Inelastic scattering of neutrons by a soliton magnetic lattice

Yu A. Izyumov and V. M. Laptev

Institute for the Physics of Metals of the Ural Science Center

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A spiral structure to which a magnetic field is applied in a plane at right angles to the wave vector or whose temperature is changed, can be continuously transformed by the natural anisotropy in the same plane into a commensurate phase. It is well known that if the symmetry of the order parameter allows a Lifshitz invariant that is linear in the gradient, the incommensurate structure near the commensurate phase is a lattice of solitons. We consider the inelastic magnetic scattering of neutrons by the normal vibration modes of this lattice which correspond to a single-band periodic potential. We have calculated the scattering cross section, for the cases of a magnetic field and of a second-order natural anisotropy, exactly in terms of elliptic functions. The exact solution enables us to trace the evolution of the inelastic scattering pattern when the field or the temperature change in the whole range where the incommensurate phase exists, including the transition to the commensurate phase itself. We show, in particular, that in the vicinity of the transition diffuse scattering occurs due to the continuous decrease in the wave vector of the structure. The specific picture of the scattering in the (ω,q)-plane enables us to verify experimentally the soliton theory of the incommensurate phase.

I. INTRODUCTION

The spiral modulation of ferro- and anti-ferromagnetic structures in crystals is, as a rule, caused by the competition between exchange interactions of opposite sign, but there are a number of examples where the spiral magnetic structures are due to anisotropic interactions described by Lifshitz invariants which are linear in the gradient.[1] Dzyaloshinskiĭ[2] has shown that in the case when the temperature or the magnetic field is changed there occurs a phase transition from an incommensurate phase which is a simple spiral kind of structure to a commensurate one, and that in the vicinity of the transition the incommensurate phase is described as a lattice of solitons. Pokrovskiĭ and Talapov,[3,4] and also Bulaevskiĭ and Khomskiĭ,[5] have studied the excitation spectrum of the soliton lattice and showed that the periodic potential determining the motion of the fluctuations in the soliton lattice is a single-band one and the excitation spectrum is described by the Lamé equation with band index $l = 1$.

The aim of the present paper is a study of the scattering of neutrons by the vibrational modes of the soliton lattice. It turned out that thanks to the single-band nature of the potential not only the problem of the spectrum of the fluctuations[3,4] but also the evaluation of the cross section for inelastic scattering by them can be solved exactly in terms of elliptic functions. The exact solution enables us to trace the evolution pattern of the inelastic scattering when one changes the magnetic field or the temperature, right up to the transition to the commensurate phase, in a similar way as we traced earlier[6] the evolution of elastic scattering by the soliton lattice.

We now consider the typical situation when the modulated structure is described by a two-component order parameter (OP) $(\eta = \xi^*)$ and by a Ginzburg-Landau functional of the following form:

$$\Phi=\frac{1}{V}\int d\mathbf{r}\Big\{r(\eta\xi)+u(\eta\xi)^2+i\sigma\Big(\eta\frac{\partial\xi}{\partial z}-\xi\frac{\partial\eta}{\partial z}\Big)+\gamma\frac{\partial\eta}{\partial z}\frac{\partial\xi}{\partial z}+\gamma_\perp\Big(\frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial y}\Big)+w(\eta^n+\xi^n)\Big\}. \quad (1.1)$$

Such a functional describes, in particular, the spiral magnetic structure in a uniaxial crystal with the wave vector along the main axis (the z axis) and an nth order anisotropy in the base plane while the components of the local magnetic moment M are connected with the OP by the relations

$$M_+=\eta,\quad M_-=\xi,\quad M_z=0. \quad (1.2)$$

The conditions for stability of such a structure require that $u>0$, $\gamma>0$, and $\gamma_\perp>0$, while the parameter σ can have any sign and the parameter $r\propto(T-T_c)$. We shall assume, without loss of generality, that the parameter w is positive.

The approximation

$$\eta=\rho e^{i\varphi},\quad \xi=\rho e^{-i\varphi},\quad \rho=\text{const}, \quad (1.3)$$

in which we assume that the modulus of the OP is independent of the coordinates, reduces (1.1) to a functional depending only on the phase $\varphi(\mathbf{r})$, while the equilibrium distribution $\bar\varphi(z)$ of the OP can be expressed in terms of a function of the amplitude[2]

$$\bar\varphi(z)=\frac{2}{n}\,\mathrm{am}\Big(\frac{v^{1/2}}{\varkappa}z,\varkappa\Big). \quad (1.4)$$

One finds the modulus $\varkappa$ of the elliptic functions from the condition that Φ be a minimum

$$E/\varkappa=(v_c/v)^{1/2},\quad v_c=n^2\pi^2\sigma^2/16\gamma^2,\quad v=n^2w\rho^{n-2}/\gamma, \quad (1.5)$$

where E (and later K) are complete elliptic integrals of the second and first kind.

