Theory of the temperature dependence of the magnon spectrum in ferromagnetic metals

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A quantum-mechanical formulation of the dynamical approach describing the fluctuations and allowing the determination of the magnon spectrum at low temperatures is proposed. A thermodynamic method of investigating the effect of the fluctuations on the equilibrium density matrix and the quasiparticle energy is developed. The dynamical approach allows the consideration of both the short-range Fermi-liquid interaction between the electrons and the long-range Coulomb interaction, interactions which essentially govern the fluctuation temperature dependence of the magnon spectrum in itinerant electron ferromagnets.

1. INTRODUCTION

The phenomenological approach, developed in Refs. 1–3, to the theory of conductive collective-electron ferromagnets leads to the following temperature dependence of the magnon spectrum at low temperatures:

$$\omega(k,\ T)=\omega(k,\ 0)\,(1-AT^2+BT^{5/2}). \tag{1.1}$$

Here the T^2 term is determined by the thermal smearing of the electron Fermi level. This effect has been quantitatively studied in specific models in a number of papers (see, for example, Refs. 3 and 4), and does not present any problems now. The case is somewhat different with the $T^{5/2}$ term, which is usually related with the effective magnon-magnon interaction, or, in current usage, with the magnetic fluctuations. Here, on the one hand, there has been developed a consistent approach to the computation of this term in the Heisenberg model of the ferromagnet,[5,6] in which the collective electrons are considered to be stationary, localized at the lattice sites. On the other hand, there are theories that take account of the role of the itinerant electrons in the Hubbard model,[7–12] and are meant to describe strong magnets, in which a band with one electron-spin direction is empty. These theories can be divided into two groups, according to the assumptions made in them. Thus, in Refs. 7–10 the authors use the normal-coordinate approximation, the inaccuracy of which leads to results that are, as we shall show, inconsistent with the Landau-Lifshitz equation within its limits of applicability. On the other hand, in the papers cited in Ref. 11 the authors use the "local" band theory approximation, in which the magnetization satisfies the Landau-Lifshitz equation. But in this case the authors, in the first place, ignore the fact that this equation is not satisfied under conditions when the higher derivatives must be taken into account, and, in the second place, neglect the spin-density-squared-nonconserving fluctuations, which, as follows from our analysis, have an appreciable effect on the temperature dependence of the magnon spectrum.

In Ref. 13, which is devoted to the investigation of weak ferromagnets, we propose a consistent dynamical approach that describes the magnetic fluctuations in the region of low temperatures without requiring additional assumptions. This approach is illustrated there in the simple approximation involving the use of the semiclassical equations of motion of the electron liquid. In this paper we present a quantum-dynamical approach. We then rid ourselves of the common assumption made in Refs. 7–13, in which the long-range electron Coulomb interaction is completely neglected, which, as shown below, makes these papers treat the electron liquid in magnets as a system of uncharged particles, or as a system in which the effective wave-length of the magnons is smaller than the Coulomb-field-shielding distance. Below we construct, for a charged electron liquid under conditions of weak nonlinearity, a theory of the temperature dependence of the magnon spectrum at low temperatures on the basis of the method of dynamical equations developed in Ref. 13. Such a theory allows us to make a consistent allowance for the long-range electron Coulomb interaction, and also allows us to reveal the hitherto undiscussed approximations implicit in the papers based on the method of normal coordinates[7–10] and on the local-band theory.[11]

2. THE EQUATIONS OF MOTION. DERIVATION OF THE DISPERSION EQUATION

Our dynamical treatment of the effects of the magnetic fluctuations will be based on the assumption that we can describe the ferromagnetic state of a metal with the aid of a complete quantum set, including the momentum **p** and the electron-spin components $\sigma = \pm 1$, using which we can, for the electron density matrix $\hat{\rho}$ and energy operator $\hat{\varepsilon}$, write

$$\langle\sigma\mathbf{p}+\hbar\mathbf{k}/2|\hat{\rho}|\sigma'\ \mathbf{p}-\hbar\mathbf{k}/2\rangle$$
$$=(2\pi)^3\delta_{\sigma\sigma'}\delta(\mathbf{k})\,n_0{}^\sigma(\mathbf{p})+\delta\rho^{\sigma\sigma'}(\mathbf{p},\ \mathbf{k},\ t),$$
$$\langle\sigma\mathbf{p}+\hbar\mathbf{k}/2|\hat{\varepsilon}|\sigma'\ \mathbf{p}-\hbar\mathbf{k}/2\rangle$$
$$=(2\pi)^3\delta_{\sigma\sigma'}\delta(\mathbf{k})\,\varepsilon_0{}^\sigma(\mathbf{p})+\delta\varepsilon^{\sigma\sigma'}(\mathbf{p},\ \mathbf{k},\ t).$$

Here $n_0^\sigma(\mathbf{p}) = n_0(\mathbf{p}) + \sigma s_0(\mathbf{p})$ and $\varepsilon_0^\sigma(\mathbf{p}) = \varepsilon_0(\mathbf{p}) - \sigma\hbar\Omega_0/2$ are respectively the distribution function and the energy of the quasiparticles without allowance for the effect of the fluctuations, the $\delta\varepsilon^{\sigma\sigma'}(\mathbf{p},\mathbf{k},t)$ are the matrix elements of the nonequilibrium self-consistent electron potential, which is a

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functional of the non-equilibrium density matrix $\delta\rho^{\sigma\sigma'}(\mathbf{p}, \mathbf{k}, t)$. We shall, neglecting the relativistic interactions, use the simplest model functional dependence

$$\delta\varepsilon^{\sigma\sigma'}(\mathbf{k}, t)=2[\delta_{\sigma\sigma'}\phi(\mathbf{k})\delta n(\mathbf{k}, t)+(\hat{\sigma})_{\sigma\sigma'}\Psi(\mathbf{k})\delta s(\mathbf{k}, t)],$$
$$\delta_{\sigma\sigma'}\delta n(\mathbf{k}, t)+(\hat{\sigma})_{\sigma\sigma'}\delta s(\mathbf{k}, t)=\int d\tau\delta\rho^{\sigma\sigma'}(\mathbf{p}, \mathbf{k}, t), \qquad (2.1)$$
$$d\tau=(2\pi\hbar)^{-3}d\mathbf{p}$$

(where the $\hat{\sigma}$ are Pauli matrices), in which we take account of the interaction of the electrons with the charge- and spin-density fluctuations, which is characterized by the functions $\phi(\mathbf{k})$ and $\Psi(\mathbf{k})$. In the function $\phi(\mathbf{k})$ we separate out the singular Coulomb part, i.e., we set $\phi(\mathbf{k}) = 4\pi e^2/\mathbf{k}^2 + \varphi(\mathbf{k})$, so that $\varphi(\mathbf{k})$ and $\Psi(\mathbf{k})$ describe the short-range inter-electron interaction. In so doing we ignore the dependence of φ and Ψ on the affiliation of the electrons with different bands. A formulation of the theory that takes account of this dependence is given in Refs. 14 and 15.

