Stability of nonlinear periodic waves in weakly dispersive media

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The instability of one-dimensional nonlinear periodic waves with respect to two- and three-dimensional perturbations in weakly dispersive media is analyzed. The analysis is aimed toward problems of waves in magnetized plasmas and of Rossby waves in a rotating fluid. A variety of two- and three-dimensional nonlinear equations describing these waves is taken into account. Equations corresponding to the so-called isotropic, anisotropic, hybrid, and vector models are analyzed. Dispersion relations for long-wave perturbations are derived for these models, which cover most of the basic types of weakly dispersive waves in magnetized plasmas. Periodic waves are generally stable (unstable) with respect to transverse perturbations if the corresponding solitons are stable (unstable) in a given model. The stability of oblique perturbations is analyzed. Such perturbations may be unstable if transverse perturbations are stable.

I. INTRODUCTION

Periodic waves rank along with solitons as the simplest elements in the theory of nonlinear steady-state waves in dispersive media.[1,2] At a given scale length (a characteristic wavelength), the amplitude of a periodic wave may be much lower than that of a soliton. Experimentally, therefore, periodic waves are generally more common than solitons. The observation of periodic waves has been reported in many studies of plasma confinement in a magnetic field. For example, periodic drift waves have been observed in Q machines (laboratory devices with an alkali-metal plasma), as described in the review by Buchel'nikova.[3]

In the present paper we are interested in the stability of nonlinear periodic waves. Specifically, we will discuss waves with a weak dispersion ("weakly dispersive waves" or, in the conventional terminology,[2] "waves in weakly dispersive media"). In a plasma, the weakly dispersive waves might be the many varieties of gradient waves, including the drift waves mentioned above and also ion acoustic and magnetosonic waves. Also weakly dispersive are long Rossby waves in a rotating fluid; these waves are analogous to drift waves in a plasma. Our purpose here is to determine the stability of these and similar waves under the assumption that the waves are one-dimensional, while the perturbations which arise are two- or three-dimensional. In this sense our approach is similar to that of Refs. 4 and 5, with the distinction that those other papers dealt with solitons, while here we are concerned with periodic waves.

To simplify the discussion we analyze some model equations, and we then determine which physical problems we can study with these equations. The nonlinear equations which we discuss here are a set of two- and three-dimensional generalizations of the Korteweg-de Vries equations. We put the equations in two groups: those which do not contain a vector nonlinearity (the so-called scalar models) and those which do contain a vector nonlinearity (vector models). We write the equations of the scalar models in the general form

$$\partial u/\partial t+u\partial u/\partial x+\partial^3 u/\partial x^3=R. \tag{1.1}$$

The function $u=u(\mathbf{r},t)$ characterizes the wave field, and $R=R[u]$ is some linear functional of u. We assume that the coordinate and the time, like the function u itself, are normalized in some appropriate way.

In the case of the two-dimensional model of Kadomtsev and Petviashvili,[6] we would have $\mathbf{r}=(x,y)$; R can be taken to be the expression

$$R=\sigma\partial v_y/\partial y, \tag{1.2}$$

where the function v_y satisfies the equation

$$\partial v_y/\partial x=\partial u/\partial y; \tag{1.3}$$

and the quantity $\sigma=\pm 1$ gives the sign of the dispersion ($\sigma=1$ for waves with a positive dispersion and $\sigma=-1$ for waves with a negative dispersion). The generalization of Eqs. (1.2) and (1.3) to the case of a three-dimensional model of the Kadomtsev-Petviashvili type are equations of the type

$$R=\sigma \operatorname{div} \mathbf{v}_\perp, \tag{1.4}$$

$$\partial \mathbf{v}_\perp/\partial x=\nabla_\perp u. \tag{1.5}$$

Equation (1.1) with R as in (1.2) or (1.4) applies primarily to media with isotropic wave properties.[6] In this sense, these equations correspond to isotropic models (two- or three-dimensional). For very anisotropic media, e.g., a magnetized plasma or a rotating fluid, we must also deal with two-dimensional equations of the type (1.1), with

$$R=\sigma\partial^3 u/\partial x\partial y^2. \tag{1.6}$$

Here, as in (1.3), we have $\sigma=\pm 1$, but the relationship between this quantity and the sign of the wave dispersion is generally not single-valued (Section 6 below). In problems involving magnetized plasmas, we also run into three-dimensional generalizations of Eqs. (1.1) and (1.6). Here we have

$$R=\sigma\Delta_\perp\partial u/\partial x, \tag{1.7}$$

where $\Delta_\perp=\partial^2/\partial y^2+\partial^2/\partial z^2$. We will call the two- and three-dimensional models described by Eqs. (1.1), (1.6),

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and (1.7) "anisotropic models." Equations of the type in (1.1) and (1.7) were first derived by Zakharov and Kuznetsov.[7]

Also of substantial interest for the case of a magnetized plasma is the so-called three-dimensional hybrid model, in which R is a combination of terms of the type in (1.2) and (1.6), i.e.,

$$R=\sigma_1\partial v_z/\partial z+\sigma_2\partial^3u/\partial x\partial y^2, \quad (1.8)$$

where v_z satisfies a relation like (1.3),

$$\partial v_z/\partial x=\partial u/\partial z, \quad (1.9)$$

and $\sigma_1 = \pm 1$, $\sigma_2 = \pm 1$.

All of the scalar models listed here can be analyzed by a common approach in a hybrid model, with subsequent changes in notation, if it is assumed that σ can take on the value 0 in addition to the values ± 1. In this approach, a two-dimensional isotropic model corresponds to the case $\sigma_1 = \pm 1$, $\sigma_2 = 0$, while a two-dimensional anisotropic model corresponds to the case $\sigma_1 = 0$, $\sigma_2 = \pm 1$. The transformation from these two-dimensional models to the analogous three-dimensional models is made by introducing the change of notation $\partial^2/\partial z^2 \to \nabla_\perp^2$ or $\partial^2/\partial y^2 \to \nabla_\perp^2$.

In problems involving a magnetized plasma and a rotating fluid, a two-dimensional equation with a vector nonlinearity of the type [cf. (1.1), (1.6)]

$$\partial u/\partial t+u\partial u/\partial x+\hat{D}(\partial u^2/\partial x^2+\partial^2u/\partial y^2)=0 \quad (1.10)$$

arises, where

$$\hat{D}=\partial/\partial x+\mu[\nabla u,\ \nabla]_z, \quad (1.11)$$

and μ is a constant coefficient. For a magnetized plasma we are also forced to deal with a large family of three-dimensional equations with a vector nonlinearity. The simplest of these equations is

$$\partial u/\partial t+u\partial u/\partial x+\hat{D}(\partial^2u/\partial x^2+\partial^2u/\partial y^2)=\sigma\partial v_z/\partial z, \quad (1.12)$$

where v_z satisfies

$$\hat{D}v_z=\partial u/\partial z. \quad (1.13)$$

Equations (1.10)–(1.13) correspond to so-called two- and three-dimensional vector models.

In Section 2 we outline our approach for studying the equations of the scalar models. In Section 3 we derive dispersion relations for the waves described by the scalar models, and we analyze these relations in Section 4. In Section 5 we discuss the effects which stem from a vector nonlinearity. In Section 6 we determine the relationship between these models and certain problems in the physics of magnetized plasmas. The results are summarized and discussed in Section 7.

Now we discuss the work which has been done previously on the stability of nonlinear periodic waves described by these model equations. To the best of our knowledge, there has been no previous study of the stability of the waves described by the anisotropic, hybrid, and vector models. On the other hand, the stability of periodic waves has been analyzed in the isotropic model (the Kadomtsev-Petviashvili model). This analysis was begun by Infeld *et al.*[8,9] (see also Ref. 10) and pursued by Kuznetsov *et al.*[11] This series of papers was preceded by a study of waves in the one-dimensional Korteweg–de Vries model, begun by Whitham[12] (see also Ref. 13). Theoretical research on the stability of one-dimensional solitons in this model has also influenced the development of the theory of the stability of periodic waves in the Kadomtsev-Petviashvili model. Noteworthy among these studies is the paper by Kadomtsev and Petviashvili,[6] which demonstrates that such solitons are stable with respect to two-dimensional perturbations in media with a negative dispersion, but are unstable in media with a positive dispersion.

The analysis in Refs. 8 and 9 dealt with long-wave perturbations; i.e., the physical results of those studies were derived through a series expansion in the small wave numbers of the perturbations. In this approximation, and if small terms on the order of the expansion parameter are ignored, the frequency of the perturbations is found to be proportional to the wave number, so that we can introduce the concept of a "characteristic perturbation velocity," which has the same meaning as the group velocity (or phase velocity) of waves with a linear dispersion law. Whitham[12] showed that long-wave perturbations in systems described by the Korteweg-de Vries equation have three real characteristic velocities. In other words, it follows from Ref. 12 that in such systems there are three branches of stable long-wave perturbations. Whitham[12] derived expressions for these characteristic velocities; this was equivalent to calculating the frequencies of the corresponding perturbation branches.

