Universality of the temperature dependence of the thermal conductivity and viscosity of liquid ^{3}He

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For temperatures higher than the degeneracy temperature of the spin excitations but lower than the average kinetic energy per atom, the thermal conductivity of liquid ^{3}He is proportional to the temperature while the viscosity depends weakly on the temperature. In a wide range of temperatures and molar volumes the thermal conductivity and viscosity are characterized by a scaling dependence on the temperature. This makes it possible to determine the values of the kinetic coefficients of ^{3}He in that region of temperatures and molar volumes in which they have still not been measured. At high temperatures the thermal conductivity and viscosity of ^{3}He increase with increase of the density of the liquid. Relations connecting the kinetic coefficients of ^{3}He are obtained. From a comparison of the properties of ^{3}He and ^{4}He the value of the average kinetic energy per ^{3}He atom is determined. The possible result of the determination of the momentum-distribution function of the particles on the basis of experimental data from measurement of the structure factor of liquid ^{3}He is predicted.

1. THE PHYSICAL PICTURE

1. Quantum effects on liquid ^{3}He are strongly pronounced at temperatures lower than the average kinetic energy E_k per atom. The energy E_k is approximately the same for liquid ^{3}He and ^{4}He, and for ^{3}He at the saturated-vapor pressure this quantity is of the order of 13 K. At the end of the paper we determine this value from a comparison of the properties of ^{3}He and ^{4}He. We note that the quantity E_k is appreciably greater than the analogous parameter $3/5\varepsilon_F \approx 3K$ of an ideal gas of the same density. A distinctive feature of ^{3}He is the existence of a large temperature window $T_F < T < E_k$, where T_F is the spin degeneracy temperature, defined in terms of the magnetic susceptibility $\chi(T)$ at $T = 0$: $\chi(0) = C_k/T_F$, where C_k is the Curie constant in the law $\chi(T) = C_k/T$ for $T \gg T_F$. When the molar volume V of the liquid changes from 36.8 to 26.2 cm^3/mole the parameter T_F changes from 0.36 to 0.185 K. In the region $T_F < T < E_k$ the Curie law is fulfilled approximately for χ, and the single-particle motion of an atom and the oscillations of the total density of the liquid have a quantum character. But the region of applicability of Fermi-liquid theory is bounded by the conditions $T_c < T < T_F$, where T_c is the temperature of the phase transition of ^{3}He to the superfluid state. For such T the gas of spin excitation is also degenerate. Since upon increase of the density n of the liquid the scales T_c and E_k increase while the parameter T_F decreases, the range of applicability of Fermi-liquid theory becomes narrower and the semiquantum window $T_F < T < E_k$ becomes wider. The important point is that there exists an enormous temperature range $T_c < T < E_k$ in which the parameter T_F is the only characteristic energy scale of the liquid, since for such T it is possible, in the zeroth approximation, to set $T_c = 0$ and $E_k = \infty$. It is clear that in such a situation the thermodynamic and kinetic properties of ^{3}He should be described by very simple, universal laws. And indeed, the analysis performed in Ref. 2 of the experimental data of Greywall[1] has shown that the spin entropy of ^{3}He has a scaling dependence on the parameter T/T_F. Moreover, it has turned out that both the low-temperature ($T < T_F$) and the high-temperature ($T > T_F$) expansions for the spin entropy S^σ/R per particle, where R is the gas constant, are limiting expressions for a more general expansion of S^σ in an asymptotic series in powers of the dimensionless parameter $\Phi = \chi T/C_k$; $0 < \Phi < 1$. The power series in T/T_F or T_F/T converge very slowly, but the series in powers of Φ converges so fast that even the first approximation $S^\sigma/R = S_0\Phi$ is in quantitative agreement with experiment. This result is important, since it has turned out that one can find simple relations by expanding observable quantities in series in observable quantities without specifying a model of the liquid. The reason for such a surprising lack of dependence of the residual entropy S_0 on the molar volume V for $T_F < T < E_k$ has remained an open question. The appreciable difference of the value $S_0 \simeq 0.97$ from ln2 has not been explained either in the framework of Andreev's theory of semiquantum liquids[3] or in the theory of ^{3}He considered by Goldstein.[4] The paramagnon model and the model of almost antiferromagnetic ^{3}He (Ref. 5) also fail to give this value of S_0. Moreover, in the framework of gas models it is difficult to explain even the fact that the spin entropy approaches a constant value. In the linear approximation, when the interaction of the spin excitations is not taken into account, it is not the entropy but the spin specific heat that approaches a constant value at high T.