In the ground state of the ferromagnet the axis of quantization (the z axis) is oriented along the spontaneous spin density $s_0 = \int d\tau s_0(p)$, and the energy $\hbar\Omega_0$ of the spin-induced splitting is given by the equation[16] $\hbar\Omega_0 = -4\Psi s_0$, where $\Psi = \Psi(0)$.

We base the dynamical theory on the equation of motion for the electron density matrix:

$$\partial\hat{\rho}/\partial t=i(\hat{\rho}\hat{\varepsilon}-\hat{\varepsilon}\hat{\rho})/\hbar,$$

which corresponds to the following system of equations:

$$\begin{aligned}&[\hbar\omega+\varepsilon_0^{\pm}(\mathbf{p}-\hbar\mathbf{k}/2)-\varepsilon_0^{\mp}(\mathbf{p}+\hbar\mathbf{k}/2)]\delta\rho^{\mp\pm}(\mathbf{p}, k)-[n_0^{\pm}(\mathbf{p}-\hbar\mathbf{k}/2)\\&-n_0^{\mp}(\mathbf{p}+\hbar\mathbf{k}/2)]\delta\varepsilon^{\mp\pm}(k)=\int(dk')\{[\delta\rho^{\pm\pm}(\mathbf{p}-\hbar\mathbf{k}'/2, k-k')\\&-\delta\rho^{\mp\mp}(\mathbf{p}+\hbar\mathbf{k}'/2, k-k')]\delta\varepsilon^{\mp\pm}(k')\\&+\delta\rho^{\mp\pm}(\mathbf{p}-\hbar\mathbf{k}'/2, k-k')\delta\varepsilon^{\mp\mp}(k')\\&-\delta\rho^{\mp\pm}(\mathbf{p}+\hbar\mathbf{k}'/2, k-k')\delta\varepsilon^{\pm\pm}(k')\},\end{aligned} \qquad (2.2)$$

$$\begin{aligned}&[\hbar\omega+\varepsilon_0^{\pm}(\mathbf{p}-\hbar\mathbf{k}/2)-\varepsilon_0^{\pm}(\mathbf{p}+\hbar\mathbf{k}/2)]\delta\rho^{\pm\pm}(\mathbf{p}, k)-[n_0^{\pm}(\mathbf{p}-\hbar\mathbf{k}/2)\\&-n_0^{\pm}(\mathbf{p}+\hbar\mathbf{k}/2)]\delta\varepsilon^{\pm\pm}(k)=\int(dk')\{[\delta\rho^{\pm\pm}(\mathbf{p}-\hbar\mathbf{k}'/2, k-k')\\&-\delta\rho^{\pm\pm}(\mathbf{p}+\hbar\mathbf{k}'/2, k-k')]\\&\times\delta\varepsilon^{\pm\pm}(k')\pm\delta\rho^{-+}(\mathbf{p}\mp\hbar\mathbf{k}'/2, k-k')\delta\varepsilon^{+-}(k')\\&\mp\delta\rho^{+-}(\mathbf{p}\pm\hbar\mathbf{k}'/2, k-k')\delta\varepsilon^{-+}(k')\}\end{aligned} \qquad (2.3)$$

for the Fourier components

$$\delta\rho^{\sigma\sigma'}(\mathbf{p}, k)=\int_{-\infty}^{+\infty}dt\exp(i\omega t)\delta\rho^{\sigma\sigma'}(\mathbf{p}, \mathbf{k}, t),$$

where $k = (\omega, \mathbf{k})$, $(dk) = (2\pi)^4 d\omega d\mathbf{k}$.

Let us first of all point out that neglecting the right-hand side of Eq. (2.2) in the zeroth approximation, we have

$$\delta\rho^{\mp\pm}(\mathbf{p}, k)=2\Pi^{\mp\pm}(\mathbf{p}, k)\Psi(\mathbf{k})\delta s^{\pm}(k), \qquad (2.4)$$

where

$$\begin{aligned}&\delta s^{\pm}(k)=\delta s^{x}(k)\pm i\delta s^{y}(k),\\&\Pi^{\sigma\sigma'}(\mathbf{p}, k)=[n_0^{\sigma'}(\mathbf{p}-\hbar\mathbf{k}/2)-n_0^{\sigma}(\mathbf{p}+\hbar\mathbf{k}/2)]\\&\times[\hbar\omega+\varepsilon_0^{\sigma'}(\mathbf{p}-\hbar\mathbf{k}/2)-\varepsilon_0^{\sigma}(\mathbf{p}+\hbar\mathbf{k}/2)]^{-1}.\end{aligned} \qquad (2.5)$$

Accordingly, neglecting the fluctuations, we obtain the usual dispersion equation[2,3,13,16]:

$$D_+(k)=1-\Psi(\mathbf{k})\Pi^{-+}(k)=0, \quad \Pi^{\sigma\sigma'}(k)=2\int d\tau\Pi^{\sigma\sigma'}(p, k), \qquad (2.6)$$

which gives the magnon frequency $\omega(\mathbf{k})$ without allowance for the effect of the fluctuations.

For our purpose—the determination of the magnon spectrum with allowance for the fluctuations—the right member of (2.2) must be taken into account. Let us divide the equation (2.2) for $\delta\rho^{-+}$ by

$$G^{-1}(\mathbf{p}, k)=\hbar\omega+\varepsilon_0^{+}(\mathbf{p}-\hbar\mathbf{k}/2)-\varepsilon_0^{-}(\mathbf{p}+\hbar\mathbf{k}/2),$$

integrate over the momenta, then multiply by $\delta s^-(-k_1)$, and carry out a statistical averaging, denoted below by $\langle ... \rangle$, over the thermodynamic-equilibrium state. As a result, we find the equation

$$\begin{aligned}&D_+(k)\langle\delta s^+(k)\delta s^-(-k_1)\rangle\\&=\int d\tau G(\mathbf{p}, k)\int(dk')\langle\delta\varepsilon^{-+}(k')\delta s^-(-k_1)\\&\times[\delta\rho^{++}(\mathbf{p}-\hbar\mathbf{k}'/2, k-k')-\delta\rho^{--}(\mathbf{p}+\hbar\mathbf{k}'/2, k-k')\\&-\Pi^{-+}(\mathbf{p}+\hbar(\mathbf{k}-\mathbf{k}')/2, k')\\&\times\delta\varepsilon^{++}(k-k')+\Pi^{-+}(\mathbf{p}-\hbar(\mathbf{k}-\mathbf{k}')/2, k')\delta\varepsilon^{--}(k-k')]\rangle,\end{aligned} \qquad (2.7)$$

in the course of the derivation of which we eliminated the components $\delta\rho^{-+}(\mathbf{p} \pm \hbar\mathbf{k}'/2, k - k')$ in (2.2) with the aid of the relation (2.4). Similarly, we can eliminate from (2.7) the quantities

$$\delta\rho^{\pm\pm}(\mathbf{p}\mp\hbar\mathbf{k}'/2, k-k'), \quad \delta\varepsilon^{\pm\pm}(k-k'),$$

using the following formulas, which follow from (2.3):

$$\begin{aligned}\delta\rho^{\pm\pm}(\mathbf{p}, k)=&(2\pi)^4\delta(k)\Delta n^{\pm}(\mathbf{p})\\&+\int(dk')\Gamma^{\pm}(\mathbf{p}, k, k')\delta s^+(k-k')\delta s^-(k'),\end{aligned} \qquad (2.8)$$

$$\delta s^z(k)=(2\pi)^4\delta(k)\Delta s+\int(dk')\Gamma^s(k, k')\delta s^+(k-k')\delta s^-(k'), \qquad (2.9)$$

$$\delta n(k)=(2\pi)^4\delta(k)\Delta n+\int(dk')\Gamma^n(k, k')\delta s^+(k-k')\delta s^-(k'). \qquad (2.10)$$