An important feature of the characteristic velocities which Whitham found[12] is that when we go from the case of periodic waves to the case of a chain of infinitely remote solitons two of the three velocities, which are identical in magnitude and opposite in sign (i.e., which have the same square), vanish. In this sense we can assume that one-dimensional perturbations of a soliton are characterized by two zero frequencies. We are then led to ask how these zero frequencies of a soliton will change in the case of two-dimensional perturbations. This question was answered by Kadomtsev and Petviashvili,[6] who showed that two-dimensional long-wave perturbations of a soliton are also characterized by two frequencies which have a common square, and the sign of this square is determined by the sign of the dispersion of the medium, in accordance with the discussion above. The result which Kadomtsev and Petviashvili found regarding the instability of long-wave perturbations in media with a positive dispersion (as mentioned above) means that there are two imaginary characteristic velocities. We thus see the relationship between the results of Whitham[12] and Kadomtsev and Petviashvili.[6]

Clearly, however, the situations discussed in Refs. 6 and 12 are different limiting cases of the overall problem of the two-dimensional stability of periodic waves. This was pointed out some time ago by Infeld *et al.*,[8,9] who derived a dispersion relation for long-wave perturbations, which turned out to be cubic in the perturbation frequency, as in the one-dimensional case. Analyzing this equation for several particular cases, Infeld *et al.*[8,9] concluded that periodic waves, like solitons, are stable in a medium with a negative dispersion, while they may be unstable in a medium with a positive dispersion.

In contrast with Refs. 8, 9, and 12, Kuznetsov *et al.*[11] used the inverse scattering technique. They reported that in Kadomtsev-Petviashvili systems one-dimensional periodic waves are unstable if the dispersion is positive, while they are stable with respect to transverse perturbations if the dispersion is negative. As we mentioned earlier, similar results follow for the case of long-wave perturbations from the earlier studies by Infeld *et al.*[8–10] (however, those earlier results were not discussed or even mentioned in Ref. 11).

Let us analyze Eqs. (1.1)–(1.13) above by expanding in small wave numbers. In this regard, our closest predecessors are Infeld *et al.*[8,9] It is clear from the discussion above that our analysis differs from Refs. 8 and 9 in that we are considering a broader set of models. This set also includes the Kadomtsev-Petviashvili model, studied in Refs. 8 and 9 and also in Ref. 11. We are thus in a position to test the results of Refs. 8 and 9 and to get an idea of how general the results of Refs. 8, 9, and 11 are. Our analysis provides support for the general dispersion relation derived in Refs. 8 and 9 and for the expressions considered in Refs. 8 and 9 for the perturbation frequencies in limiting cases. However, we also analyze some other types of oblique perturbations, and we find that periodic waves described by Kadomtsev-Petviashvili models may be unstable with respect to oblique perturbations in media with a negative dispersion. This result means that the conclusions which Infeld *et al.*[8,9] and Kuznetsov *et al.*[11] reached regarding the stability of such waves in media with a negative dispersion apply only in particular cases.

2. APPROACH FOR STUDYING THE NONLINEAR EQUATIONS FOR SCALAR MODELS

We assume

$$u=u_0+\tilde{u}. \tag{2.1}$$

The function u_0 describes an initial one-dimensional steady-state wave which is propagating along the x axis at a velocity v_0; i.e., we have $u_0=u_0(\xi)$, where $\xi=x-v_0t$, and the function $\tilde{u}$ characterizes a two-dimensional $\tilde{u}=\tilde{u}(\xi,y,t)$ or three-dimensional $\tilde{u}=\tilde{u}(\xi,y,z,t)$ perturbation of the wave. According to (1.1), the function u_0 satisfies the equation

$$(u_0''-v_0u_0+u_0^2/2)'=0, \tag{2.2}$$

where the prime indicates a derivative with respect to ξ. Equation (2.2) is the steady-state Korteweg-de Vries equation. It describes two physically distinct types of wave, solitons and periodic waves.[1,2] In the case of interest here, that of periodic waves, we can take u_0 to be the function[1,2]

$$u_0=b\,\mathrm{dn}_s^2(\zeta,\,s), \tag{2.3}$$

where dn is the Jacobi elliptic function of modulus s^2, $\zeta=(b/12)^{1/2}\xi$, and $b=3v_0/(2-s^2)$. We assume $v_0>0$, $b>0$, $0\leqslant s^2\leqslant 1$. According to (1.1), the perturbed function $\tilde{u}$ satisfies the equation

$$(\hat{Q}\tilde{u})'\equiv(\tilde{u}''-v_0\tilde{u}+u_0\tilde{u})'=-\partial u/\partial t+R, \tag{2.4}$$

where $\partial/\partial t\equiv(\partial/\partial t)_\xi$, $R=R[\tilde{u}]$. In accordance with the discussion in Section 1, we restrict the discussion to the case in which R is as in (1.8), and v_z is as in (1.9).

Let us specify the space-time dependence of the perturbations. We assume (cf. Refs. 8 and 9)

$$(\tilde{u},\,R,\,v_z)=(\bar{u},\,\bar{R},\,\bar{v}_z)\exp\left[-i(\Omega t-\mathbf{K}\mathbf{r})\right], \tag{2.5}$$

where $\bar{u}=\bar{u}(\xi)$, $\bar{R}=\bar{R}(\xi)$, and $\bar{v}_z=\bar{v}_z(\xi)$ are periodic functions (their period is the same as that of the initial wave); Ω is the perturbation frequency; $\mathbf{K}=(K_x,K_y,K_z)$; $\mathbf{r}=(\xi,y,z)$; K_y and K_z are the wave numbers of the perturbations along y and z; and K_x is the average longitudinal wave number of the perturbation, also known as the Bloch quasimomentum. From (2.4), (1.8), and (1.9) we then find the following equations for the functions $\bar{u}$, $\bar{R}$, and $\bar{v}_z$:

$$(\hat{Q}\bar{u})'=i(\Omega-K_x\hat{Q}+K_x^3)\bar{u}-2iK_x\bar{u}''+3K_x^2\bar{u}'+\bar{R}, \tag{2.6}$$

$$\bar{R}=i\sigma_1K_z\bar{v}_z-\sigma_2K_y^2(\bar{u}'+iK_x\bar{u}), \tag{2.7}$$

$$\bar{v}_z'+iK_xv_z=iK_z\bar{u}. \tag{2.8}$$

By analogy with Refs. 8 and 9, we assume that Ω and $\mathbf{K}$ are small parameters, and we solve Eqs. (2.6)–(2.8) by expanding in Ω and $\mathbf{K}$. We write $\bar{u}$, $\bar{R}$, and $\bar{v}_z$ in the form

$$\bar{u}=u_1+u_2+u_3+\ldots,\quad \bar{R}=R_3+\ldots,\quad \bar{v}_z=v_{z2}+v_{z3}+\ldots\,. \tag{2.9}$$

The function u_1 satisfies the relation

$$(\hat{Q}u_1)'=0. \tag{2.10}$$

In the next approximation we find from (2.6)–(2.8)

$$(\hat{Q}u_2)'=i(\Omega u_1-2K_xu_1''), \tag{2.11}$$

$$v_{z2}'=iK_zu_1. \tag{2.12}$$

Finally, in the highest-order approximation which we will consider here, we find the following system of equations for u_3, R_3, and v_{z3} from (2.6)–(2.8):

$$(\hat{Q}u_3)'=i[(\Omega-K_x\hat{Q})u_2-2K_xu_2'']+3K_x^2u_1'+R_3, \tag{2.13}$$

$$R_3=i\sigma_1K_zv_{z2}-\sigma_2K_y^2u_1', \tag{2.14}$$

$$v_{z3}'=i(K_zu_2-K_xv_{z2}). \tag{2.15}$$

All of the unknown functions u_i ($i=1,\,2,\,3$) and v_{zi} ($i=2,\,3$) are periodic in ξ. It can be seen from (2.12) that the condition that v_{z2} be periodic is equivalent to the vanishing of the average value of u_1 over the period of the initial wave:

$$\langle u_1\rangle=0. \tag{2.16}$$

Here and below, angle brackets are used to represent this average,

$$\langle\cdots\rangle\equiv\oint(\cdots)\,d\zeta/\oint d\zeta$$

and the integrals are taken over the period of the initial wave. Analogously, we find from (2.15) and the condition for the periodicity of v_{z3}

$$K_z\langle u_2\rangle-K_x\langle v_{z2}\rangle=0. \tag{2.17}$$

The functions $\hat{Q}u_i$ are also periodic. Consequently, we find from (2.13) and (2.14)

$$\Omega\langle u_2\rangle-K_x\langle\hat{Q}u_2\rangle+\sigma_1K_z\langle v_{z2}\rangle=0. \tag{2.18}$$

Yet another useful integral relation (an orthogonality condition) can be found by taking an average of (2.12) with a weight of u_0; using (2.14), we would thus have

$$i\langle u_0[(\Omega-K_x\hat{Q})u_2-2K_xu_2''+\sigma_1K_zv_{z2}]\rangle + (3K_x^2-\sigma_2K_y^2)\langle u_0u_1'\rangle=0. \tag{2.19}$$

Our problem thus reduces to one of calculating the functions u_1, u_2, and v_{z2} from Eqs. (2.10)–(2.12) under the auxiliary conditions (2.16)–(2.19).