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One can easily write the Lagrange equation for the fluctuation $\delta\varphi$ of the OP for a given equilibrium distribution $\bar{\varphi}$. If we write $\delta\varphi$ in the form

$$\delta\varphi(\mathbf{r},\ t)=e^{i\omega t}\exp\{i(q_x x+q_y y)\}\psi(z), \tag{1.6}$$

the quantity ψ will satisfy the Lamé equation with band index $l=1$:[3]

$$d^2\psi/du^2=[2\varkappa^2\,\mathrm{sn}^2(u,\ \varkappa)-\varkappa^2-\varepsilon]\psi, \tag{1.7}$$

where $u=(v^{1/2}/\varkappa)z$ is the dimensionless coordinate and

$$\varepsilon=\frac{1}{\gamma}\frac{\varkappa^2}{v}[\mu\omega^2-\gamma_\perp(q_x^2+q_y^2)]. \tag{1.8}$$

The parameter μ occurs in the expression for the kinetic energy

$$T=\frac{1}{V}\int d\mathbf{r}\mu\dot{\eta}\dot{\xi}.$$

The spectrum of the eigenvalues ε is implicitly determined from two equations[3] which can conveniently be written in terms of the Jacobi functions sn u, cn u, dn u and the zeta function $Z(u)$. For the first band $(0<\varepsilon<\varkappa'^2\equiv + -\varkappa^2, |q|<(\varkappa/v^{1/2})Q_0\equiv\pi/2K)$ these equations have the form

$$\varepsilon=\varkappa'^2\,\mathrm{sn}^2(\alpha,\ \varkappa'),\qquad q=Z(\alpha,\ \varkappa')+\pi\alpha/2KK', \tag{1.9}$$

and for the second band $(1<\varepsilon<\infty, |q|>(\varkappa/v^{1/2})Q_0)$

$$\varepsilon=\frac{1}{\mathrm{sn}^2(\alpha,\varkappa')},$$

$$q=Z(\alpha,\varkappa')+\frac{\mathrm{cn}(\alpha,\varkappa')\,\mathrm{dn}(\alpha,\varkappa')}{\mathrm{sn}(\alpha,\varkappa')}+\frac{\pi\alpha}{2KK'}. \tag{1.10}$$

In both cases α is a real parameter $(-K'<\alpha<K')$ and by eliminating it we obtain the dispersion curve ($q\equiv(\varkappa/v^{1/2})q_z$ is the dimensionless momentum). All elliptic functions are written in standard notation.[7] We show the shape of the spectrum in Fig. 1, where the wave functions for the limiting points a, b, and c are

$$\psi_a(u)\sim\mathrm{dn}\,u,\quad \psi_b(u)\sim\mathrm{cn}\,u,\quad \psi_B(u)\sim\mathrm{sn}\,u. \tag{1.11}$$

These expressions are special cases of the general expression for the eigenfunction of the Lamé equation:

$$\psi(u)=e^{iqu}\theta_4\left(\frac{\pi}{2K}(u-u_0)\right)\Big/\theta_4\left(\frac{\pi}{2K}u\right), \tag{1.12}$$

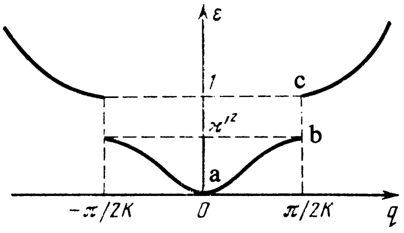

FIG. 1. Vibration spectrum of the soliton lattice (in the dimensionless quantities ε and q).

where $u_0=i\alpha+K$ for the first band and $u=i\alpha$ for the second band.

Formulae (1.6) and (1.12) give the wave function of the normal vibrational modes of the soliton lattice and we can use them to evaluate the cross section for inelastic scattering by these modes. It is useful first to introduce the density of states for one-dimensional motion in the direction of the z axis, $N(\varepsilon)=dq/\varepsilon$. Using Eqs. (1.9) and (1.10) to eliminate α from the expression $N(\varepsilon)=(dq/d\alpha)(d\varepsilon/d\alpha)$ we get for the first and second bands the simple formulae

$$N_1(\varepsilon)=\frac{E/K-\varepsilon}{2[\varepsilon(1-\varepsilon)(\varkappa'^2-\varepsilon)]^{1/2}}$$

$$N_2(\varepsilon)=\frac{\varepsilon-E/K}{2[\varepsilon(\varepsilon-1)(\varepsilon-\varkappa'^2)]^{1/2}}, \tag{1.13}$$

showing the square root singularity at the boundaries of both bands. If there is no anisotropy (at $\varkappa=0$) they go over into the well known formula $N(\varepsilon)\,1/2\varepsilon^{1/2}$ for the one-dimensional motion of free particles and have, close to the phase transition to the commensurate structure (as $\varkappa\to1$), the asymptotic form

$$N_1(\varepsilon)=\frac{1}{2\ln(4/\varkappa')}\frac{1}{[\varepsilon(\varkappa'^2-\varepsilon)]^{1/2}},\quad N_2(\varepsilon)=\frac{1}{2(\varepsilon-1)^{1/2}}. \tag{1.14}$$

The small logarithmic factor arises here from the well known asymptotic form of the complete elliptic integral $K(\varkappa)\approx\ln(4/\varkappa)$. The density of states of the three-dimensional motion of the model studied by us was evaluated in Ref. 8. We need in what follows just the density of states of the one-dimensional motion.

