.1):

$$x+iy=\begin{cases} a+2a\sum_{n=1}^{\infty}(-1)^n e^{-n\xi-in\theta}, & \xi>0, \\ -a-2a\sum_{n=1}^{\infty}(-1)^n e^{n\xi+in\theta}, & \xi<0. \end{cases} \tag{A2.2}$$

We assume that a uniform electric field $\mathbf{E}_0$ with a potential

$$\varphi_0=-\mathbf{E}_0\mathbf{r}=-E_0(x\cos\alpha+y\sin\alpha) \tag{A2.3}$$

is given far from the inclusions (in the host of conductivity σ_1). Using (A2.2), we find that the potential φ_0 becomes

$$\varphi_0=\begin{cases} -E_0\left[a\cos\alpha+2a\sum_{n=1}^{\infty}(-1)^n e^{-n\xi}\cos(n\theta+\alpha)\right], & \xi>0, \\ E_0\left[a\cos\alpha+2a\sum_{n=1}^{\infty}(-1)^n e^{n\xi}\cos(n\theta-\alpha)\right], & \xi<0. \end{cases} \tag{A2.4}$$

We seek the potential outside the inclusions [$|\xi|\leqslant\xi_0$, where ξ_0 is defined in (25)] in the form

$$\varphi_e=\varphi_0+\sum_{n=1}^{\infty}(a_n \operatorname{ch} n\xi\sin n\theta+b_n \operatorname{sh} n\xi\cos n\theta), \quad |\xi|\leqslant\xi_0 \tag{A2.5}$$

with φ_0 from (A2.3), (A2.4). Inside the inclusion on the right ($\xi\gtrsim\xi_0$) we have

$$\varphi_i^{(1)}=a_0^{(1)}+\sum_{n=1}^{\infty}(a_n^{(1)}e^{-n\xi}\sin n\theta+b_n^{(1)}e^{-n\xi}\cos n\theta), \quad \xi\geqslant\xi_0. \tag{A2.6}$$

Inside the inclusion on the left ($\xi\leqslant-\xi_0$) we have

$$\varphi_i^{(2)}=a_0^{(2)}+\sum_{n=1}^{\infty}(a_n^{(2)}e^{n\xi}\sin n\theta+b_n^{(2)}e^{n\xi}\cos n\theta), \quad \xi\leqslant-\xi_0. \tag{A2.7}$$

The boundary conditions on the inclusion at the right are standard: continuity of the potential and of the normal component of the current density,

$$\xi=\xi_0:\ \varphi_e=\varphi_i^{(1)}, \quad \sigma_1\,\partial\varphi_e/\partial\xi=\sigma_2\,\partial\varphi_i^{(1)}/\partial\xi, \tag{A2.8}$$

where σ_2 is the conductivity of the inclusion. Similar boundary conditions must hold at the inclusion at the left ($\xi=-\xi_0$).

Imposing the boundary conditions at $\xi=\pm\xi_0$, we find the expansion coefficients in (A2.5)–(A2.7):

$$a_0^{(2)}=-a_0^{(1)}=aE_{0x}, \quad a_n^{(2)}=a_n^{(1)}=\frac{\sigma_1}{\sigma_1-\sigma_2}a_n e^{2n\xi_0},$$

$$b_n^{(2)}=-b_n^{(1)}=-\frac{\sigma_1}{\sigma_1-\sigma_2}b_n e^{2n\xi_0},$$

$$a_n=4aE_{0y}\delta_0\frac{(-1)^n e^{-2n\xi_0}}{1-\delta_0 e^{-2n\xi_0}}, \quad b_n=-4aE_{0x}\delta_0\frac{(-1)^n e^{-2n\xi_0}}{1+\delta_0 e^{-2n\xi_0}},$$

$$E_{0x}=E_0\cos\alpha, \quad E_{0y}=E_0\sin\alpha, \quad \delta_0=\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}. \tag{A2.9}$$

Finding the asymptotic expression for φ_e from (A2.5) ($\xi\approx 2ax/r^2$ and $\theta\approx\pi-2ayr^{-2}$ in the limit $r\to\infty$), and comparing the result with (22), we find the principal values of the tensor $\hat{\Lambda}^{(2)}$ [see (26)].

In discussing the spectrum of local oscillations in the LC model, we seek the potential in the form in (A2.5)–(A2.7) with $\varphi_0=0$, $a_0^{(1)}=a_0^{(2)}=0$. Imposing the boundary conditions at $\xi=\pm\xi_0$, we find a homogeneous system of equations for the expansion coefficients in (A2.5)–(A2.7). From the condition that the system be soluble, we find a relation which leads to expressions (47) with $\sigma_2/\sigma_1=-\omega^2/\Omega^2$.

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