3. DERIVATION OF A DISPERSION RELATION FOR SCALAR MODELS

Using the periodicity condition and the auxiliary condition (2.16), we find from (2.10) the following expression for u_1:

$$u_1=Au_0', \tag{3.1}$$

where A is an arbitrary constant. Substituting (3.1) into (2.11), (2.12), we find

$$(\hat{Q}u_2)'=iA(\Omega u_0'-2K_xu_0'''), \tag{3.2}$$

$$v_{z2}'=iAK_zu_0'. \tag{3.3}$$

Solving (3.2), we find the following expression for u_2:

$$u_2=iu_0'\left(\Omega\int\frac{f\,d\xi}{u_0'^2}-AK_x\xi\right). \tag{3.4}$$

Here

$$f=(Au_0^2+Bbu_0+Cb^2)/2, \tag{3.5}$$

B and C are integration constants and the integral is to be understood as a function of its upper limit. Analogously, Eq. (3.3) and the auxiliary condition (2.17) imply

$$v_{z2}=iAK_z(u_0-\langle u_0\rangle)+\langle u_2\rangle K_z/K_x. \tag{3.6}$$

Using (3.4), we find that the condition that v_{z3} be periodic implies

$$\alpha_{11}A+\Omega(\alpha_{12}B+\alpha_{13}C)=0. \tag{3.7}$$

Here

$$\alpha_{11}=\Omega I_1+2K_xv_0/(2-s^2),\quad \alpha_{12}=I_{-1},\quad \alpha_{13}=I_{-3}, \tag{3.8}$$

and the quantities I_m are defined by

$$I_m=\left\langle \mathrm{sn}^2\,\mathrm{cn}^2\frac{d}{d\,\mathrm{dn}}\left(\frac{1}{\mathrm{dn}}\frac{d\,\mathrm{dn}^m}{d\,\mathrm{dn}}\right)\right\rangle, \tag{3.9}$$

where $\mathrm{sn}=\mathrm{sn}(\zeta,s)$, $\mathrm{cn}=\mathrm{cn}(\zeta,s)$ are the Jacobi elliptic functions. Analogously, using (3.4) and (3.6), we find from (2.18)

$$\alpha_{21}A+\Omega(\alpha_{22}B+\alpha_{23}C)=0, \tag{3.10}$$

where

$$\begin{gathered}\alpha_{21}=\Lambda\Omega I_3+2\sigma_1K_z^2K_xv_0\lambda/(2-s^2),\\ \alpha_{22}=\Lambda I_1-K_x^2v_0/(2-s^2),\quad \alpha_{23}=\Lambda I_{-1},\\ \Lambda=\Omega K_x+\sigma_1K_z^2,\quad \lambda=E(s)/K(s);\end{gathered} \tag{3.11}$$

and $K(s)$ and $E(s)$ are the complete elliptic integrals of the first and second kinds. Finally, we find from (2.19)

$$\alpha_{31}A+\Omega(\alpha_{32}B+\alpha_{33}C)=0. \tag{3.12}$$

Here

$$\begin{gathered}\alpha_{31}=\Omega^2K_x[(2-s^2)I_3+2\lambda]+\Omega K_x^2\frac{v_0(1-s^2)}{2-s^2}\\ +3\sigma_1K_z^2\left(\lambda\Omega I_3+2K_x\frac{v_0\langle\mathrm{dn}^4\rangle}{2-s^2}\right)-\frac{2}{5}\sigma_2K_y^2K_x\frac{v_0^2s^4I_5}{(2-s^2)^2},\\ \alpha_{32}=\Omega K_x[1+(2-s^2)I_1]+3\sigma_1K_z^2v_0\lambda I_1,\\ \alpha_{33}=\Omega K_x(2-s^2)I_{-1}+\frac{3K_x^2v_0}{2-s^2}+3\sigma_1K_z^2\lambda I_{-1}.\end{gathered} \tag{3.13}$$

From (3.7), (3.10), and (3.12) we find the dispersion relation

$$|\alpha_{ik}|=0,\quad i,k=1,2,3. \tag{3.14}$$

Substituting α_{ik} from (3.8), (3.11), and (3.13), and carrying out some manipulations, we find the following cubic equation in Ω (cf. Ref. 8):

$$\begin{gathered}(a_3\Omega^2-d_1v_0^2\sigma_2K_y^2)(K_x\Omega+\sigma_1K_z^2)\\ +a_2\Omega K_xv_0(K_x\Omega+4\sigma_1K_z^2/3)\\ +a_0v_0^3K_x^4+(b_0\sigma_1K_z^2-d_0v_0\sigma_2K_y^2)(K_xv_0)^2\\ +c_0v_0\sigma_1^2K_z^4=0.\end{gathered} \tag{3.15}$$

Here

$$\begin{gathered}a_3=4\lambda(2-s^2)d/s^4,\quad a_2=3\lambda^2+s^4I_3^2,\quad a_0=-4(1-s^2)^2/(2-s^2)^2,\\ b_0=4\{\lambda[4\lambda(2-s^2)-5(1-s^2)]-s^4(1-s^2)I_1I_3\}/3(2-s^2),\\ c_0=4g^2/3s^4,\quad d_1=4gh/15s^4(2-s^2),\\ d_0=2h[3\lambda+I_3(2-s^2)]/15(2-s^2)^2,\end{gathered} \tag{3.16}$$

where

$$\begin{gathered}d=(1-\lambda)[\lambda-(1-s^2)],\\ g=2(2-s^2)\lambda-3\lambda^2-(1-s^2),\\ h=2(1-s^2+s^4)\lambda-(2-s^2)(1-s^2),\end{gathered} \tag{3.17}$$

and the quantities I_1 and I_3 are related to s^2 and λ by

$$I_1=[2\lambda-(1-s^2)]/s^4,\quad I_3=[\lambda(2-s^2)-2(1-s^2)]/s^4, \tag{3.18}$$

according to (3.9).

Using (3.17) and (3.18), we can also express all of the coefficients in (3.16) in terms of s^2 and λ. Using the definition of λ and the asymptotic expressions for the complete elliptic integrals $K(s)$ and $E(s)$, we find

$$\begin{gathered}\lambda\approx2\{\ln[16/(1-s^2)]\}^{-1},\quad s\to1;\\ \lambda\approx1-s^2/2-s^4/16,\quad s\to0.\end{gathered} \tag{3.19}$$

We can thus find limiting expressions for the coefficients in (3.16) in the limits $s\to1$ and $s\to0$. In the limit $s\to1$ we find

$$\begin{gathered}a_3=a_2=4\lambda^2,\quad a_0=-4(1-s^2)^2,\\ b_0=c_0=16\lambda^2/3,\quad d_1=d_0=8\lambda^2/15.\end{gathered} \tag{3.20}$$

When $s\to0$, on the other hand, we find from (3.16)

$$\begin{gathered}a_3=2,\quad a_2=3,\quad a_0=-1,\quad b_0=2,\\ c_0=3s^4/16,\quad d_1=3s^4/32,\quad d_0=15s^4/64.\end{gathered} \tag{3.21}$$

With $K_y=0$ (or $\sigma_2=0$) we find from (3.15)

$$\begin{gathered}a_3\Omega^2(K_x\Omega+\sigma_1K_z^2)+a_2\Omega K_xv_0(K_x\Omega+4\sigma_1K_z^2/3)\\ +a_0v_0^3K_x^4+b_0\sigma_1v_0^2(K_xK_z)^2+c_0v_0\sigma_1^2K_z^4=0.\end{gathered} \tag{3.22}$$

A dispersion relation of this type (with $v_0 = 1$) was derived by Infeld *et al.*[8] Although we are using the same notation (α_3, α_2, etc.) as in Ref. 8 for the coefficients in Eq. (3.22), our coefficients differ from those of Ref. 8 by numerical factors.

If we set $K_y = K_z = 0$ (or $\sigma_1 = \sigma_2 = 0$) in (3.15), i.e., we assume that the perturbations are one-dimensional, this equation reduces to

$$a_3\Omega^2+a_2v_0\Omega^2K_x+a_0v_0^3K_x^3=0. \quad (3.23)$$

Using (3.16) for the coefficients α_3, α_2, and α_0, we can put this dispersion relation in the form

$$\left[(1-\lambda)\Omega+\frac{s^2}{2-s^2}K_xv_0\right]\left\{[\lambda-(1-s^2)]\Omega+\frac{s^2(1-s^2)}{2-s^2}K_xv_0\right\}$$
$$\times\left(\lambda\Omega-\frac{1-s^2}{2-s^2}K_xv_0\right)=0, \quad (3.24)$$

in agreement with the results found by Whitham.[12] [Infeld *et al.*[8,10] also went through the transformation from (3.22) to (3.24).]