2. GENERAL EXPRESSION FOR THE INELASTIC SCATTERING CROSS SECTION

The cross section for the magnetic scattering of neutrons with an energy transfer ω and a momentum transfer $\mathbf{Q}$ can be expressed in terms of the Fourier transform of the magnetic-moment correlator:[1]

$$\frac{d^2\sigma}{d\mathbf{Q}d\omega}\propto\sum_{\substack{\gamma,\gamma'\\ \alpha,\beta}}K_{\gamma\gamma'}(\delta_{\alpha\gamma}-e_\alpha e_\gamma)(\delta_{\beta\gamma'}-e_\beta e_{\gamma'})$$
$$\times\langle M_\alpha(\mathbf{r},t)M_\beta(\mathbf{r}',0)\rangle_{Q,\omega}, \tag{2.1}$$

where

$$K_{\gamma\gamma'}=\overline{\sigma_\gamma\sigma_{\gamma'}}=\delta_{\gamma\gamma'}+i\sum_{\gamma''}\varepsilon_{\gamma\gamma'\gamma''}p_{\gamma''}^0,$$

$\mathbf{e}=\mathbf{Q}/Q$ is the unit scattering vector $\mathbf{p}^0$ the polarization vector of the incident neutron beam, and $\varepsilon_{\gamma\gamma'\gamma''}$ the unit antisymmetric tensor.

The spectrum of the fluctuations was determined in the preceding section for the case of a limitingly large "easy plane" type anisotropy, neglecting the departure of the magnetic moment from the basal plane so that for the consideration of the cross section in that limit we must also use expression (1.2) for the magnetic moment in terms of the OP. The fluctuations in the OP reduced in the approximation (1.3) to phase fluctuations $\delta\varphi$. The part of Eq. (2.1) which is quadratic in $\delta\varphi$ describes the inelastic scattering by the excitations of the soliton lattice. The scattering cross section is then determined by the correlator $\langle\delta\varphi(\mathbf{r},t)\delta\varphi(\mathbf{r}',0)\rangle$. To evaluate that quantity it is convenient to expand an arbitrary fluctuation in terms of the functions (1.6):

$$\delta\varphi(\mathbf{r},t)=\iint d\mathbf{q}\,d\omega\alpha(\mathbf{q},\omega)[\delta(\omega-\omega_{\mathbf{q}})+\delta(\omega+\omega_{\mathbf{q}})] \times e^{i\omega t}\exp\{i(q_x x+q_y y)\}\psi_{(\varkappa/v^{1/2})q_z}\left(\frac{v^{1/2}}{\varkappa}z\right), \quad (2.2)$$

where $\psi_q(u)$ is an eigenfunction of the Lamé equation and $\omega_{\mathbf{q}}$ the corresponding eigenvalue (it is found from Eq. (1.8) for ε). We express the energy functional (1.1) in terms of the amplitudes $\alpha(\mathbf{q},\omega)$ of this expansion. In the harmonic approximation

$$\Phi=\Phi_0+\int d\mathbf{q}2S(\mathbf{q})\mu\omega_{\mathbf{q}}^2\{|\alpha(\mathbf{q},\omega_{\mathbf{q}})|^2+|\alpha(\mathbf{q},-\omega_{\mathbf{q}})|^2\}, \quad (2.3)$$

where $S(\mathbf{q})$ is the normalization integral of the eigenfunctions $\psi_q(u)$:

$$\int\psi_q^*(u)\psi_{q'}(u)du=S(\mathbf{q})\delta(q+q'). \quad (2.4)$$

The averaging over the statistical ensemble with energy (2.3) is a Gaussian path integral over the quantities $\alpha(q,\omega)$, so that one can easily write down an expression for the correlator in which we are interested:

$$\langle\delta\varphi(\mathbf{r},t)\delta\varphi(\mathbf{r}',0)\rangle=\int d\mathbf{q}\frac{kT}{2\mu\omega_{\mathbf{q}}^2}(e^{-i\omega_{\mathbf{q}}t}+e^{i\omega_{\mathbf{q}}t}) \times\exp\{iq_x(x-x')+iq_y(y-y')\} \frac{1}{S(\mathbf{q})}\psi_{(\varkappa/v^{1/2})q_z}\left(\frac{v^{1/2}}{\varkappa}z\right)\psi_{-(\varkappa/v^{1/2})q_z}\left(\frac{v^{1/2}}{\varkappa}z'\right). \quad (2.5)$$

Working out the sum in Eq. (2.1) over the vector indexes α and β and using (2.5) we can write the cross-section for inelastic scattering by OP fluctuations in the following form:

$$\frac{d^2\sigma_{\text{inel}}}{dQd\omega}\sim\rho^2\int d\mathbf{q}\frac{kT}{2\mu\omega_{\mathbf{q}}^2S(\mathbf{q})}\delta(Q_x-q_x)\delta(Q_y-q_y)[\delta(\omega-\omega_{\mathbf{q}}) +\delta(\omega+\omega_{\mathbf{q}})]\{(1+e_z^2-2e_z(\mathbf{ep}_0))F^2(q_z,Q_z)+(1+e_z^2 +2e_z(\mathbf{ep}_0))F^2(-q_z,-Q_z)+[(e_x+ie_y)^2+(e_x-ie_y)^2] \times F(q_z,Q_z)F(-q_z,-Q_z)\}, \quad (2.6)$$

where the form factor of the normal vibrational mode of the soliton lattice is

$$F(q_z,Q_z)=\int dze^{-iQ_zz}e^{i\bar{\varphi}(z)}\psi_{(\varkappa/v^{1/2})q_z}\left(\frac{v^{1/2}}{\varkappa}z\right). \quad (2.7)$$

The remaining calculation of the cross section thus reduces to evaluating the quantity $F(q_z,Q_z)$. The result depends in an essential way on the anisotropy index n through Eq. (1.4) which determines the equilibrium structure. First of all we consider the case $n=1$ corresponding to a magnetic field applied to the basal plane. The calculation of expression (2.7) for $n=1$ and also for $n=2$ (second-order anisotropy in the basal plane) is completely rigorous for all values $0\leqslant\varkappa\leqslant1$; for $n>2$ it is possible to obtain an asymptotic expansion in the limit of small $\varkappa$.