2. In this paper we consider kinetic phenomena in liquid ^{3}He. It is shown that in the interval $0.005 < T < 2.5$ K the thermal conductivity $\varkappa$ and viscosity η of ^{3}He at constant volume of the liquid, like χ and S^σ, are characterized by a universal, scaling dependence of T. This implies that one can define normalization parameters $\varkappa_0$, η_0 and temperature scales $T_\varkappa$ and T_η in such a way that the dimensionless func-

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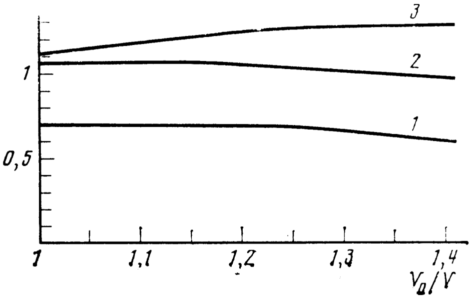

FIG. 1. Dependence of the spin-entropy, viscosity, and thermal-conductivity temperature scales T_S, T_η, and $T_\varkappa$, in units of the magnetic scale T_F, on V_0/V; $V_0 = 36.84$ cm^3/mole; 1) $T_\varkappa/T_F$, 2) T_η/T_F, 3) T_S/T_F.

tions $\varkappa/\varkappa_0$ and η/η_0 will not depend on the density n of the liquid:

$$\frac{\varkappa}{\varkappa_0}=A_\varkappa\left(\frac{T}{T_\varkappa}\right),\quad \frac{\eta}{\eta_0}=A_\eta\left(\frac{T}{T_\eta}\right),$$

$$\frac{S^\sigma}{RS_0}=A_S\left(\frac{T}{T_S}\right),\quad \frac{\chi T}{C_k}=\Phi\left(\frac{T}{T_F}\right). \tag{1}$$

The quantities $T_\varkappa$, T_η, T_S, T_F, $\varkappa_0$, and η_0, however, are strong functions of the density. The important point is that in experiment all the scales have almost the same dependence on the molar volume, i.e., there is indeed one scale—T_F. Figure 1 shows the dependence of the parameters T_S/T_F, $T_\varkappa/T_F$, and T_η/T_F on V. It can be seen that this dependence is indeed very weak. We note that this is evidence for the consistency and correctness of the experimental data that we analyzed in order to obtain the values of the scales T_S, T_F, $T_\varkappa$, and T_η. It turned out that not all the experimental data could be made consistent; only some of them satisfy the rather rigid constraints associated with the universality of the dependences of the thermodynamic and kinetic characteristics of ^{3}He on T and n at $T_c < T < E_k$. All the parameters in (1) are fixed by the values of the leading terms in the asymptotic forms of the observable quantities at $T_c < T < T_F$ and $T_F < T < E_k$:

$$\Phi=\frac{T}{T_F},\quad A_S=\frac{T}{T_S},\quad A_\varkappa=\frac{T_\varkappa}{T},\quad A_\eta=\frac{T_\eta^2}{T^2}\quad \text{for}\quad T\ll T_F,$$
$$\Phi=1,\quad A_S=1,\quad A_\eta=1,\quad A_\varkappa=\frac{T}{T_\varkappa}\quad \text{for}\quad T\gg T_F. \tag{2}$$

We have confined ourselves to considering the interval $0.005 < T < 2.5$ K, and do not take corrections $\sim T_c/T$ and T/E_k into account. The low-temperature asymptotic behavior of χ, S, η, and $\varkappa$ corresponds to the Fermi-liquid region of T, and has been investigated in the original basic papers of Landau,[6] Pomeranchuk,[7] and Abrikosov and Khalatnikov.[8] The region of low T has been considered within the framework of the paramagnetic model of Rice,[9] while Béal-Monod,[10] by supplementing this model with sum rules for χ, has analyzed the high-temperature limit. The paramagnon model, while not explaining the approach of the magnetic susceptibility to a Curie law at $T_F < T < E_k$, nevertheless predicts a scaling dependence for $\varkappa$ and η on T (Ref. 9). This is fully understandable. Although this gas model contains as an upper scale the ideal-gas parameter ϵ_F, in the model the small parameter $\lambda\varepsilon_F$ is introduced artificially. The limit $\lambda\to 0$ corresponds to an almost ferromagnetic gas, and the parameter $\lambda\varepsilon_F$ plays the role of T_F. Any model that leads to the inequality $T_c < T_F < E_k$ will possess this property. Therefore, in the present paper, as in Ref. 2, we have declined to consider a concrete model of ^{3}He and we take the liquid to be as it is in experiment. The aim of the paper is to discover universal laws, and not to give a rigorous justification of them.