Here we have introduced the following notation:

$$\begin{aligned}\Gamma^{\pm}(\mathbf{p}, k, k')=&2\Pi^{\pm\pm}(\mathbf{p}, k)[\phi(\mathbf{k})\Gamma^n(k, k')\pm\Psi(\mathbf{k})\Gamma^s(k, k')]\\&\pm4\Psi(\mathbf{k}-\mathbf{k}')\Psi(\mathbf{k}')G(\mathbf{p}\mp\hbar\mathbf{k}'/2, k-k')\\&\times[\Pi^{\pm\pm}(\mathbf{p}, k)-\Pi^{+-}(\mathbf{p}\pm\hbar(\mathbf{k}-\mathbf{k}')/2, k')],\end{aligned} \qquad (2.11)$$

$$\begin{aligned}\Gamma^s(k, k')=&4\Psi(\mathbf{k}-\mathbf{k}')\Psi(\mathbf{k}')\{S_1(k-k', k', k)\\&\times[1-\phi(\mathbf{k})\Pi_n(k)]-\phi(\mathbf{k})\Pi_s(k)S_2(k-k', k', k)\}/D(k),\end{aligned} \qquad (2.12)$$

$$\begin{aligned}\Gamma^n(k, k')=&-4\Psi(\mathbf{k}-\mathbf{k}')\Psi(\mathbf{k}')\{S_2(k-k', k', k)\\&\times[1-\Psi(\mathbf{k})\Pi_n(k)]-\Psi(\mathbf{k})\Pi_s(k)S_1(k-k', k', k)\}/D(k),\end{aligned} \qquad (2.13)$$

$$\begin{aligned}S_{1,2}(k-k', k', k)=&\frac{1}{2}\int d\tau G(\mathbf{p}, k-k')\{\Pi^{--}(\mathbf{p}-\hbar\mathbf{k}'/2, k)\\&-\Pi^{+-}(\mathbf{p}-\hbar\mathbf{k}/2, k')\pm\Pi^{++}(\mathbf{p}+\hbar\mathbf{k}'/2, k)\mp\Pi^{+-}(\mathbf{p}+\hbar\mathbf{k}/2, k')\},\end{aligned} \qquad (2.14)$$

$$D(k)=[1-\Psi(\mathbf{k})\Pi_n(k)][1-\phi(\mathbf{k})\Pi_n(k)] -\phi(\mathbf{k})\Psi(\mathbf{k})\Pi_s^2(k),$$

$$\Pi_{n,s}(k)=2\int d\tau\Pi_{n,s}(\mathbf{p},k)=\int d\tau[\Pi^{++}(\mathbf{p},k)\pm\Pi^{--}(\mathbf{p},k)].$$

Let us emphasize that, in the formulas (2.8)–(2.10), we have separated the steady-state and spatially homogeneous $k=0$ contributions, whose form in the state of thermodynamic equilibrium will be established in the next section. Bearing this separation in mind, and taking account of the indeterminacy of the kernals in (2.11)–(2.13) at $k=0$, we shall determine them below in the sense requiring that we discard at $k=0$ the contribution of the integrals in (2.8)–(2.10).

Taking the relation (2.1) into account, and substituting the expressions (2.8)–(2.10) into (2.7), we arrive at the following dispersion equation:

$$D_+(k)+\delta D(k)-\int (dk')T(k,k-k',-k')(s^+s^-)_{k'}=0, \quad (2.15)$$

which establishes the dependence of the magnon spectrum on the transverse magnetic fluctuations, due both to the explicit dependence of (2.15) on the spectral fluctuation density $(s^+s^-)_k$ (where

$$\langle\delta s^+(k)\delta s^-(-k_1)\rangle=(2\pi)^4\delta(k-k_1)\times(s^+s^-)_k\,)$$

and to the corresponding dependence of the fluctuation-related corrections $\Delta n^\pm(\mathbf{p})$ and Δs (see the following section), which determine the function

$$\delta D(k)=2\Psi(\mathbf{k})\int d\tau G(\mathbf{p},k) \times[4\Psi\Pi^{-+}(\mathbf{p},k)\Delta s-\Delta n^+(\mathbf{p}-\hbar\mathbf{k}/2) +\Delta n^-(\mathbf{p}+\hbar\mathbf{k}/2)]. \quad (2.16)$$

Here we have taken into account the fact that, as demonstrated in the next section, $\Delta n=0$. For the kernel in (2.15) we obtain the expression

$$\begin{aligned}T(k,k',k'') &=8\Psi(\mathbf{k}-\mathbf{k}')[\Psi(\mathbf{k}'-\mathbf{k}'')\Psi(\mathbf{k}'')V(k'-k'',k'',k',k)\\ &+\Psi(\mathbf{k}')\Gamma^s(k',k'')S_1(k,k'-k,k')\\ &-\phi(\mathbf{k}')\Gamma^n(k',k'')S_2(k,k'-k,k')], \quad (2.17)\end{aligned}$$

where

$$\begin{aligned}V(k'-k'',k'',k',k) &={}^1/_2\int d\tau G(\mathbf{p},k'-k'')\{G(\mathbf{p}+\hbar(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')/2,k)\\ &\times[\Pi^{++}(\mathbf{p}+\hbar\mathbf{k}''/2,k')-\Pi^{+-}(\mathbf{p}+\hbar\mathbf{k}'/2,k'')]\\ &+G(\mathbf{p}-\hbar(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')/2,k)\\ &\cdot[\Pi^{--}(\mathbf{p}-\hbar\mathbf{k}''/2,k')-\Pi^{+-}(\mathbf{p}-\hbar\mathbf{k}'/2,k'')]\}. \quad (2.18)\end{aligned}$$

Here we have, assuming the fluctuation effects to be weak, limited ourselves in the left member of (2.15) to the consideration of the terms linear in $(s^+s^-)_k$. Furthermore, bearing in mind the investigation below of the properties of the low-frequency and long-wave excitations, we neglected in (2.17) the imaginary part of $\Pi^{-+}(\mathbf{p},k)$, and used the relation $\mathrm{Re}\,\Pi^{-+}(\mathbf{p},k)=\mathrm{Re}\,\Pi^{+-}(\mathbf{p},-k)$.