3. SOLITON LATTICE PRODUCED IN A MAGNETIC FIELD

Let there therefore be a magnetic field applied along the $-\mathbf{x}$ axis and let there be no natural anisotropy in the basal plane. The field H is then connected with the parameter w from the functional (1.1) through the relation $H=2w$. The soliton lattice caused by the field is characterized by the elliptic modulus $\varkappa$ which can be found from the equation

$$E/\varkappa=(H_c/H)^{1/2}, \qquad H_c=(\pi^2\sigma^2/8\gamma)\rho, \quad (3.1)$$

where H_c is the critical field at which the spiral structure changes into the ferromagnetic (commensurate) structure induced by the field.

When $n=1$, by virtue of the definition (1.4)

$$e^{i\varphi(z)}=2\,\mathrm{cn}^2(u,\varkappa)-1+2i\,\mathrm{cn}(u,\varkappa)\,\mathrm{sn}(u,\varkappa). \quad (3.2)$$

Using this expression and Eq. (1.12) we see that the integrand in (2.7) is periodic along the real axis with period $2K$ so that it can be expanded in a Fourier series

$$e^{i\bar{\varphi}(z)}\theta_4\left(\frac{\pi}{2K}(u-u_0)\right)\Big/\theta_4\left(\frac{\pi}{2K}u\right)=\sum_{l=-\infty}^{\infty}a_l\exp\left\{il\frac{\pi}{K}u\right\}.$$

Thanks to this expression it is possible to write (2.7) in the form

$$F(q_z,Q_z)=\sum_l a_l\delta(Q_z-q_z-lk), \quad (3.3)$$

where $k=v^{1/2}\pi/\varkappa K$ is the wave vector of the structure. The series coefficients a_l are found by taking the inverse Fourier transforms, with the integration over the interval $-K\leqslant u\leqslant K$ reduced to a contour integral over a quasiperiodicity cell of the integrand[7] (Fig. 2):

$$a_l=\frac{1}{2K}\int_c du\left\{\frac{2\,\mathrm{cn}^2u-1}{1-\exp\{(\pi/2K)(2lK'+iu_0)\}} +i\frac{2\,\mathrm{cn}\,u\,\mathrm{sn}\,u}{1+\exp\{(\pi/2K)(2lK'+iu_0)\}}\right\} \times\left[\theta_4\left(\frac{\pi}{2K}(u-u_0)\right)\Big/\theta_4\left(\frac{\pi}{2K}u\right)\right]\exp\left\{-i\frac{\pi}{K}ul\right\}. \quad (3.4)$$

The integrand has a third-order pole in the point $u=iK'$.

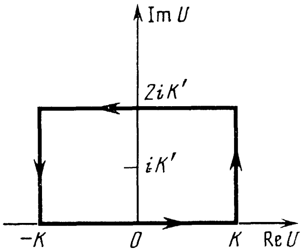

FIG. 2. Contour of integration in (3.4).

After evaluating the integral by the residue theorem we get

$$F(q_z, Q_z) = i2\left(\frac{\pi}{2K}\right)^{1/2} \frac{\theta_1(\pi u_0/2K)}{(\varkappa\varkappa')^{1/2}}\left[\frac{\mathrm{cn}^2 u_0}{\varkappa^2 \mathrm{sn}^2 u_0} + \frac{1}{v} Q_z^2\right]$$
$$\times \sum_l \frac{\exp\{(\pi/2K)(2lK'+iu_0)\}}{\mathrm{sh}\{(\pi/2K)(2lK'+iu_0)\}} \delta(Q_z - q_z - kl). \qquad (3.5)$$

After long transformations of the elliptic functions we can write the scattering cross section in the following form:

$$\frac{d^2\sigma_{\rm inel}}{d\mathbf{Q}\,d\omega} \infty \rho^2 \int d\mathbf{q} \frac{kT}{2\mu\omega_\mathbf{q}^2}[\delta(\omega-\omega_\mathbf{q}) + \delta(\omega+\omega_\mathbf{q})]$$
$$\times \delta(Q_x - q_x)\delta(Q_y - q_y)$$
$$\times \left(\frac{\pi}{K\varkappa^2}\right)^2 \frac{\left(\varepsilon - \frac{\varkappa^2}{v} Q_z^2\right)^2}{|E/K - \varepsilon|} \sum_{l=-\infty}^{\infty} J_l(q_z)\delta(Q_z - q_z - lk), \qquad (3.6)$$

where

$$J_l(q_z) = \begin{cases} S_{2l}^2(1-e_x^2) + C_{2l}^2(1-e_y^2) + 2S_{2l}C_{2l}e_z(\mathbf{e}\mathbf{p}_0), & |q_z| < Q_0, \\ C_{2l}^2(1-e_x^2) + S_{2l}^2(1-e_y^2) + 2S_{2l}C_{2l}e_z(\mathbf{e}\mathbf{p}_0), & |q_z| > Q_0, \end{cases} \qquad (3.7)$$

$$S_p = \mathrm{sh}^{-1}\left(\frac{\pi}{2K}(pK'+\alpha)\right), \quad C_p = \mathrm{ch}^{-1}\left(\frac{\pi}{2K}(pK'+\alpha)\right). \qquad (3.8)$$

In obtaining this expression we used Eqs. (1.9) and (1.10). We note that when $n = 1$ the wave vector corresponding to the break in the spectrum is $Q_0 = k/2$.