In accordance with the discussion in Section 1, dispersion relation (3.22) describes wave perturbations in an isotropic model, while Eq. (3.15) with $(\sigma_1,\sigma_2)\neq 0$ describes them in a hybrid model. If, on the other hand, we set $K_z = 0$ (or $\sigma_1 = 0$) in (3.15), we find

$$\Omega(a_3\Omega^2+a_2\Omega K_xv_0-d_1v_0^2\sigma_2K_y^2)+v_0^3K_x(a_0K_x^2-d_0\sigma_2K_y^2)=0. \quad (3.25)$$

This dispersion relation describes wave perturbations in an anisotropic model. We can also consider the case of purely transverse perturbations, i.e., perturbations for which we have $K_x = 0$. In this case, our general dispersion relation (3.15) reduces to a quadratic equation in Ω:

$$\Omega^2=-v_0(C_1\sigma_1K_z^2-v_0C_2\sigma_2K_y^2), \quad (3.26)$$

where

$$C_1=g^2/3(2-s^2)\lambda d, \quad C_2=gh/15(2-s^2)^2\lambda d. \quad (3.27)$$

The coefficients C_1 and C_2 are positive for all s^2 on the interval under consideration here, $0\leqslant s^2\leqslant 1$. In the limit $s^2\to 1$ we have

$$C_1={}^4/_3, \quad C_2={}^2/_{15}, \quad (3.28)$$

while at $s\ll 1$ we have

$$C_1=3s^4/32, \; C_2=3s^4/64. \quad (3.29)$$

Let us analyze these dispersion relations.

4. ANALYSIS OF THE STABILITY OF WAVES IN SCALAR MODELS

4.1. Longitudinal perturbations. ($K_y = K_z = 0$). According to (3.24), longitudinal perturbations of periodic waves are characterized by frequencies $\Omega = \Omega_{1,2,3}$, where (cf. Refs. 12, 8, and 9)

$$\Omega_1=\frac{1-s^2}{(2-s^2)\lambda}K_xv_0, \quad \Omega_2=-\frac{s^2(1-s^2)}{(2-s^2)[\lambda-(1-s^2)]}K_xv_0,$$
$$\Omega_3=-\frac{s^2}{(2-s^2)(1-\lambda)}K_xv_0. \quad (4.1)$$

All three roots are seen to be real, so that these waves are stable against longitudinal perturbations.

In the limiting case of highly nonlinear waves, $s\to 1$, we find from (4.1)

$$\Omega_{1,2}=\pm K_xv_0(1-s^2)/\lambda, \quad \Omega_3=-K_xv_0. \quad (4.2)$$

We see that in this limiting case the frequencies $\Omega_{1,2}$ have identical squares (cf. the discussion in Section 1 of the results of Ref. 12). These roots are small in comparison with Ω_3 and vanish at $s = 1$. We also see that in our approximation the large root Ω_3 does not depend on the parameters s and λ. It is found from small-oscillation equation (2.4) when we neglect nonlinearity and dispersion; i.e., it is found from an equation

$$\partial\tilde{u}/\partial t=v_0\tilde{u}'. \quad (4.3)$$

In the limit of slightly nonlinear waves, $s\ll 1$, we find

$$\Omega_1=K_xv_0/2, \quad \Omega_2=\Omega_3=-K_xv_0 \quad (4.4)$$

instead of (4.2) from (4.1). A distinctive feature of this case is that two of the three perturbation frequencies are equal. This property, like the properties of solutions (4.2), discussed above, will be used in our analysis of oblique perturbations in Subsection 4.3 below.

The one-dimensional stability of periodic waves described by a Korteweg–de Vries equation, which we are treating in this section of the paper, has also been discussed by Kuznetsov and Mikhailov,[14] who used the inverse scattering method. They offered a proof that such waves are stable.[14] That conclusion agrees with the results which we discussed above and which should be credited to Whitham,[13] although there is no mention of Whitham's results in Ref. 14.

4.2. Transverse perturbations. Now, in contrast with Subsection 4.1, we set $K_x = 0$, but we allow K_y, $K_z \neq 0$, so we are dealing with transverse perturbations. Such perturbations are described by dispersion relation (3.26). We know that in the limit $s\to 1$ the nonlinear periodic waves described by a function of the type in (2.3) correspond to a sequence of solitons which are spaced quite far apart.[1,2] In this case, dispersion relation (3.26) reduces to the dispersion relations of soliton perturbations, which have been analyzed previously.

For an isotropic model (for Kadomtsev-Petviashvili models) Eq. (3.26) yields

$$\Omega^2=-v_0C_1\sigma_1K_z^2. \quad (4.5)$$

According to Ref. 6, in a model of this type solitons are stable if $\sigma_1 = -1$ (in a medium with a negative dispersion), while they are unstable if $\sigma_1 = 1$ (with a positive dispersion). It can be seen from (4.5) and from the condition $C_1 > 0$, mentioned earlier, that the same comment applies in the case of periodic waves if we are speaking in terms of the transverse perturbations of these waves with which we are concerned in the present subsection. This circumstance was pointed out in Refs. 8, 9, and 11. In the limiting cases $s = 1$ and $s\ll 1$, expression (4.5) with (3.27) and (3.28) can be written

$$\Omega^2=-\frac{4}{3}v_0\sigma_1K_z^2\begin{cases}1, & s=1\\ 9s^4/128, & s\ll 1\end{cases}. \quad (4.6)$$

The upper equation in (4.6) agrees with Ref. 6. The perturbation frequency (or growth rate) is quite sensitive to the oscillatory part of the amplitude of the initial wave, which is characterized by the parameter s; the frequency decreases in proportion to s^2 in the limit $s \to 0$.

For an anisotropic model, (3.26) leads to

$$\Omega^2 = v_0^2 C_2 \sigma_2 K_y^2. \tag{4.7}$$

In a model of this type, the condition for the stability of periodic waves, like that for solitons, with respect to transverse perturbations is also determined exclusively by the sign function σ_2 (it does not depend on the magnitude of the parameter s). According to (4.7) and the condition $C_2 > 0$, when $\sigma_2 = -1$, there is an instability, while at $\sigma_2 = 1$ the solution is stable. Consequently, if σ_2 characterizes the sign of the wave dispersion (as we will see in Section 6, this may not be the case!), the stability picture is precisely the opposite of that in the case of an anisotropic model. By analogy with (4.6), we find from (4.7), (3.27), and (3.28)

$$\Omega^2 = \frac{2}{15} v_0^2 \sigma_2 K_y^2 \begin{cases} 1, & s=1 \\ 45s^4/128, & s \ll 1. \end{cases} \tag{4.8}$$

In the case $s = 1$ this result agrees with Refs. 4, 15, and 16, which dealt with the stability of solitons. We see that, as in an isotropic model, we have $(s \ll 1)\, |\Omega| \sim s^2$ for slightly nonlinear waves.

It follows from the complete equation (3.26) that in a hybrid model periodic waves are stable with respect to transverse perturbations for all s and K_z/K_y if $\sigma_1 = -1$, $\sigma_2 = 1$, while they are unstable if $\sigma_1 = 1$, $\sigma_2 = -1$. If, on the other hand, we have $\sigma_1 = \sigma_2$, the stability condition depends on both K_z/K_y and s (and also on v_0). Let us examine the case $\sigma_1 = \sigma_2 = -1$ in more detail. From (3.26) we find, in place of (4.6) and (4.8),

$$\Omega^2 = v_0 \begin{cases} \frac{4}{3}\left(K_z^2 - \frac{v_0}{10} K_y^2\right), & s=1 \\ \frac{3}{32}\left(K_z^2 - \frac{v_0}{2} K_y^2\right) s^4, & s \ll 1 \end{cases}. \tag{4.9}$$

The s dependence of the stability criterion is seen to be of only a quantitative nature. For qualitative estimates we can ignore this dependence, writing the instability condition as (cf. Ref. 16)

$$K_y \gtrsim K_z / v_0^{1/2}. \tag{4.10}$$

If $\sigma_1 = \sigma_2 = 1$, the signs of the right sides of (4.9) are switched. In this case, the instability should occur at $K_z \gtrsim K_y v_0^{1/2}$.

4.3. Oblique perturbations. Since the general dispersion relation (3.15), for oblique perturbations is quite complicated, we consider the limiting cases of these perturbations corresponding to values of the parameter s^2 in the interval $1 - s^2 \ll 1$ (perturbations of highly nonlinear waves) or $s^2 \ll 1$ (perturbations of weakly nonlinear waves).

4.3.1. Oblique perturbations of highly nonlinear waves ($1 - s^2 \ll 1$). Using (3.20), we find that at $1 - s^2 \ll 1$ dispersion relation (3.15) becomes [cf. (4.2), (4.6), (4.8), (4.9)]

$$(\Omega^2 + {}^4/_3 v_0 \sigma_1 K_z^2 - {}^2/_{15} v_0^2 \sigma_2 K_y^2)(\Omega K_x + K_x^2 v_0 + \sigma_1 K_z^2) = K_x^4 v_0^3 (1-s^2)/\lambda^2. \tag{4.11}$$

For small values of K_z and K_y, such that we have

$$K_z^2,\ K_y^2 v_0 \ll K_x^2 v_0, \tag{4.12}$$

Eq. (4.11), like the dispersion relation (3.24), for longitudinal perturbations has two small roots $\Omega_{1,2}$ and one large root Ω_3 [cf. (4.2)]. In this case the terms with K_z^2 and K_y^2 have only a slight effect on the large root Ω_3, so that Ω_3 can be assumed given to be approximately by the second relation in (4.2), while the effect of these terms on the small roots $\Omega_{1,2}$ may be extremely important. When these terms are taken into account, we find from (4.11) the following equation, instead of the first equation in (4.2):

$$\Omega_{1,2}^2 = v_0 \left[\frac{(1-s^2)^2}{\lambda^2} K_x^2 v_0 - \frac{4}{3} \sigma_1 K_z^2 + \frac{2}{15} v_0 \sigma_2 K_y^2\right]. \tag{4.13}$$

It is clear from a comparison of (4.13) with (4.6), (4.8), and (4.9) that for very nonlinear waves (4.13) may be regarded as a generalization of the expression for the square of the frequency of transverse perturbations to the case $K_x \neq 0$. In addition, Eq. (4.13) makes it possible to study the transition from Whitham's one-dimensional results[12] to the two-dimensional soliton results of Refs. 6 and 15, as we mentioned in Section 1.