3. DERIVATION OF THE THERMODYNAMIC AVERAGES

In the preceding section we obtained a magnon dispersion equation containing the average quantities and fluctuations characterizing the stationary and spatially homogeneous state of a magnet. In this section we shall determine these quantities in the thermodynamic-equilibrium state. For this purpose, we shall use the standard approach that takes account of the fluctuations (see, for example, Refs. 17 and 2), modifying it in such a way that we can determine the densities of the quantities in phase space, densities which characterize the equilibrium density matrix

$$\langle\sigma\mathbf{p}|\hat\rho|\sigma'\mathbf{p}'\rangle=(2\pi\hbar)^3\delta_{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{p}')n^\sigma(\mathbf{p})$$

and the quasiparticle energy

$$\langle\sigma\mathbf{p}|\hat\varepsilon|\sigma'\mathbf{p}'\rangle=(2\pi\hbar)^3\delta_{\sigma\sigma'}\delta(\mathbf{p}-\mathbf{p}')\varepsilon^\sigma(\mathbf{p}),$$

where $n^\sigma(\mathbf{p})=n(\mathbf{p})+\sigma s(\mathbf{p})$. This approach is based on the use of the thermodynamic potential Φ, which is considered to be a functional of the density matrix $\hat\rho$, the quasiparticle energy $\hat\varepsilon$, the chemical potential η, and the temperature T, and can be written in the form

$$\Phi[\hat\rho,\hat\varepsilon,\eta,T]=\Phi_0[\hat\varepsilon,\eta,T]+2\Psi s^2 -\mathrm{Sp}\int d\tau\hat\rho(p)[\hat\varepsilon(p)-\varepsilon_0(p)]+\Delta\Phi[\hat\rho,\hat\varepsilon,T]. \quad (3.1)$$

Here

$$\Phi_0[\hat\varepsilon,\eta,T]=-\varkappa T\sum_\sigma\int d\tau\ln\left[1+\exp\frac{\eta-\varepsilon^\sigma(\mathbf{p})}{\varkappa T}\right] \quad (3.2)$$

coincides in outward appearance with the thermodynamic potential of an electron gas, $\varepsilon_0(\mathbf{p})$ being a given function of the momentum. The term on the right-hand side of (3.1) containing s^2, where $s=1/2\int d\tau[n^+(\mathbf{p})-n^-(\mathbf{p})]$, corresponds to the exchange interaction energy, and, finally,[2]

$$\Delta\Phi[\hat\rho,\hat\varepsilon,T]=\hbar\int(dk)\,\mathrm{cth}\frac{\hbar\omega}{2\varkappa T}\mathrm{Im}[\ln\tilde D_+(k)-\tilde D_+(k)] \quad (3.3)$$

describes the contribution of the transverse magnetic fluctuations. In this case $\tilde D_+(k)=1-\Psi(k)\tilde\Pi^{-+}(k)$ can be expressed in terms of the fluctuations $\tilde\Pi^{-+}(k)$ and $\tilde G(p,k)$, which differ from the functions Π^{-+} and $G(\mathbf{p},k)$ introduced above by the substitutions $\varepsilon_0^\sigma(p)\to\varepsilon^\sigma(\mathbf{p})$ and $n_0^\sigma(\mathbf{p})\to n^\sigma(\mathbf{p})$.

Minimizing (3.1) with respect to $n^\pm(\mathbf{p})$ and $\varepsilon^\pm(\mathbf{p})$, we obtain the following relations for the determination of the energy and density matrix of the electrons:

$$\begin{aligned}\varepsilon^\sigma(\mathbf{p})&=\varepsilon_0(\mathbf{p})+2\Psi\sigma s+\Delta_f\varepsilon^\sigma(\mathbf{p}),\\ n^\sigma(\mathbf{p})&=n_F[\varepsilon^\sigma(\mathbf{p}),\eta]+\Delta_f n^\sigma(\mathbf{p}),\end{aligned} \quad (3.4)$$

where $n_F(\varepsilon,\eta)$ is the Fermi distribution function for the electrons with energy ε and chemical potential η, and the quantities

$$\Delta_f\varepsilon^\sigma(\mathbf{p})=4\sigma\int(dk)\Psi^2(\mathbf{k})G(\mathbf{p}+\sigma\hbar\mathbf{k}/2,k)(s^+s^-)_k, \quad (3.5)$$

$$\Delta_f n^\sigma(\mathbf{p})=-4\int(dk)\Psi^2(\mathbf{k})G(\mathbf{p}+\sigma\hbar\mathbf{k}/2,k) \times[n_0^\sigma(\mathbf{p})-n_0^{-\sigma}(\mathbf{p}-\hbar\mathbf{k})](s^+s^-)_k \quad (3.6)$$

stem from the fluctuation contribution (3.3) to the thermodynamic potential. Owing to the weakness of the fluctuation effects, we have ignored in the formulas (3.5) and (3.6) the deviation of $\varepsilon^\sigma(\mathbf{p})$ and $n^\sigma(\mathbf{p})$ from $\varepsilon_0^\sigma(\mathbf{p})$ and $n_0^\sigma(\mathbf{p})$, which is legitimate because of the proportionality of the integrands in (3.5) and (3.6) to the spectral density of the magnetic fluctuations[2]:

$$(s^+s^-)_k = -[\hbar/2\Psi(\mathbf{k})]\operatorname{cth}(\hbar\omega/2\varkappa T)\operatorname{Im}D_+^{-1}(k).$$

Further, the equilibrium fermion-number density is defined, as usual,[18] as

$$n = -\left(\frac{\partial\Phi}{\partial\eta}\right)_{\hat{\rho},\hat{\varepsilon},T} = -\left(\frac{\partial\Phi_0}{\partial\eta}\right)_{\hat{\varepsilon},T} = \sum_\sigma \int d\tau n_F[\varepsilon^\sigma(\mathbf{p}),\eta], \tag{3.7}$$

which furnishes the equation for the determination of the chemical potential η, an equation which takes account, in accordance with (3.4), of the effect of the fluctuations. The chemical potential η_0 of the electrons is, when the fluctuations are ignored, determined by the condition

$$n_0 = \sum_\sigma \int d\tau n_F[\varepsilon_0^\sigma(\mathbf{p}),\eta_0].$$

In this case, bearing in mind the conservation of the fermion number, we find that $\Delta n = n - n_0 = 0$.

The expression, following from (3.1), for the entropy is the sum of the electron and fluctuation contributions. The electron contribution in this case has the usual form[18]:

$$S_e = -\sum_\sigma \int d\tau\{n_F[\varepsilon^\sigma(\mathbf{p}),\eta]\ln n_F[\varepsilon^\sigma(\mathbf{p}),\eta] + (1-n_F[\varepsilon^\sigma(\mathbf{p}),\eta])\ln(1-n_F[\varepsilon^\sigma(\mathbf{p}),\eta])\}, \tag{3.8}$$

the contribution of the fluctuations here being taken into account by the electron distributions. Correspondingly, for the fluctuation contribution to the entropy we obtain

$$S_f = -2\int (dk)\{[1+N(\omega)]\ln[1+N(\omega)] - N(\omega)\ln N(\omega)\}\operatorname{Im}\frac{\partial}{\partial\omega}[\ln D_+(k) - D_+(k)], \tag{3.9}$$

where $N(\omega) = [\exp(\hbar\omega/\varkappa T) - 1]^{-1}$ is the Bose distribution function.