In the framework of our adopted model Eq. (3.6) gives an exact expression for the cross section for a spiral structure in a magnetic field. In order to get the explicit ω — and $\mathbf{Q}$-dependences of the cross section it is necessary to use Eqs. (1.9) and (1.10) to eliminate the parameter α from the expressions for S_{2l} and C_{2l}. This can be done in analytical form in the two limiting cases, $\varkappa \to 0$ and $\varkappa \to 1$, corresponding to a weak and a strong field.

As $\varkappa \to 0$ one obtains easily the asymptotic behavior of the quantities determining the scattering intensity in the lth magnetic Brillouin zone. We have

$$\mathrm{ch}\left\{\frac{\pi}{2K}(2lK'+\alpha)\right\} \approx \mathrm{sh}\left\{\frac{\pi}{2K}(2lK'+\alpha)\right\}$$
$$\approx \frac{1}{2}\left(\frac{4}{\varkappa}\right)^{2l}\left(\frac{1+\mathrm{th}(\pi\alpha/2K)}{1-\mathrm{th}(\pi\alpha/2K)}\right)^{1/2} \qquad (l>0)$$

and a similar expression for $l < 0$. We can obtain the analytical q_z-dependence of α and ε only in two limiting cases:

far from the gap in the spectrum, when $\delta \equiv |1 - q_z^2/Q_0^2| \gg \varkappa^2$:

$$\mathrm{th}\frac{\pi\alpha}{2K} = \frac{q_z}{Q_0} + O(\varkappa^4), \quad |q_z| < Q_0,$$
$$\mathrm{cth}\frac{\pi\alpha}{2K} = \frac{q_z}{Q_0} + O(\varkappa^4), \quad |q_z| > Q_0, \qquad (3.9)$$
$$\varepsilon = \frac{q_z^2}{Q_0^2}\left[1 - \frac{\varkappa^2}{2} - \frac{3\varkappa^4}{32} - \frac{\varkappa^4}{8}\frac{1}{1-q_z^2/Q_0^2}\right],$$

and close to the gap when $\delta \lesssim \varkappa^2$:

$$\left|\mathrm{th}\frac{\pi\alpha}{2K}\right| = 1 - \frac{\delta}{4} - \frac{1}{4}\left(\delta^2 + \frac{\varkappa^4}{4}\right)^{1/2}, \quad |q_z| < Q_0,$$
$$\left|\mathrm{cth}\frac{\pi\alpha}{2K}\right| = 1 + \frac{\delta}{4} + \frac{1}{4}\left(\delta^2 + \frac{\varkappa^4}{4}\right)^{1/2}, \quad |q_z| > Q_0, \qquad (3.10)$$
$$\varepsilon = 1 - \frac{\varkappa^2}{2} \pm \left(\delta^2 + \frac{\varkappa^4}{4}\right)^{1/2}.$$

We consider separately the contributions to the scattering cross section (3.6) from two regions of $\mathbf{q}$-space corresponding to scattering by modes with momenta far from the momentum at the break in the spectrum and to momenta close to it. Using the available asymptotic behavior the first contribution is given by the expression

$$\left.\frac{d^2\sigma_{\rm inel}}{d\mathbf{Q}\,d\omega}\right|_{\delta\gg\varkappa^2} \infty \rho^2 \int_{\delta\gg\varkappa^2} d\mathbf{q} \frac{kT}{2\mu\omega_\mathbf{q}^2}[\delta(\omega-\omega_\mathbf{q}) + \delta(\omega+\omega_\mathbf{q})]$$
$$\times \delta(Q_x - q_x)\delta(Q_y - q_y)$$
$$\times\left\{\frac{\varkappa^4}{16}\frac{Q_0^2 q_z^2}{(Q_0^2 - q_z^2)^2}\left[(1-e_x^2) + \left(\frac{q_z}{Q_0}\right)^2\right.\right.$$
$$\left.\times(1-e_y^2) + 2\left(\frac{q_z}{Q_0}\right)e_z(\mathbf{e}\mathbf{p}_0)\right]\delta(Q_z - q_z)$$
$$+ \sum_{l=1}^{\infty}\frac{1}{16}\left(\frac{\varkappa^2}{16}\right)^{2(l-1)}\frac{(q_z^2 - Q_z^2)^2}{Q_0^2}\left[\frac{1}{(q_z+Q_0)^2}(1+e_z^2+2e_z(\mathbf{e}\mathbf{p}_0))\right.$$
$$\times \delta(Q_z - q_z - lk)$$
$$\left.\left.+ \frac{1}{(q_z - Q_0)^2}(1+e_z^2 - 2e_z(\mathbf{e}\mathbf{p}_0))\delta(Q_z - q_z + lk)\right]\right\}. \qquad (3.11)$$