It follows from (4.13) that highly nonlinear waves become unstable if

$$\frac{4}{3} \sigma_1 K_z^2 - \frac{2}{15} v_0 \sigma_2 K_y^2 > \frac{(1-s^2)^2}{\lambda^2} v_0 K_x^2. \tag{4.14}$$

The instabilities which arise under condition (4.14) are the same as those discussed in Subsection 4.2 for the case $s = 1$. In this sense, condition (4.14) may be thought of as the condition for the applicability of the approximation of purely transverse perturbations with $s = 1$. Whether the instabilities described by the models which we are discussing here will be manifested in a medium with a given dispersion relation is determined by the same factors as discussed in Subsection 4.2.

We now consider perturbations with K_z and K_y much larger than those corresponding to condition (4.12). It follows from (4.11) that the roots $\Omega_{1,2}$ are small in comparison with Ω_3, increase with increasing K_z and K_y, and may become comparable to Ω_3 at sufficiently large values of K_z and K_y. In this case, the solutions of (4.11) are found by expanding in the small parameter $(1 - s^2)^2/\lambda^2$. In lowest order, we find $\Omega = \Omega_{1,2,3}^0$, where (cf. Refs. 8 and 9)

$$(\Omega_{1,2}^{(0)})^2 = -v_0 ({}^4/_3 \sigma_1 K_z^2 - {}^2/_{15} v_0 \sigma_2 K_y^2), \tag{4.15}$$

$$\Omega_3^{(0)} = -K_x v_0 - \sigma_1 K_z^2 / K_x. \tag{4.16}$$

Let us assume that the instabilities associated with condition (4.14) do not occur, in accordance with the situation in which the left side of inequality (4.14) is negative:

$${}^4/_3 \sigma_1 K_z^2 - {}^2/_{15} v_0 \sigma_2 K_y^2 < 0. \tag{4.17}$$

According to (4.15), the roots $\Omega_{1,2}^{(0)}$ are real in this case: $(\Omega_{1,2}^{(0)})^2 > 0$. Let us also consider values of K_z and K_y such that one of the roots, say $\Omega_1^{(0)}$, is equal to the root $\Omega_3^{(0)}$; i.e., $\Omega_1^{(1)} = \Omega_3^{(0)}$. If follows from (4.15) and (4.16) that this case arises when

$$(K_x v_0+\sigma_1 K_z^2/K_x)^2=-v_0({}^4/_3\sigma_1 K_z^2-{}^2/_{15}v_0\sigma_2 K_y^2). \quad (4.18)$$

We now consider the small term on the right side of (4.11); we find that this term leads to

$$\Omega_{1,3}=\Omega_3^{(0)}+\delta, \quad (4.19)$$

where the small correction δ is given by

$$\delta^2=-(1-s^2)^2 K_x^4 v_0^3/2\lambda^2(K_x^2 v_0+\sigma_1 K_z^2). \quad (4.20)$$

In the case of an isotropic model ($\sigma_2=0$ or $K_y=0$), inequality (4.17) means that the situation corresponds to media with a negative dispersion: $\sigma_1<0$. In this case we find from (4.18) and (4.20)

$$K_x^2 v_0=3K_z^2, \quad (4.21)$$

$$\delta^2=-\frac{3}{4}\frac{(1-s^2)}{\lambda^2}K_x^2 v_0^2. \quad (4.22)$$

We see that the condition $\delta^2<0$ holds, so that highly nonlinear periodic waves in a medium with a negative dispersion, describable by a Kadomtsev-Petviashvili model, are unstable against oblique perturbations. This result is qualitatively different from that for the case of purely transverse perturbations, discussed in Subsection 4.2.

For an anisotropic model ($\sigma_1=0$ or $K_z=0$), inequality (4.18) corresponds to waves with a positive dispersion: $\sigma_2>0$. In this case we have, instead of (4.21) and (4.22)

$$K_x^2={}^2/_{15}K_y^2, \quad (4.23)$$

$$\delta^2=-\frac{1}{2}\frac{(1-s^2)^2}{\lambda^2}K_x^2 v_0^2. \quad (4.24)$$

By analogy with (4.22), we have $\delta^2<0$ again in this case. This result means that waves are unstable in media which have positive dispersion and which are describable by an anisotropic model for the oblique perturbations which we are considering here. Again, we have a qualitative difference from the properties of purely transverse perturbations.

4.3.2. Oblique perturbations of slightly nonlinear waves ($s^2\ll1$). In the case $s^2\ll1$ we find the following dispersion relation from (3.15) and (3.21):

$$(\Omega+K_x v_0)^2(\Omega K_x-{}^1/_2K_x^2v_0+\sigma_1K_z^2)$$
$$=-{}^3/_{32}s^4v_0[\sigma_1^2K_z^4-{}^1/_2\sigma_2K_y^2v_0(\Omega K_x+{}^5/_2K_x^2v_0+\sigma_1K_z^2)]. \quad (4.25)$$

In the approximation $s=0$ we then find three real roots, which are mentioned in Refs. 8 and 9 [and cf. (4.4)]:

$$\Omega_1^{(0)}=\Omega_2^{(0)}=-K_xv_0, \quad \Omega_3^{(0)}=K_x({}^1/_2v_0-\sigma_1K_z^2/K_x^2). \quad (4.26)$$

In calculating the corrections to these roots which come from the terms with s^4 on the right side of (4.25), we should distinguish between the cases with $\Omega_1^{(0)}\neq\Omega_3^{(0)}$ and $\Omega_1^{(0)}=\Omega_3^{(0)}$. Let us find the corrections to the roots $\Omega_{1,2}^{(0)}$ in the former case. Setting $\Omega_{1,2}=\Omega_{1,2}^{(0)}+\delta$, we find the following equation for δ:

$$\delta^2={}^3/_{32}s^4v_0({}^3/_2K_x^2v_0-\sigma_1K_z^2)^{-1}$$
$$\times[\sigma_1K_z^2-{}^1/_2\sigma_2K_y^2v_0({}^3/_2K_x^2v_0+\sigma_1K_z^2)]. \quad (4.27)$$

For an isotropic model, with $\sigma_2=0$ or $K_y=0$, this equation reduces to

$$\delta^2={}^3/_{32}s^4v_0K_z^2({}^3/_2K_x^2v_0-\sigma_1K_z^2)^{-1}. \quad (4.28)$$

We see that in this model the perturbations of the type under consideration here, like transverse perturbations, can grow only in media with a positive dispersion: $\sigma_1>0$. For the anisotropic model ($\sigma_1=0$ or $K_z=0$), we find from (4.27)

$$\delta^2=-({}^3/_{64})s^4v_0^2\sigma_2K_y^2. \quad (4.29)$$

This case corresponds to an instability of waves in media with $\sigma_2>0$, in qualitative distinction from the case of the transverse perturbations discussed in Subsection 4.2 but in analogy with the oblique perturbations of highly nonlinear waves which were discussed in Subsection 4.3.1.

Expression (4.27) becomes inapplicable if the condition

$$\frac{3}{2}K_x^2v_0=\sigma_1K_z^2 \quad (4.30)$$

holds, as it may in media with a positive dispersion: $\sigma_1>0$. This case corresponds to the equality of all three roots in (4.26). In place of (4.27) we then find the following result from (4.25) for the isotropic model:

$$\delta^3=-(27s^4/128)v_0^3K_x^3. \quad (4.31)$$

Comparison of (4.31) with (4.28) clearly shows that the case in which all three roots are equal corresponds to perturbations with the maximum growth rate.

5. VECTOR NONLINEARITY

Since the problem is complicated, we will consider only transverse perturbations in this section of the paper; i.e., we set $K_x=0$ (cf. Sections 2–4).