Similarly, the expression for the spin density s is also equal to the sum of the electron

$$s_e = {}^1/_2 \int d\tau\{n_F[\varepsilon^+(p),\eta] - n_F[\varepsilon^-(p),\eta]\} \tag{3.10}$$

and fluctuation

$$\Delta_f s = \frac{1}{2}\int d\tau[\Delta_f n^+(p) - \Delta_f n^-(p)'] = -\frac{2}{\hbar}\int (dk)\Psi(k)\frac{\partial D_+(k)}{\partial\omega}(s^+s^-)_k \tag{3.11}$$

contributions, with the electron distributions $n_F[\varepsilon^\sigma(p),\eta]$ in (3.8) and (3.10) taking account of the effect of the fluctuations on the energy and the chemical potential of the electrons.

If we are interested in only the contribution of the magnons, then it is sufficient to take

$$\operatorname{Im}\partial[\ln D_+(k)]/\partial\omega = -\pi\delta[\omega-\omega(k)],$$

where the magnon frequency $\omega(\mathbf{k})$ is determined by (2.6). Then from (3.1), (3.8), and (3.10) we obtain the following well-known expressions[6]:

$$\Delta\Phi = (2\pi)^{-3}\int dk\{\varkappa T\ln[1-\exp(-\hbar\omega/\varkappa T)] + \hbar\omega/2\}_{\omega=\omega(k)},$$

$$S_f = (2\pi)^{-3}\int dk\{[1+N(\omega)]\ln[1+N(\omega)] - N(\omega)\ln N(\omega)\}_{\omega=\omega(k)},$$

$$\Delta_f s = -(2\pi)^{-3}\int dk[N(\omega) + {}^1/_2]_{\omega=\omega(k)},$$

which give the contributions of the magnons to the thermodynamic potential, the entropy, and the spin density.

To determine the thermodynamic averages of $\Delta n^\pm(p)$ and Δs, which determine, in accordance with (2.16), the function $\delta D(k)$, we use perturbation theory, which allowed us to limit ourselves in (3.5) and (3.6) to the approximation linear in the spectral density of the magnetic fluctuations. Retaining in the first addends in the formulas (3.4) the terms linear in

$$\Delta\varepsilon^\sigma(p) = \Delta_f\varepsilon^\sigma(p) + \sigma 2\Psi\Delta s, \qquad \Delta\eta = \eta - \eta_0$$

we obtain

$$n^\sigma(\mathbf{p}) = n_0^\sigma(\mathbf{p}) + \Delta n^\sigma(\mathbf{p}) = n_0^\sigma(\mathbf{p}) + \Delta_f n^\sigma(\mathbf{p}) + \Pi^{\sigma\sigma}(\mathbf{p})[\Delta\varepsilon^\sigma(\mathbf{p}) - \Delta\eta], \tag{3.12}$$

where $n_0^\sigma(p)$, $\Delta n^\sigma(p)$, and Δs coincide with the corresponding quantities used in the preceding section, and the quantities $\Pi^{\sigma\sigma}(\mathbf{p}) = \Pi^{\sigma\sigma}(\mathbf{p},0)$ occurring here correspond to that $k = 0$ limit which is obtained by letting first $\omega \to 0$, and then $\mathbf{k} \to 0$.

As follows from the formula (3.12), the fluctuation corrections $\Delta n^\sigma(\mathbf{p})$ to the electron density matrix, which are of interest to us here, can be expressed in terms of the quantities $\Delta\eta$ and Δs. Taking the relations (3.5) and (3.6) into account, we have

$$\Delta n^\sigma(\mathbf{p}) = -\Pi^{\sigma\sigma}(\mathbf{p})[\Delta\eta - \sigma 2\Psi\Delta s] + \sigma 4\int (dk)\Psi^2(\mathbf{k})G(\mathbf{p}+\sigma\hbar\mathbf{k}/2,k) \times[\Pi^{\sigma\sigma}(\mathbf{p}) - \Pi^{-+}(\mathbf{p}+\sigma\hbar\mathbf{k}/2,k)](s^+s^-)_k. \tag{3.13}$$

We obtain for the determination of $\Delta\eta$ and Δs the equations

$$\Pi_n\Delta\eta - 2\Psi\Pi_s\Delta s = \int d\tau[\Pi^{++}(\mathbf{p})\Delta_f\varepsilon^+(\mathbf{p}) + \Pi^{--}(\mathbf{p})\Delta_f\varepsilon^-(\mathbf{p})'], \tag{3.14}$$

$$\Pi_s\Delta\eta/2 + (1-\Psi\Pi_n)\Delta s = \Delta_f s + {}^1/_2\int d\tau[\Pi^{++}(\mathbf{p})\Delta_f\varepsilon^+(\mathbf{p}) - \Pi^{--}(\mathbf{p})\Delta_f\varepsilon^-(\mathbf{p})], \tag{3.15}$$

where $\Pi_{n,s} = \Pi_{n,s}(0)$, the first of which was found through the expansion of (3.7), while the second arose in the integration of (3.13). The solution of Eqs. (3.14) and (3.15) yields

$$\Delta\eta = -8[\Pi_n + \Psi(\Pi_s^2 - \Pi_n^2)]^{-1}\int (dk)\Psi^2(\mathbf{k}) \times[(1-\Psi\Pi_n)S_2(k,-k,0) - \Psi\Pi_s S_1(k,-k,0)](s^+s^-)_k, \tag{3.16}$$

$$\Delta s=4[\Pi_n+\Psi(\Pi_s^2-\Pi_n^2)]^{-1}\int(dk)\Psi^2(\mathbf{k})$$
$$\times[\Pi_nS_1(k,-k,0)+\Pi_sS_2(k,-k,0)](s^+s^-)_k. \quad (3.17)$$

Using the limiting values, introduced in the preceding section, of the quantities $\Gamma^{\pm}$, Γ^s, and Γ^n, we obtain for the fluctuation corrections (3.13), (3.16), and (3.17) the expressions

$$\Delta n^\sigma(\mathbf{p})=\lim_{k\to 0}\int(dk')\Gamma^\sigma(\mathbf{p},k,-k')(s^+s^-)_{k'},$$

$$\Delta\eta=-\lim_{k\to 0}\int(dk')\Gamma^n(k,-k')\phi(\mathbf{k}')(s^+s^-)_{k'},$$

$$\Delta s=\lim_{k\to 0}\int(dk')\Gamma^s(k,-k')(s^+s^-)_{k'},$$

with the aid of which we can represent the function (2.16) in the following form:

$$\delta D(k)=-\int(dk')\lim_{k''\to 0}T(k,k'',-k')(s^+s^-)_{k'}. \quad (3.18)$$

Here $k''=0$ corresponds to that limit in which we let first $\omega''\to 0$ and then $\mathbf{k}''\to 0$.