The second contribution is given by the formula

$$\left.\frac{d^2\sigma_{\rm inel}}{d\mathbf{Q}\,d\omega}\right|_{\delta\lesssim\varkappa^2} \infty \rho^2 \int_{\delta\lesssim\varkappa^2} d\mathbf{q} \frac{kT}{2\mu\omega_\mathbf{q}^2}[\delta(\omega-\omega_\mathbf{q}) + \delta(\omega+\omega_\mathbf{q})]$$
$$\times \delta(Q_x - q_x)\delta(Q_y - q_y)$$
$$\times\left\{A_-(\delta)(1+e_z^2+2e_z(\mathbf{e}\mathbf{p}_0))\delta(Q_z - q_z) + A_+(\delta)\right.$$
$$\times(1+e_z^2 - 2e_z(\mathbf{e}\mathbf{p}_0))\delta(Q_z - q_z + k)$$
$$+ \left(1 - \frac{Q_z^2}{Q_0^2}\right)^2 A_+(\delta)(1+e_z^2+2e_z(\mathbf{e}\mathbf{p}_0))\delta(Q_z - q_z - k)$$
$$+ \sum_{l=2}^{\infty}\left(\frac{\varkappa^2}{16}\right)^{2(l-1)}\left(1 - \frac{Q_z^2}{Q_0^2}\right)^2\left[A_+(\delta)(1+e_z^2+2e_z(\mathbf{e}\mathbf{p}_0))\right.$$
$$\times\delta(Q_z - q_z - lk)$$
$$\left.\left.+ \left(\frac{16}{\varkappa^2}\right)^2 A_-(\delta)(1+e_z^2 - 2e_z(\mathbf{e}\mathbf{p}_0))\delta(Q_z - q_z + lk)\right]\right\},$$
$$A_\pm(\delta) = 1 \pm \delta/(\delta^2 + {}^1/_4\varkappa^4)^{1/2}. \qquad (3.12)$$

We note that (3.12) is valid only when $q_z \approx Q_0$; the expression for $q_z \approx -Q_0$ is obtained from (3.12) by the formal substitution $\mathbf{p}_0 \to -\mathbf{p}_0$, $k \to -k$.

Expressions (3.11) and (3.12) describe the scattering cross section in weak fields $H \ll H_c$, since we have $\varkappa^2 = (\pi^2/4)(H/H_c)$ for $\varkappa \ll 1$. The formulae introduced here show the existence of inelastic-scattering peaks in different magnetic Brillouin zones and a reduction in their intensity proportional to H^{2l}, when the number l of the zone increases. The pat-

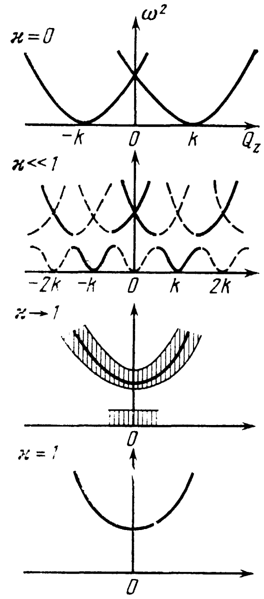

FIG. 3. Evolution of the pattern of inelastic scattering in the case $n = 1$ when $\varkappa$ changes from 0 $(H = 0)$ to 1 $(H = H_c)$.

tern of the scattering in the (ω^2, Q_z) plane which follows from these formulae is shown in Fig. 3. The lines show here the locations of the inelastic peaks. The heavy lines correspond arbitrarily to unity, intensity and the dashed ones to intensities reduced by factor $\varkappa^4$, $\varkappa^8$, and so on. For small $\varkappa$, the sections having unity intensity correspond to the dispersion curve centered on the wave vectors $Q_z = \pm k$. On the curves centered in the neighboring magnetic-lattice points lattice ($Q_z = 0$, $\pm 2k$) the intensity is $\propto \varkappa^4$, and so on. Near the points where there is a break in the spectrum ($Q_z = \pm k/2$) the intensity changes smoothly thanks to the branch hybridization described by Eq. (3.12).

When $\varkappa$ increases the wave vector k of the structure decreases and the whole pattern is compressed towards the ordinate axis. In the limit as $\varkappa \to 1$ there occurs a complicated distribution of the scattering intensity in regions of the (ω, Q_z) plane. The whole of the asymptotic behavior is in an essential way determined by the known asymptotic behavior of the elliptic integral K, in particular, the wave vector $k = \pi q_0/\ln(4/\varkappa')$, where the parameter $q_0 = (\pi/4)(|\sigma|/\gamma)$ is of the order of the wavevector of the spiral for $\varkappa = 0$. The asymptotic form of Eqs. (1.9) and (1.10) which parametrically determine the dispersion laws in the two zones has the form

$$\frac{q_z}{q_0} = \frac{\alpha}{\ln(4/\varkappa')} + O(\varkappa'^2), \quad \varepsilon = O(\varkappa'^2), \quad (|q_z| < Q_0) \tag{3.13}$$

and

$$\frac{q_z}{{}^1\!/_2\pi q_0} = \operatorname{ctg}\alpha + \frac{\alpha}{\ln(4/\varkappa')} + O(\varkappa'^2),$$
$$\varepsilon = 1 + \left(\frac{q_z}{q_0}\right)^2 - \frac{\pi q_z}{q_0}\frac{\operatorname{arcctg}(q_z/q_0)}{\ln(4/\varkappa')}, \quad |q_z| > Q_0. \tag{3.14}$$