5.1. Derivation of a dispersion relation in vector models. From (1.12) and (1.13) we find the following equations to replace (2.6):

$$\hat{D}_0\hat{Q}\bar{u}=i\Omega\bar{u}+K_y^2\hat{D}_0\bar{u}+iK_z\sigma\bar{v}_z, \quad (5.1)$$

$$\hat{D}_0v_z=iK_z\bar{u}, \quad (5.2)$$

where

$$\hat{D}_0=\partial/\partial\xi+i\psi', \quad (5.3)$$

$$\psi=\varkappa(u_0-\langle u_0\rangle), \quad \varkappa=K_y\mu. \quad (5.4)$$

Using (5.1), we find that the analog of Eq. (2.10) in the case of a vector model is the equation

$$\hat{D}_0\hat{Q}u_1=0. \quad (5.5)$$

Correspondingly, in second order we have the following equations for u_2 and v_{z2} [cf. (2.11), (2.12)]:

$$\hat{D}_0\hat{Q}u_2=i\Omega\sigma u_1, \quad (5.6)$$

$$\hat{D}_0v_{z2}=iK_zu_1. \quad (5.7)$$

In third order we have the equations [cf. (2.13)–(2.15)]

$$\hat{D}_0\hat{Q}u_3=i\Omega u_2+K_y^2\hat{D}_0u_1+iK_z\sigma v_{z2}, \quad (5.8)$$

$$\hat{D}_0v_{z3}=iK_zu_2. \quad (5.9)$$

From Eqs. (5.7)–(5.9) and the conditions for the periodicity of the functions v_{z2}, v_{z3}, and u_3 we find the following integral relations, which replace (2.16)–(2.18) in the case of vector models with $K_x=0$:

$$\langle e^{i\psi}u_i\rangle=0, \quad i=1, 2, \tag{5.10}$$

$$\langle e^{i\psi}v_{z2}\rangle=0. \tag{5.11}$$

An orthogonality conditional analogous to (2.19) is found from (5.8) by taking an average of this equation with a weight $(e^{i\psi}-1)/i\varkappa$:

$$i\Omega\left\langle\frac{e^{i\psi}-1}{i\varkappa}u_2\right\rangle-K_y^2\langle u_0'u_1\rangle+iK_z\sigma\left\langle\frac{e^{i\psi}-1}{i\varkappa}v_{z2}\right\rangle=0. \tag{5.12}$$

Returning to Eq. (5.5), and using relation (5.10) with $i=1$, we conclude that expression (3.1) for the function u_1 remains in force. Solving Eq. (5.6) with (3.1) and (5.3), we find expression (3.4) for u_2, with the following function f:

$$f=b^2(Ah_1+Bh_2+C)/2. \tag{5.13}$$

Here

$$h_1=\frac{2}{b^2\varkappa^2}(1-i\psi-e^{-i\psi}), \quad h_2=\frac{1}{i\varkappa b}(1-e^{-i\psi}), \tag{5.14}$$

and B and C are integration constants, as in Section 3. By analogy with (5.7), in (5.11) we find

$$v_{z2}=iAK_z(1-\langle e^{i\psi}\rangle e^{-i\psi})/i\varkappa. \tag{5.15}$$

The condition for the periodicity of the function u_2, relation (5.10) with $i=2$, and orthogonality condition (5.12) constitute a system of equations for the constants A, B, and C which is analogous to system (3.7), (3.10), (3.12), with the following coefficients α_{ik}:

$$\begin{gathered}\alpha_{11}=\Omega\langle\langle h_1\rangle\rangle, \quad \alpha_{12}=\langle\langle h_2\rangle\rangle, \quad \alpha_{13}=I_{-3},\\ \alpha_{21}=\Omega\langle\langle h_1h_2^*\rangle\rangle, \quad \alpha_{22}=\langle\langle|h_2|^2\rangle\rangle, \quad \alpha_{23}=\alpha_{12}^*,\\ \alpha_{31}=\Omega^2I_v+\frac{4}{2-s^2}\left(\sigma qK_z^2v_0+K_y^2v_0^2\frac{h}{15(2-s^2)}\right),\\ \alpha_{32}=\Omega\alpha_{21}^*, \quad \alpha_{33}=\Omega\alpha_{11}^*.\end{gathered} \tag{5.16}$$

Here $\langle\langle\cdots\rangle\rangle$ has the meaning

$$\langle\langle X\rangle\rangle=\left\langle\mathrm{sn}^2\,\mathrm{cn}^2\frac{d}{d\,\mathrm{dn}}\frac{1}{\mathrm{dn}}\frac{d}{d\,\mathrm{dn}}\left(\frac{X}{\mathrm{dn}^3}\right)\right\rangle, \tag{5.17}$$

$$q=\langle|h_2|^2\rangle=\frac{1}{b^2\varkappa^2}(1-|\langle e^{i\psi}\rangle|^2), \tag{5.18}$$

$$I_v=\langle\langle|h_1|^2\rangle\rangle, \tag{5.19}$$

and the asterisk means complex conjugation. The notation is otherwise the same as in Sections 2–4.

The dispersion relation $\det\alpha_{ik}=0$ takes the form

$$p\Omega^2=-\frac{4}{2-s^2}\left(\sigma qK_z^2v_0+K_y^2v_0^2\frac{h}{15(2-s^2)}\right), \tag{5.20}$$

where

$$p=I_v+\frac{\alpha_{13}|\alpha_{21}|^2+\alpha_{22}|\alpha_{11}|^2-(\alpha_{11}^*\alpha_{12}\alpha_{21}+\text{c.c.}}{|\alpha_{12}|^2-\alpha_{22}\alpha_{13}}. \tag{5.21}$$

Equation (5.20) generalizes the dispersion relation for a scalar hybrid model with $K_x=0$ in (3.26) to the case of a vector nonlinearity. The transformation from (5.20) to Eq. (3.26) is made by expanding in ψ in Eqs. (5.21) and (5.18) for p and q and by then letting μ (or $\varkappa$) go to zero. On the other hand, Eq. (5.20) is a generalization of the dispersion relation for perturbations of the corresponding type of soliton, which was derived in Ref. 4. To go from (5.20) to that dispersion relation we should take the limit $s\to1$. In this case, we have $\alpha_{13}\to\infty$, α_{12} finite, while the quantities α_{11}, α_{21}, and α_{22}—like I_v—are small, on the order of reciprocal of the period of the initial wave. In this case, expression (5.21) reduces to the form

$$p=I_v. \tag{5.22}$$

In addition, we need to take the limit $s\to1$ in expression (5.19) for I_v and also in (5.18) for q and (3.17) for h. For this purpose we need to (first) make the standard substitutions

$$\mathrm{dn}\to1/\mathrm{ch}, \qquad \mathrm{cn}\to1/\mathrm{ch}, \qquad \mathrm{sn}\to\mathrm{sh}/\mathrm{ch}, \tag{5.23}$$

where ch and sh are the hyperbolic cosines and sine, and (second) switch from averages over the period ($\langle\cdots\rangle$) to an ordinary integration over the period, thereby explicitly identifying the small factor $1/L$, where

$$L=\oint d\zeta. \tag{5.24}$$

We then find

$$p=\chi_0/L, \quad q=\chi_1/L, \quad h=\chi_2/L, \tag{5.25}$$

where

$$\begin{gathered}\chi_0=\frac{4}{\varkappa^2}\int(1-\cos\psi)G(\zeta)\,d\zeta,\\ \chi_1=\frac{4}{\varkappa^2}\int(1-\cos\psi)\,d\zeta, \quad \chi_2=\frac{15}{\varkappa^2}\int\left(\frac{\partial\psi}{\partial\zeta}\right)^2d\zeta,\end{gathered} \tag{5.26}$$

$\psi=\varkappa/\mathrm{ch}^2\zeta$, and the function G is defined by

$$G(\zeta)=\frac{15}{2\,\mathrm{ch}^2\zeta}-\frac{5}{2\,\mathrm{ch}^4\zeta}-1-\frac{15\zeta\,\mathrm{sh}\,\zeta}{2\,\mathrm{ch}^5\zeta}. \tag{5.27}$$

The meaning of the quantities χ_0, χ_1, and χ_2 is analogous to that of α_{11}, $\alpha_{\parallel}^{(1)}$, and $\alpha_{\perp}$ of Ref. 4, while the function $G(\zeta)$ is the same as that introduced in Ref. 4. Using (5.25), we can reduce dispersion relation (5.20) to the form

$$\chi_0\Omega^2=-4(\sigma K_z^2v_0\chi_1+K_y^2v_0^2\chi_2), \tag{5.28}$$

which is the same, aside from changes in notation, as the corresponding particular case of Eq. (4.8) in Ref. 4.

5.2. Analysis of the dispersion relation in vector models. It was pointed out in Ref. 4 that when the vector nonlinearity is large the coefficient χ_0 becomes negative. In this case we find a stability picture which is radically different from that discussed in Section 4. In particular, in the case of transverse perturbations with $K_z=0$ we find the following dispersion relation from (5.28):

$$\Omega^2=4v_0\chi_2K_y^2/|\chi_0|. \tag{5.29}$$

Equation (5.29) is evidence of the suppression of the instability predicted by the two-dimensional anisotropic model for a large vector nonlinearity. This result was found in Ref. 4 in an analysis of the stability of solitons. Since the case $s\to1$ describes both solitons and highly nonlinear periodic waves, it is clear that such waves are stable in the limit of a large vector nonlinearity. What happens when the parameter $\varkappa$

increases for the case of slightly nonlinear waves, with $s \ll 1$? Setting $\varkappa \sim 1$, we find that under the condition $s \ll 1$ expression (5.21) for p reduces to

$$p=8/(3+2\varkappa^2), \tag{5.30}$$

while q and h are characterized by

$$q=s^4/8, \quad h=15s^4/8. \tag{5.31}$$

It follows that under the condition $\varkappa > 1$, with small values of s, such that $\varkappa s^2 < 1$ (more on this below), we find the following dispersion relation in place of (4.9):

$$\Omega^2=-v_0(\rho K_y)^2(v_0K_y^2+\sigma K_z^2)/8. \tag{5.32}$$

Here $\rho \equiv 3\mu v_0 s^2/4$ is the scale of the particle oscillations when they drift in crossed fields in the case of a plasma, or it is an equivalent length scale in the case of a rotating fluid.