The formula (3.18) allows us to write the dispersion equation (2.15) in the form

$$D_+(k)-\int(dk')[T(k,k-k',-k')+T(k,k''=0,-k')]$$
$$\times(s^+s^-)_{k'}=0. \quad (3.19)$$

Notice that this formula follows directly from (2.7) if we do not separate out in the formulas (2.8)–(2.10) the contributions $\sim\delta(k)$. But then the limit $\lim_{k''\to 0}T(k,k'',-k')$ remains indeterminate, in contrast to the above analysis, which is free of such an indeterminacy.

The formulas obtained in this section allow us to determine in explicit form the temperature dependences of the fluctuation corrections to the energy, the density matrix, the chemical potential of the quasiparticles, the equilibrium spin density, as well as the mean-square spin density, for which we have the approximate expression

$$S_l^2(T)\approx s_0^2+2s_0\Delta s+\int(dk)(s^+s^-)_k\approx-\gamma_{ij}\int(dk)k_ik_j(s^+s^-)_k,$$

where we have taken into account the fact that, up to terms $\sim\mathbf{k}^2$,

$$\Gamma^s(0,-k)|_{\omega=\omega(\mathbf{k})}=-(2s_0)^{-1}(1+\gamma_{ij}k_ik_j).$$

Using the approximate expression

$$(s^+s^-)_k=4\pi s_0\left[N(\omega)+\frac{1}{2}\right]\delta[\omega-\omega(\mathbf{k})] \quad (3.20)$$

for the spectral density of the fluctuations, we have

$$\begin{aligned}&\Delta\varepsilon^\sigma(\mathbf{p})=[a_\varepsilon(\mathbf{p})-\sigma a_\Omega(\mathbf{p})]T^{5/2},\\&\Delta n^\sigma(\mathbf{p})=a_n(\mathbf{p})T^{5/2}+\sigma a_s(\mathbf{p})T^{5/2},\\&\Delta\eta=a_\eta T^{5/2},\quad \Delta s=a_sT^{5/2},\\&S_l(T)-S_l(0)=a_lT^{5/2}.\end{aligned} \quad (3.21)$$

Here the coefficients $a_{\varepsilon,\Omega}(\mathbf{p})$, $a_{n,s}(\mathbf{p})$, $a_{\eta,s}$, and a_l are given by Eqs. (3.5), (3.13), (3.16), and (3.17), with their integrands expanded in powers of k up to, and including terms $\sim k^2$. Let us note that the first two equations in (3.21) correspond to the result obtained in a phenomenological analysis of the temperature dependence of the electron energy in conducting ferromagnets.[2,3] The last equation in (3.21), which describes the temperature dependence of the square of the spin density, is obtained in Ref. 19 for the particular case of strong magnets.

4. TEMPERATURE DEPENDENCE OF THE MAGNON SPECTRUM

Let us proceed to discuss the dependence of the magnon spectrum in ferromagnetic metals on temperature. For this purpose, let us, taking account of (3.20), write the solution to the dispersion equation (3.19) in the following form:

$$\omega(\mathbf{k},T)=\omega(\mathbf{k})-2s_0\Omega_0\int\frac{d\mathbf{k}'}{(2\pi)^3}\left[N(\omega)+\frac{1}{2}\right]$$
$$\times[T(k,k-k',-k')+T(k,0,-k')]_{\substack{\omega=\omega(\mathbf{k})\\ \omega'=\omega(\mathbf{k}')}}. \quad (4.1)$$

The first term in the right member of (4.1) contains, as follows from Eq. (2.6), a small term that depends on the temperature according to the law T^2, and is due to the effect of the Fermi excitations of the electrons, while the second term describes the effect of the magnetic fluctuations on the magnon spectrum.

Notice that, by setting $\mathbf{v}=\partial\varepsilon_0(\mathbf{p})/\partial\mathbf{p}\to 0$ in the relation (2.17), we obtain for the combination of kernels T in (4.1) in the limit of magnets with localized electrons the result

$$[T(k,k-k',-k')+T(k,0,-k')]_{\substack{\omega=\omega(\mathbf{k})\\ \omega'=\omega(\mathbf{k}')}}$$
$$=\frac{\omega(\mathbf{k})+\omega(\mathbf{k}')-\omega(\mathbf{k}-\mathbf{k}')}{2s_0^2\Omega_0}, \quad (4.2)$$

(where $\omega(\mathbf{k})=4s_0[\Psi(\mathbf{k})-\Psi]/\hbar$, which is in full accord with the theory of the Heisenberg magnets.[5,6]

In the general case, expanding the combination of kernels T in powers of k and k', and retaining the terms $\sim kk'$ and $\sim\mathbf{k}^2\cdot\mathbf{k}'^2$, we have

$$[T(k,k-k',-k')+T(k,0,-k')]_{\substack{\omega=\omega(\mathbf{k})\\ \omega'=\omega(\mathbf{k}')}}$$
$$=(\alpha_{ij}k_ik_j'+t_{ijkl}k_ik_jk_k'k_l')/s_0^2\Omega_0. \quad (4.3)$$

The first term in the right-hand side of (4.3), which is determined by the magnon stiffness α_{ij} ($\omega(\mathbf{k})=\alpha_{ij}k_ik_j$), describes, for example, the amplitude of the four-magnon scattering processes, and is analogous to a term that arises in the theory of the Heisenberg magnets.[5,6] As shown in Refs. 19 and 20, this term stems from the approximate conservation of the square of the spin density in ferromagnetic metals, and can be obtained with the use as the dynamical equations of the Landau-Lifshitz equation, which follows from (2.2) and (2.3) in the long-wave limit. The second term, which is proportional to $\mathbf{k}^2\cdot\mathbf{k}'^2$, describes in accordance with Eq. (4.1) the fluctuation-governed temperature dependence of the magnon spectrum. In this case the tensor t_{ijkl} is, according to (2.17), essentially determined by the effects of the long-range electron Coulomb interaction and by the fluctuations due to the variation of the square of the spin density, and manifesting themselves in the deviation of the kernel (2.12) from the value $-(2s_0)^{-1}$. Let us note that allowance for such effects falls outside the limits of applicability of the Landau-Lifshitz equation.