As we have $k \to 0$ when $\varkappa \to 1$ we can change in the sum over l in Eq. (3.6) to an integral which can be removed thanks to the δ-function of the momenta. We can get rid of the integration over q_z in (3.12) through $\delta(\omega \pm \omega_q)$ after introducing the density of states in the spectrum:

$$N(\omega^2) = \frac{dq_z}{d\omega^2} = \frac{\mu}{\gamma}\frac{\varkappa}{v^{1/2}} N(\varepsilon), \tag{3.15}$$

where the quantity $N(\varepsilon)$ in both zones is given by Eq. (1.13). As a result we get for the cross section for scattering by the modes belonging to the first and second zones, respectively

$$\frac{d^2\sigma_{\text{inel}}}{d\mathbf{Q}\,d\omega} \propto \rho^2 \frac{kT}{\mu|\omega|} N_1(\omega^2) \frac{\pi Q_z^4}{q_0^3} \left\{ \frac{1-e_x^2}{\operatorname{sh}^2(\pi Q_z/2q_0)} + \frac{1-e_y^2}{\operatorname{ch}^2(\pi Q_z/2q_0)} + \frac{2e_z(\mathbf{e}\mathbf{p}_0)}{\operatorname{sh}(\pi Q_z/2q_0)\operatorname{ch}(\pi Q_z/2q_0)} \right\}, \tag{3.16}$$

$$\begin{aligned}
\frac{d^2\sigma_{\text{inel}}}{d\mathbf{Q}\,d\omega} \propto{}& \rho^2 \int d\mathbf{q} \frac{kT}{2\mu\omega_q^2}(1-e_x^2)\delta(\mathbf{Q}-\mathbf{q}) \\
&\times[\delta(\omega-\omega_{\mathbf{q}})+\delta(\omega+\omega_{\mathbf{q}})] + \rho^2 \frac{kT}{\mu|\omega|} N_2(\omega^2) \\
&\times \frac{\pi}{\ln(4/\varkappa')} \frac{[q_0^2+q_z^2(\omega)-Q_z^2]^2}{[q_0^2+q_z^2(\omega)]q_0^3} \\
&\times\Big[(1-e_x^2)\operatorname{sh}^{-2}\left(\frac{\pi}{2}\frac{Q_z-q_z(\omega)}{q_0}\right) \\
&+(1-e_y^2)\operatorname{ch}^{-2}\left(\frac{\pi}{2}\frac{Q_z-q_z(\omega)}{q_0}\right) \\
&+2e_z(\mathbf{e}\mathbf{p}_0)\operatorname{sh}^{-1}\left(\frac{\pi}{2}\frac{Q_z-q_z(\omega)}{q_0}\right) \\
&\times\operatorname{ch}^{-1}\left(\frac{\pi}{2}\frac{Q_z-q_z(\omega)}{q_0}\right)\Big].
\end{aligned} \tag{3.17}$$

Here

$$q_z(\omega) = \left[\frac{\mu}{\gamma}\omega^2 - q_0^2 - \frac{\gamma_\perp}{\gamma}(Q_x^2+Q_y^2)\right]^{1/2}. \tag{3.18}$$

Formula (3.16) shows that there is in the (ω^2, Q_z)-plane a continuous intensity distribution around $Q_z = 0$ with a width $2q_0/\pi$ in the variable Q_z and around $\omega^2 = (\delta/\mu)(Q_x^2 + Q_y^2)$ one with a width $(\gamma v/\mu)(\varkappa'^2/\varkappa^2)$ in the variable ω^2. The intensity of this contribution tends to zero as $(\ln^{-1}(4/\varkappa'))$. A similar contribution is contained also in Eq. (3.17) but it occurs in the vicinity of the singular contribution which is approximately described by the first term. In that term we neglected a decrease by a factor $\sim (\ln^{-1}(4/\varkappa'))$ due to the transition of part of the scattering to the second diffusive term. When $\varkappa = 1$ the whole expression (3.16) and the second term in (3.17) vanish so that the scattering is completely described by the first term of (3.17) and corresponds to the inelastic scattering by excitations of the commensurate ferromagnetic phase which has a frequency spectrum

$$\mu\omega_q^2 = \frac{\pi^2\sigma^2}{4\gamma} + \gamma q_z^2 + \gamma_\perp(q_x^2+q_y^2). \tag{3.19}$$

Figure 3 shows the evolution of the inelastic scattering in the whole range where the incommensurate phase exists—from the spiral structure in zero field to the ferromagnetic structure in the critical field.

4. SOLITON LATTICE RESULTING TO NATURAL ANISOTROPY

When $n>1$ the last term in the functional (1.1) describes the natural anisotropy in the base plane. Unfortunately we can obtain an exact analytical evaluation of the form-factor (2.7) only for $n=2$ when

$$e^{i\bar{\varphi}(z)}=\mathrm{cn}(u,\ \varkappa)+i\,\mathrm{sn}(u,\ \varkappa). \tag{4.1}$$