With $K_z = 0$, expression (5.32) holds out to the limits of applicability of the general starting assumptions of our problem (Section 2). In the case $K_z \neq 0$, in which the contribution of q is important, we should note that, according to (5.18), under the condition $\rho K_y \gtrsim 1$ we have

$$q=[1-J_0(K_y\rho)]/\varkappa^2, \tag{5.33}$$

where J_0 is the Bessel function. In the limit $K_y\rho \gg 1$ we have $q_1 = 1/\varkappa^2$, and instead of (5.32) we find

$$\Omega^2=-v_0(v_0\rho^2K_y^4/8+\sigma K_z^2). \tag{5.34}$$

It can be seen from (5.32), (5.34) that in the case of slightly nonlinear waves ($s \ll 1$) the sign of Ω^2 does not change, as it does in the case of highly nonlinear waves [cf. (5.29)].

6. SOME PROBLEMS OF THE PHYSICS OF MAGNETIZED PLASMAS WHICH LEAD TO THE MODEL EQUATIONS DISCUSSED HERE

There are some specific wave modes in a magnetized plasma which can be described by the model equations which we have been discussing here. Using the dispersion relations found above, we can determine which of the corresponding types of periodic waves are stable, and which are unstable, with respect to the simplest case, of transverse perturbations.

6.1. Waves describable by the three-dimensional isotropic model. Although a magnetized plasma is a very anisotropic medium, there are branches of slightly dispersive waves in it which can be described by a three-dimensional isotropic model. Among these waves are low-frequency magnetosonic waves (with frequencies below the ion cyclotron frequency). In terms of physical variables, these waves are described by the equations

$$\frac{2}{c_A}\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[(1-\nu^2)\frac{c^2}{\omega_{pe}^2}\frac{\partial^2 h}{\partial x^2}-\varepsilon h+\frac{3}{2}h^2\right]$$
$$=-\left(\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right), \tag{6.1}$$
$$\frac{\partial v_y}{\partial x}=\frac{\partial h}{\partial y}, \quad \frac{\partial v_z}{\partial x}=\frac{\partial h}{\partial z}.$$

Here $h = \widetilde{B}_z/B_0$, where B_0 is the equilibrium magnetic field, assumed to be directed along z, and $\widetilde{B}_z$ is the z component of the perturbed magnetic field; $(v_y, v_z) = (v_{yi}, v_{zi})/c_A$, where v_{yi} and v_{zi} are components of the perturbed velocity of the ions; $\varepsilon = 1 - c_A^2/U^2$ is the dimensionless magnetosonic dielectric constant of the plasma; U is the wave propagation velocity along x; $\nu = \alpha(m_i/m_e)^{1/2}$, where α is the angle between the x axis and the wave propagation direction (we assume $\alpha \ll 1$); c_A and c are the Alfvén velocity and the velocity of light; ω_{pe} is the electron plasma frequency; and m_e and m_i are the electron and ion masses.

A change of variables reduces Eq. (6.1) to (1.1), (1.5). We find $\sigma = -1$ at $\nu < 1$ and $\sigma = 1$ at $\nu > 1$. It is thus clear from Sections 3 and 4 that waves with $\nu < 1$ are stable, while waves with $\nu > 1$ are unstable, with respect to transverse perturbations.

Yet another wave which can be described by a three-dimensional isotropic model is the lower-hybrid-drift wave, also called the high-frequency drift wave. Some time ago, these waves were studied in detail in the linear and quasilinear approximations in connection with the instabilities caused by a transverse current in a highly inhomogeneous plasma[17,18] (so-called lower-hybrid-drift instabilities). A nonlinear theory of lower-hybrid-drift waves has also been discussed in a recent paper.[19] The frequencies of these waves typically lie between the ion cyclotron frequency ω_{Bi} and the lower-hybrid frequency ω_{LH}, where $\omega_{LH} \equiv (m_i/m_e)^{1/2}\omega_{Bi}$. Lower-hybrid-drift waves propagate across the magnetic field, in the direction of the inhomogeneity of the plasma density, the y axis, at a velocity $U \approx \omega_{Bi}/\varkappa_n$, where $\varkappa_n = \partial \ln n_0/\partial y$ is the scale length of the gradient of the equilibrium plasma density n_0. G. D. Aburdzhaniya, V. P. Lakhin, and one of the present authors (A.B.M.) recently showed that such waves are described by the following equations in the nonlinear case:

$$\frac{1}{U}\frac{\partial \Phi}{\partial t}+\frac{\partial}{\partial x}\left(\frac{U^2}{\omega_{LH}^2}\frac{\partial^2\Phi}{\partial x^2}-\varepsilon\Phi+\frac{3}{2}\Phi^2\right)=-\left(\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right),$$
$$\frac{\partial v_y}{\partial x}=\frac{\partial \Phi}{\partial y}, \quad \frac{\partial v_z}{\partial z}=\frac{m_i}{m_e}\frac{\partial \Phi}{\partial z}. \tag{6.2}$$

Here $\Phi = e_i\varphi/m_iU^2$, where φ is the electrostatic potential of the wave field; $v_y = V_{yi}/U$; $v_z = -V_{ze}/U$, where V_{ze} is the longitudinal velocity of the electrons; $\varepsilon = 1 - \omega_{Bi}/\varkappa_n U$ is the dimensionless dielectric constant corresponding to lower-hybrid-drift waves; and e_i is the ion charge.

Equations (6.2) reduce to (1.1), (1.5) in the case $\sigma = -1$. It is thus clear, in accordance with (4.5), that nonlinear lower-hybrid-drift waves are stable.

6.2. Waves described by a three-dimensional anisotropic model. One particular wave in a magnetized plasma which can be described by a three-dimensional anisotropic model is quite well known: the ion acoustic wave, for whose analysis this model was originally proposed.[7] In terms of physical variables, the nonlinear equation for such waves is[7]

$$\frac{\partial v_z}{\partial t}+c_s\frac{\partial}{\partial z}\left[1+\frac{1}{2}(\rho_0^2+d_e^2)\Delta_\perp+\frac{d_e^2}{2}\frac{\partial^2}{\partial z^2}+\frac{v_z}{2c_s}\right]v_z=0. \tag{6.3}$$

Here v_z is the longitudinal velocity of the ions, c_s is the ion acoustic velocity, ρ_0 is the ion Larmor radius calculated from the electron temperature, and d_e is the electron Debye length. A change of variables reduces (6.3) to (1.1) with R as in (2.6) and with $\sigma = -1$. The corresponding waves are therefore unstable, as is clear from (4.7). This result was derived some time ago[15] for the case $s = 1$, i.e., the case of solitons.

In a magnetized plasma with hot ions ($T_i > T_e$, where T_i is the ion temperature) there exists a branch of electron acoustic waves.[20] The nonlinear equation for these waves is analogous to (6.3), so that one-dimensional electron acoustic waves, like ion acoustic waves, are unstable.

6.3. Waves described by a hybrid model. The stability of solitons of high-frequency magnetosonic waves propagating across the magnetic field was studied in Ref. 5. According to Ref. 5, these waves are described by the nonlinear equation

$$\frac{2}{c_A}\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(\frac{c^2}{\omega_{pe}{}^2}\frac{\partial^2 h}{\partial x^2}-\varepsilon h+\frac{3}{2}h^2\right)-\frac{c^2}{\omega_{pi}{}^2}\frac{\partial^3 h}{\partial x\partial z^2}=\frac{\partial v_y}{\partial y},$$

$$\frac{\partial v_y}{\partial x}=\frac{\partial h}{\partial y}. \tag{6.4}$$

The notation here is the same as in (6.1). Equation (6.4) reduces to Eqs. (1.1), (1.8), and (1.9) with $\sigma_1 = -1$ and $\sigma_2 = 1$. It follows from (3.26) that these waves are stable.

The following nonlinear equation was derived some time ago[16] for drift waves:

$$\frac{1}{V_*}\frac{\partial\Phi}{\partial t}-\frac{\partial}{\partial x}\left(\varepsilon\Phi-q\frac{\Phi^2}{2}-\rho_0{}^2\Delta_\perp\Phi\right)=-\frac{c_s{}^2}{V_*{}^2}\frac{\partial v_z}{\partial z},$$

$$\frac{\partial v_z}{\partial x}=\frac{\partial\Phi}{\partial z},\qquad \Delta_\perp=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{6.5}$$

Here $\Phi = e_i\varphi/T_e$, $v_z = V_{zi}/V_*$, V_* is the electron drift velocity, $\varepsilon = 1 - V_*/U$, $q = \partial \ln T_e/\partial \ln n_0$, and T_e is the electron temperature. Equation (6.5) reduces to Eqs. (1.1), (1.8), and (1.9) with $\sigma_1 = -1$ and $\sigma_2 = -1$. We thus have an instability if $K_y > K_z$.