In the model of metals with the isotropic dispersion laws $\varepsilon_0(\mathbf{p}) = \mathbf{p}^2/2m$ for electrons and $\omega(\mathbf{k}) = \alpha \mathbf{k}^2$ for magnons, we arrive, after substituting (4.3) into Eq. (4.1), at the following expression for the coefficient:

$$B=-\frac{4\pi}{3}\zeta\left(\frac{5}{2}\right)\frac{t_{iijj}}{\alpha s_0}\left(\frac{\varkappa}{4\pi\hbar\alpha}\right)^{5/2}, \qquad (4.4)$$

which determines, in accordance with Eq. (1.1), the fluctuation-governed temperature dependence of the magnon spectrum at low temperatures $\varkappa T < \hbar\omega_{\max}$ (where $\omega_{\max}$ is the highest magnon frequency). Here

$$\frac{\Omega_0}{9}t_{iijj}=\frac{13}{3}\alpha_H^2-7\alpha_H\alpha+\frac{1}{3}(8-\Psi\Pi_n)\alpha^2+\alpha\varepsilon+\Omega_0(\delta+\delta_H)$$
$$+\left(\frac{\Pi_n}{\Psi}+\Pi_s^2-\Pi_n^2\right)^{-1}$$
$$\times\left\{\left(\frac{9}{2}\alpha_H^2-6\alpha_H\alpha+2\alpha^2\right)(\Pi_n^2-\Pi_s^2)-\frac{2\alpha-3\alpha_H}{2\Psi m}\hbar\Pi_s\right.$$
$$\left.-\frac{\hbar^2}{12\Psi^2m^2}[2-\Psi\Pi_n-\Psi^2(4\Pi_n^2-\Pi_s^2)-3\Psi^3\Pi_n(\Pi_n^2-\Pi_s^2)]\right\}, \qquad (4.5)$$

where we have used the notation

$$\alpha=\alpha_H+\frac{2}{3}\frac{\Psi}{\Omega_0}\int d\tau v^2\left[\frac{2s_0(\mathbf{p})}{\hbar\Omega_0}+\Pi_n(\mathbf{p},0)\right],$$
$$\alpha_H=-\Omega_0\frac{\partial\ln\Psi(\mathbf{k})}{\partial(\mathbf{k}^2)}\bigg|_{\mathbf{k}=0}$$
$$\delta=\frac{4}{3}\frac{\Psi}{\Omega_0^3}\int d\tau v^4\left[\frac{2s_0(\mathbf{p})}{\hbar\Omega_0}+\Pi_n(\mathbf{p},0)-\frac{\hbar^2\Omega_0^2}{12}\frac{\partial^2\Pi_n(\mathbf{p},0)}{\partial\varepsilon_0^2}\right], \qquad (4.6)$$
$$\delta_H=-\frac{5}{3}\Omega_0\frac{\partial\ln\Psi(\mathbf{k})}{\partial(\mathbf{k}^4)}\bigg|_{\mathbf{k}=0},\quad \varepsilon=-\frac{2}{3}\frac{\Psi}{\Omega_0}\int d\tau\, v^2\Pi_n(\mathbf{p},0).$$

In this case we assume that the characteristic wavelength of the magnons is greater than the Debye radius of the electrons $(kr_D)^2, \varkappa T r_D{}^2/\hbar\alpha \ll 1$. We see here the qualitative difference between our approach and the approach used in Refs. 7–13, which totally neglects the long-range electron Coulomb interaction, and therefore applies only to the description of the role of the magnons with wavelength greater than the Coulomb-field-shielding distance.

Let us illustrate here the results obtained above for the temperature dependence of the magnon frequency in two particular cases: the cases of strong and weak ferromagnets. In the limit of a strong ferro-magnet (i.e., for $n_0^-(p) = 0$), we have

$$\frac{1}{9}t_{iijj}=-\delta_H-\frac{\hbar\alpha_H^2}{4\eta}-\frac{\hbar^2\alpha_H}{4m\eta}\left[1+\frac{6}{5}\frac{\eta}{\hbar\Omega_0}-\frac{32}{15}\left(\frac{\eta}{\hbar\Omega_0}\right)^2\right]$$
$$-\frac{\hbar^3}{12m^2\eta}\left[1-\frac{6}{5}\frac{\eta}{\hbar\Omega_0}+\frac{56}{75}\left(\frac{\eta}{\hbar\Omega_0}\right)^2-\frac{96}{175}\left(\frac{\eta}{\hbar\Omega_0}\right)^3\right] \qquad (4.7)$$

where the chemical potential η of the electrons (holes) is measured from the bottom (top) of the partially filled electron band.

In the particular case of, say, nickel, we find after setting, in accordance with Refs. 11 and 21,

$$m\approx5.5m_0,\quad \hbar\Omega_0\approx0.8\text{ eV},\quad \eta\approx0.44\text{ eV},\ \hbar\alpha\approx0.391\text{ eV}\cdot\text{Å}^2,$$
$$\hbar\alpha_H\approx6\cdot10^{-3}\text{ eV}\cdot\text{Å}^2,\quad \hbar\delta_H\approx7.8\cdot10^{-2}\text{ eV}\cdot\text{Å}^4$$

(m_0 is the free-electron mass) that $B = 0.986\times10^{-8}\,\text{K}^{-5/2}$.

In the case of a weak magnet ($2s_0/n \ll 1$, substituting into Eq. (4.4) the expression $t_{iijj} = 72(\varepsilon_F/\hbar\Omega_0)^2(\alpha+2\alpha_H)\alpha/\Omega_0$, which follows from (4.5) and (4.6), we arrive at the expression

$$B=-\frac{\sqrt{\pi}}{16}\zeta\left(\frac{5}{2}\right)\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2\frac{\alpha+2\alpha_H}{\alpha-\alpha_H}\left(\frac{\varkappa}{\hbar\omega_{max}}\right)^{5/2}, \qquad (4.8)$$

where $\alpha = \alpha_H + \hbar^2\Omega_0/24m\varepsilon_F$ and ε_F is the Fermi energy. For example, in the particular case of the weak ferromagnet MnSi, setting, in accordance with Ref. 22, $\hbar\omega_{\max}\approx3\times10^{-3}$ eV, $\hbar\Omega_0/\varepsilon_F\approx0.4$, and $\alpha_H = 0$, we find $B\approx-3.3\times10^{-6}$ $\text{K}^{-5/2}$.

Comparing the terms AT^2 and $BT^{5/2}$, which, according to (1.1), determine the Fermi-excitation- and magnetic-fluctuation-governed temperature dependence of the magnon spectrum, we see that in strong magnets, where $A = \pi^2\varkappa^2/6am\eta\Omega_0$ (Ref. 4), these terms are, generally speaking, of the same order of magnitude in a broad range of temperatures. In this case the temperature dependence of the magnon spectrum is determined by the fluctuation effects only in the region of sufficiently high temperatures $T > T_s = A^2/B^2$. For example, in the case of nickel estimates yield $A\approx1.2\times10^{-7}\,\text{K}^{-2}$ and $T_s\approx150$ K.

For the case of weak magnets, it is found in Ref. 4 that $A = 2(\pi\varkappa/\hbar\Omega_0)^2$. In this case the temperature region, $T > T_s$, where the effect of the magnon fluctuations is decisive is substantially broader than in the case of strong magnets: $\varkappa T_s \approx 3\cdot10^{-3}(\hbar\Omega_0/\varepsilon_F)^4\hbar\omega_{\max}$ (e.g., in MnSi we have $T_s\approx3\times10^{-3}$ K).