The calculations for the scattering cross section lead to the following formula which is formally analogous to (3.6):

$$\frac{d^2\sigma_{\text{inel}}}{d\mathbf{Q}\,d\omega}\propto\rho^2\int d\mathbf{q}\,\frac{kT}{2\mu\omega_{\mathbf{q}}^2}[\delta(\omega-\omega_{\mathbf{q}})+\delta(\omega+\omega_{\mathbf{q}})]\times\delta(Q_x-q_x)\,\delta(Q_y-q_y)\times\left(\frac{\pi}{2\varkappa K}\right)^2\frac{\varkappa^2Q_z^2}{v}\,\frac{1}{|E/K-\varepsilon|}\sum_{l=-\infty}^{\infty}I_{2l+1}(q_z)\times\delta(Q_z-q_z-(2l+1)k), \tag{4.2}$$

where

$$I_{2l+1}=\begin{cases}C_{2l+1}^2(1-e_x^2)+S_{2l+1}^2(1-e_y^2)+2C_{2l+1}S_{2l+1}e_z(\mathbf{e}\mathbf{p}_0), & |q_z|<Q_0\\ S_{2l+1}^2(1-e_x^2)+C_{2l+1}^2(1-e_y^2)+2C_{2l+1}S_{2l+1}e_z(\mathbf{e}\mathbf{p}_0), & |q_z|>Q_0\end{cases}$$

while in this case $Q_0=k$.

Consideration of the asymptotic behavior as $\varkappa\to0$ leads to the following results. A gap appears in the spectrum for $Q_z=0,\ \pm2k,\ \pm4k$. The dispersion curves centered on the reciprocal-lattice sites $0,\ \pm2k,\ \pm4k$ do not manifest themselves in the scattering. It is clear from (4.2) that the scattering occurs only by odd lattice sites. The intensities on the branches of the dispersion curves centered on the sites $\pm k,\ \pm3k,\ \pm5k$ are, respectively, of order of magnitude $1,\varkappa^4,\varkappa^8$. As $\varkappa\to1$ there occurs diffuse scattering in the (ω^2,Q_z)-plane as in the case of a magnetic field when $n=1$.

In the case $n=2$ the anisotropy parameter v defined in (1.5) is temperature-independent so that it is impossible to study the evolution of the scattering corresponding to a formal change of $\varkappa$ in the range $[0,1]$ by changing the temperature. This possibility occurs for higher-order anisotropies $n=4,6$ when v changes with temperature through a change in the OP modulus ρ. One can use the following representation of the equilibrium phase function:[6]

$$\bar{\varphi}(z)=kz+\sum_{p=1}^{\infty}\frac{2}{n}\,\mathrm{ch}^{-1}\left(\frac{\pi K'}{K}p\right)\sin(pnkz) \tag{4.3}$$

and expand the factor $e^{i\bar{\varphi}(z)}$ for small $\varkappa$ in Eq. (2.7) for the form factor in powers of the harmonic terms. After this the integral in (2.7) can be evaluated and appears as a power series in $\varkappa$. One can verify that inelastic scattering occurs only by the dispersion branches centered on the sites $\pm k$, $\pm k\,2Q_0,\ldots,\pm k+2lQ_0$ with intensities of the order $\varkappa^{4l}$. We show the intensity distribution in the (ω^2Q_z)-plane in Fig. 4. The intersection of the branches shown in the figure does not lead to their hybridization and the appearance of new gaps in the spectrum. This is all a consequence of the fact that the potential has a finite number of bands.

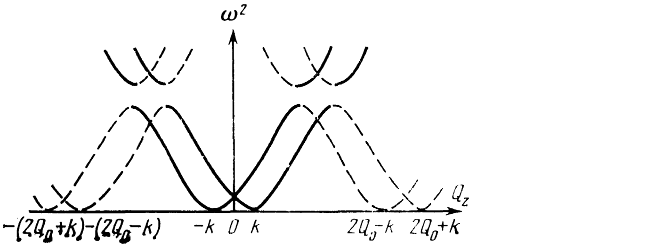

FIG. 4. Schematic picture of the inelastic scattering in the case of nth order anisotropy in the base plane.

When the temperature is lowered ρ increases and the effective anisotropy parameter $v\to v_c$, which corresponds to $\varkappa\to1$. In that limit the pattern of the inelastic scattering evolves with temperature in exactly the same way as in the case of an external magnetic field (see the lower part of Fig. 3).

If we forgo the approximation $\rho=$ const., there must appear in the spectrum of the fluctuations of the two-component OP an infinite number of gaps in the points where the branches shown in Fig. 3 intersect because of their hybridization.

An experimental study of the inelastic scattering of neutrons by spiral structures caused by relativistic interactions and a comparison of the results with the present theory would enable us to verify the validity of the concept of the incommensurate structure as a soliton lattice. We have given earlier in Ref. 6 the picture of the elastic scattering by such structures. Possible objects for a study of elastic and inelastic scattering by a soliton lattice might, for instance, be the isomorphic cubic crystals MnSi and FeGe in which the spiral structures are caused by relativistic interactions described by Lifshitz invariants.

In the comparison with experiments it is necessary to take into account the role of damping, in particular, for the low-frequency part of the spectrum near the Goldstone modes. In the present paper we neglect the damping in the system as otherwise we would not get an exact solution. We have studied in Ref. 10 the scattering of neutrons by excitations of a system described by two-component OP with the functional (1.1) without using the approximation $\rho=$ const. Without this approximation we cannot come close to the boundary of the transition from the incommensurate to the commensurate phase, but on the other hand we can introduce damping through a dissipative function. Our analysis shows that the dissipative terms conserving the total z-component of the spin are unimportant for the Goldstone modes which have frequencies linear in the wavevector, whereas the damping is quadratic. However, terms which do not conserve the total spin component can transfer the Goldstone mode at sufficiently high intensity to the regime of an overdamped oscillator. It is thus necessary to reduce as much as possible their role by first of all removing the defects from the crystal.

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