G. D. Aburdzhaniya and the present authors recently generalized Eq. (6.5) to the case of waves with an arbitrary ratio $U/c_A\alpha$. We found an equation like (6.5), but with

$$\rho_0{}^2\to\rho_0{}^2(1-U^2/c_A{}^2\alpha^2)^{-1}. \tag{6.6}$$

In this case we find Eqs. (1.1), (1.8), and (1.9) with $\sigma_1 = 1$ and $\sigma_2 = 1$. This case corresponds to an instability for arbitrary values of K_z and K_y, including $K_y = 0$; this is qualitatively different from the waves of the type (6.5). We might also note that in the two-dimensional case ($\partial/\partial z = 0$) an equation of the following structure can be found from (6.5) and (6.6):

$$\frac{1}{V_*}\frac{\partial\Phi}{\partial t}-\frac{\partial}{\partial x}\left[\varepsilon\Phi-q\frac{\Phi^2}{2}-\rho_0{}^2\left(1-\frac{U^2}{c_A\alpha^2}\right)^{-1}\Delta_\perp\Phi\right]=0. \tag{6.7}$$

The dispersion of these waves is negative if $U<\alpha c_A$ or positive if $U>\alpha c_A$. At both $U>\alpha c_A$ and $U<\alpha c_A$, however, we find nonlinear equations of the type in (1.1) and (1.6) with $\sigma = -1$ from (6.7). This example illustrates the assertion of Section 1, that the sign of σ in a two-dimensional anisotropic model (or the sign of σ_2 in a hybrid model) may, in general, be different from the sign of the wave dispersion.

6.4. Waves describable by models with a vector nonlinearity. When a vector nonlinearity is taken into account in the problem of drift waves, we find a system of equations analogous to (6.5) with the formal substitution[4]

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial x}+\frac{cT_e}{e_iUB_0}[\nabla\Phi,\nabla]_z. \tag{6.8}$$

In this case we find model equations of the type in (1.12) and (1.13) with $\sigma = -1$. We can therefore use the results derived in Section 5 in this problem. In particular, we conclude that if the vector nonlinearity is sufficiently pronounced one-dimensional periodic drift waves will be stable against two-dimensional perturbations ($K_y \neq 0$, $K_z \neq 0$). An analogous conclusion was reached in Ref. 4 for drift solitons.

In addition, many other problems involving gradient oblique ion acoustic and electron acoustic waves in a magnetized plasma (see Refs. 21–23 for examples of these waves) reduce to two- and three-dimensional model equations of the types in (1.10)–(1.13), with certain modifications.

7. DISCUSSION OF RESULTS

It is clear from the discussion in the Introduction that the problem with which we are concerned here, that of the stability of periodic waves in weakly dispersive media, has a rather long and substantial history. Our predecessors and we have derived fairly extensive results, as outlined in the Introduction. At this point, however, having presented several specific formulas in the preceding sections, we can characterize in more detail the set of results available. These results may be classified as methodological and physical. We will discuss the two types of results separately.

7.1. Methodological results. We have analyzed the problem of the stability of periodic waves in weakly dispersive media, describable by the set of nonlinear model equations in (1.1)–(1.13), against long-wave perturbations. We have assumed that the initial waves are one-dimensional, while the perturbations are two- or three-dimensional. Separating these model equations into two groups—scalar and vector—we derived dispersion relations (3.15) and (5.20), the first of which describes long-waves perturbations in scalar models, while the second does the same for vector models. These dispersion relations are the basic methodological results of the present study.

The dispersion relation (3.15) describes long-wave perturbations in the hybrid model, and the dispersion relations which follow from it, (3.22) and (3.25), describe perturbations in the isotropic and anisotropic models, respectively. Other important particular cases of Eq. (3.15) are dispersion relations (3.24) for longitudinal perturbations and (3.26) for transverse perturbations. Dispersion relation (3.24) should basically be credited to Whitham,[12] as was pointed out some time ago in Refs. 8 and 9. We credit Infeld *et al.*[8,9] with priority in the derivation of the dispersion relation (3.22). Their dispersion relation is written in a different notation, and the difference makes it difficult to establish the exact correspondence between these equations. Nevertheless, a comparison of the general structure of the two equations and of their particular consequences leads us to believe that these equations are identical. The same authors should be credited with deriving a dispersion relation for transverse perturbations in isotropic model (4.5) for the particular case of our Eq. (3.26) with $\sigma_2 = 0$ (or $K_y = 0$).

Extremely important from the methodological standpoint for this problem is the fact that dispersion relations are derived without assuming that the wave numbers of the perturbations are small. In other words, these dispersion relations give a complete description of the problem of the stability in a particular model. Kuznetsov *et al.*[11] have reported deriving a dispersion relation of this type for the Kadomtsev-Petviashvili model. However, the correspondence between their dispersion relation and the long-wave dispersion relation in (3.22) was not studied in Ref. 11. Consequently, in this paper we are unable to compare our methodological results with those of Ref. 11.

Included in our methodological results is the extension of the method of a series expansion in small wave numbers of the perturbations of the initial wave to the case of anisotropic, hybrid, and vector models. (This method, originally proposed by Rowlands,[24] has been used by Infeld *et al.*[8,9] to study the stability of waves in the Kadomtsev-Petviashvili model.)

7.2. Physical results. In addition to deriving these dispersion relations, it was our purpose to study the physical properties of the perturbations which are described by these equations, primarily to determine the conditions under which perturbations can grow in time, i.e., conditions corresponding to instability of the initial waves. The physical results obtained in this direction are characterized by the equations given in Section 4 (for the scalar models) and Subsection 5.2 (for vector models). Yet another purpose here has been to determine the correspondence between these models and actual physical situations. In Section 6 we supplemented the list of problems in the physics of magnetized plasmas with some new examples, which reduce to standard models; these are further results of the present study.

One of the central problems of the theory of the two- and three-dimensional stability of periodic waves is the question of whether the stability is unambiguously related to the nature of the dispersion of the medium, which we raised back in the Introduction. We have shown here that the stability is determined not only by the sign of the dispersion but also by the particular model and the particular type of perturbation. In agreement with Refs. 8, 9, and 11, it follows from our analysis that in the Kadomtsev-Petviashvili model and for the case of transverse perturbations, waves are stable in media with a negative dispersion and unstable in media with a positive dispersion [see (4.5) and (4.6)]. In examining oblique perturbations of waves in media with a negative dispersion, on the other hand, we found that these perturbations can grow [see Eq. (4.22)] when a certain resonant relation between the longitudinal and transverse wave numbers is satisfied [see (4.21)] and if the waves are highly nonlinear ($1 - s^2 \ll 1$, but $s^2 \neq 1$). Although oblique waves were also analyzed in Refs. 8 and 9, the discussion there dealt with situations different from that discussed here (the cases discussed in those other papers were $s^2 = 1$ and s^2 very different from unity!). Consequently the conclusion reached regarding the stability of oblique waves in Refs. 8 and 9 applies to a particular case and does not contradict our own conclusion regarding stability.

In the case of the anisotropic model, we find the following picture: Transverse perturbations in this model are stable if the dispersion is positive or unstable if the dispersion is negative [see (4.7) and (4.8)]. Oblique perturbations, on the other hand, may grow even if the dispersion is positive; in contrast with the isotropic model, discussed above, in this case both highly nonlinear and slightly nonlinear waves may be unstable [see (4.24) and (4.29)].

In the case of the hybrid model, the stability of periodic waves is essentially a three-dimensional problem. In this case, whether perturbations grow depends on both the signs of σ_1 and σ_2 (which characterize the dispersive media) and the relations among all three wave numbers [see (4.9), (4.10), (4.18), (4.20), and (4.27)].

Another specific feature of the picture of the stability of periodic waves results from a vector nonlinearity, which may be important at high wave amplitudes. The primary result which we derived in this case is that a vector nonlinearity, as in the case of solitons, can suppress instabilities which have been predicted to occur in the absence of a such a nonlinearity [see (5.38)].

It is clear from the discussion above that a study of the stability of weakly dispersive periodic waves should begin with an identification of the nature of the two- or three-dimensional equations which describe these waves. If these equations reduce to the equations which we have discussed above or equations our predecessors have discussed,[8,9,11] it becomes possible to make use of the existing results. Otherwise, additional analysis becomes necessary.

Finally, we note, as in the papers by our predecessors, we have ignored effects which stem from the particular geometry of the experimental devices in which plasma is confined. It frequently happens (in Ref. 3, for example) that these devices are cylindrical, so that the plasma confined in them is axisymmetric. In this case the azimuthal structure of the wave is characterized by a discrete set of wave numbers, the smallest being on the order of the reciprocal of the transverse dimension of the device. Consequently, if the initial waves are periodic in the azimuthal direction, as they were in Ref. 3, oblique perturbations of such waves with small quasimomenta may be forbidden by these geometric circumstances. Under these conditions, only transverse perturbations will be possible. However, the finite dimensions of the experimental device may also suppress such perturbations. Consequently, the instabilities of periodic waves which we have been discussing in this paper should be thought of as possible but not inevitable.

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