Let us give here, for the case of a weak ferromagnet, the explicit expressions for the coefficients determining, in accordance with (3.21), the temperature dependence of the parameters of the equilibrium state:

$$a_\varepsilon(\mathbf{p})=-\frac{\varepsilon_F}{2}\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2\frac{a_l}{s_0},\quad a_\Omega(\mathbf{p})=\frac{\hbar\Omega_0}{2}\left(1-\frac{v^2}{3\gamma\Omega_0^2}\right)\frac{a_l}{s_0},$$
$$a_n(\mathbf{p})=\frac{\varepsilon_F}{8}\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2\left[\Pi_n(\mathbf{p},0)+2\varepsilon_F\frac{\partial\Pi_n(\mathbf{p},0)}{\partial\varepsilon_0}\right]\frac{a_l}{s_0},$$
$$a_s(\mathbf{p})=\frac{s_0(\mathbf{p})}{s_0}a_s,$$
$$a_\eta=-\frac{\varepsilon_F}{8}\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2\frac{a_l}{s_0}, \qquad (4.9)$$
$$a_s=-\frac{\sqrt{\pi}}{16}\zeta\left(\frac{3}{2}\right)\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2 s_0\left(\frac{\varkappa}{\hbar\omega_{max}}\right)^{3/2},$$
$$a_l=-\frac{3\sqrt{\pi}}{32}\zeta\left(\frac{5}{2}\right)\left(\frac{\hbar\Omega_0}{\varepsilon_F}\right)^2\frac{s_0\gamma\omega_{max}}{\alpha}\left(\frac{\varkappa}{\hbar\omega_{max}}\right)^{5/2},$$

where $\gamma_{ij}=\gamma\delta_{ij}$, $\gamma=12(\varepsilon_F/\hbar\Omega_0)^2\alpha\Omega_0=\alpha/4\omega_{\max}$. Comparison of the fluctuation corrections given by Eqs. (3.21), (4.9) with the corresponding Fermi-excitation-related corrections[2,3] $\sim T^2$ shows that the temperature dependences of the energy, the chemical potential, the component $n(\mathbf{p})$ of the density matrix of the quasiparticles, and the mean square spin density S_l^2 in the temperature range $\varkappa T_s<\varkappa T<\hbar\omega_{\max}$ are, as found above in the analysis of the magnon frequency, determined by the effect of the fluctuations. The magnon-governed temperature dependence ($\sim T^{3/2}$) of the density-matrix component $s(\mathbf{p})$ and the spin density s is decisive in the entire low-temperature region $\varkappa T<\hbar\omega_{\max}$ (cf. Ref. 23).

Let us compare the results obtained above for the temperature dependence of the magnon frequency with the results obtained in Refs. 7–13, in which the effect of the magnon-magnon interaction on the spin-wave spectrum in ferromagnetic metals is treated without allowance for the long-range electron Coulomb interaction. Let us first of all note that, in our approach, to the models based on the Hubbard Hamiltonian, and used in Refs. 7–11 and 12, correspond the equalities $\phi(\mathbf{k})=\mp\Psi(\mathbf{k})=\text{const}$, the analyses in these papers being limited to the case of strong magnets. To the model proposed in Ref. 13, in which the electron charge density oscillations are neglected, corresponds the passage to the limit $\phi(\mathbf{k})\rightarrow 0$ in the present approach.

To obtain from the general relations (2.17) and (4.1) the results of Refs. 7–10 regarding the temperature dependence, $\sim T^{5/2}$, of the magnon spectrum, we must use, besides the indicated equalities corresponding to the neglect of the long-range Coulomb interaction, additional approximations not discussed in those papers. Thus, the results of Izuyama,[7] Kawasaki,[8] Edwards and Fisher,[9] and Morkowski[10] do not fully take account of the interaction of the quasiparticles with the charge and spin densities, and are obtained in our approach if, in determining the kernels T, we neglect the terms containing $\delta\varepsilon^{\sigma\sigma}(k)$ in the equations (2.3) for the diagonal components of the density matrix. This neglect, exposed by us, is due to the unjustified approximate introduction of the normal coordinates of the magnons in Refs. 7–10. In so doing, instead of the first term in the parentheses in our formula (4.3), Izuyama,[7] Kawasaki,[8] Edwards and Fisher,[9] and Morkowski[10] obtained a term differing from it by the replacement $\alpha_{ij}\rightarrow\alpha-\hbar/2m$, and this contradicts the Landau-Lifshitz equation. Let us note in this connection that the use of this term in Morkowski's paper[10] leads to an incorrect expression for the magnon-damping constant resulting from the four-magnon scattering processes. The correct results are arrived at through the above-indicated substitution.

The deviation of the expression obtained by Korenman *et al.*[11] for the coefficient B in (1.1) from the expression that follows from our equations (2.17) and (4.1) upon the neglect of the long-range Coulomb interaction is due to the use in Ref. 11 of the local-band theory approximation, in which the Landau-Lifshitz equation is postulated for the magnetization. On the other hand, as follows from the relations (2.17) and (4.1), the fluctuation-governed temperature dependence of the magnon spectrum requires that we make a consistent allowance for the higher derivatives, the consideration of which falls outside the limits of applicability of the Landau-Lifshitz equation. Further, the use in Ref. 11 of the Landau-Lifshitz equation does not allow the description of the effects connected with the variation of the square of the spin density, effects which manifest themselves in the deviation of the kernel $\Gamma^s(k,k')$ from the quantity $-(2s_0)^{-1}$. As shown in Refs. 19 and 20, in the limit of long-wave fluctuations, this deviation is described by terms that are quadratic in $\mathbf{k}$ and/or $\mathbf{k}'$, and have, for example, in strong magnets, in which a band with one electron-spin direction is empty $n_0^-(\mathbf{p})=0$), the form[19]

$$(2s_0)^{-1}\gamma_{ij}(\mathbf{k}-\mathbf{k}')_i k_j',\quad \gamma_{ij}=(2s_0\Omega_0^2)^{-1}\int d\tau\, v_i v_j n_0^+(\mathbf{p}).$$

According to Eqs. (2.17) and (4.1), those terms which do not conserve the square of the spin density play an important role in the determination of the temperature dependence, $\sim T^{5/2}$, of the magnon spectrum.

Concerning the paper by Corrias and Pascuale,[12] let us note that, in summing the diagrammatic series for the magnon frequency, these authors used unestablished approximations involving the arbitrary neglect of a series of diagrams, and this may be the cause of the deviation of the expression obtained by them for B from the expression that is obtained from the relations (2.17) and (4.1) with the use of the condition $\phi(\mathbf{k})=\Psi(\mathbf{k})=\text{const}$. Summarizing all the foregoing, we can assert that the above-proposed quantum-dynamical approach to the theory of the temperature dependence of the magnon spectrum enables us to rid ourselves of the additional unjustified assumptions of Refs. 7–13, and consistently take account of the role of the long-range Coulomb interaction and the magnetic fluctuations that do not conserve the square of the spin density.

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