3. In order to explain the dependences $\varkappa \propto T$ and $\eta = \eta_0$ at $T > T_F$, we recall the derivation of the law $\varkappa = \varkappa_0$ for the thermal conductivity of metals at $T > \omega_D$, where ω_D is the characteristic phonon frequency, on the basis of the gas-kinetic formula $\varkappa \propto C_V v_F l$, where l is the mean free path, v_F is the Fermi velocity, and C_V is the specific heat. Since in metals heat is transported by electrons, for which $C_V \propto T$, while their mean free path is determined by collisions with classical (for $T > \omega_D$) phonons, it follows that $l \propto 1/T$ and $\varkappa$ does not depend on T. The dependence $l \propto 1/T$ is connected with the fact that the mean free path of the electrons is inversely proportional to the phonon-gas density, which depends linearly on T at $T > \omega_D$. We note that the phonons themselves give a constant constribution to the specific heat, but do not make a noticeable contribution to the specific heat at $T > \omega_D$. A similar situation is realized in ^{3}He at $T > T_F$. The heat is transported by single-particle excitations, and their mean free path is determined by collisions with classical (at $T > T_F$) spin fluctuations, which have a temperature-independent entropy and density but do not themselves make a contribution to the thermal conductivity. This circumstance simplifies the problem substantially, since we are essentially concerned with the motion of a single particle in a field of static spin fluctuations. In the region $T_F < T < E_k$, in which the spin disorder is already a maximum but fluctuations of the total density of the liquid have still not developed, the mean free path of the single-particle excitations depends weakly on T. For such values of T we can assume that $T_F = 0$ and $E_k = \infty$, and the temperature can appear in the expression for l only in the form of a correction: $l \approx a(1 + T_F/T)$, where a is the interatomic spacing—the natural length parameter in terms of which l can be expressed. In experiment,[1,2] the nonspin entropy, i.e., the entropy associated with the motion of the "center of gravity" of an atom, is linear in the temperature, and since the characteristic velocity of an atom does not depend on T, we have $\varkappa \propto T$. Analogously, the viscosity η does not depend on T. We note that the laws $\varkappa \propto T$ and $\eta = \eta_0$ for high T were obtained in the framework of a concrete model in Ref. 10. We have convinced ourselves that these laws are determined by two rather general factors: There exist collective excitations with constant entropy, which do not contribute to the kinetic phenomena but do determine the mean free path of the single-particle excitations, which have an entropy that depends linearly on T and make a decisive contribution to these phenomena. In the standard Fermi-liquid approach

the problem of determining the kinetic coefficients of ^{3}He at $T>T_F$ appears to be very complicated, since for such T, at first sight, the density of the single-particle excitations is large and we cannot confine ourselves to taking only their pair collisions into account. A specific feature of ^{3}He is that an overwhelming fraction of the single-particle excitations are coupled into the soft diffusion spin mode, i.e., figuratively speaking, are precipitated out. Only a small fraction of them determine the transport phenomena. For the thermal conductivity and viscosity of ^{3}He this statement agrees quantitatively with experiment, but for the spin-diffusion coefficient D there is only qualitative agreement.

4. Below it will be shown that both the low-temperature η, $\varkappa$, and D and the high-temperature $\varkappa$ and η are asymptotically determined entirely by global constants—the interatomic spacing a and the parameter T_F. The atomic mass does not appear explicitly in the expressions for the kinetic coefficients:

$$\begin{gathered}\varkappa\propto k_B^2T/a\hbar,\quad \eta\propto\hbar/a^3\quad\text{for } T\gg T_F,\\ \varkappa\propto k_B^2T_F^2/a\hbar T,\quad \eta\propto\hbar T_F^2/a^3T^2,\quad D\propto k_Ba^2T_F^3/\hbar T^2\quad\text{for } T\ll T_F.\end{gathered}\tag{3}$$

These expressions illustrate a fundamental difference between normal and quantum liquids. For quantum liquids, which do not freeze at $T=0$, it is possible to realize the inequality $T<E_k$ while preserving at the same time strongly pronounced single-particle motion of the atoms. Since, dimensionally, the parameter E_k is inversely proportional to the atomic mass m, it follows that in the limit $E_k=\infty$, which is the limit we are considering, m (together with E_k) drops out of the final expressions for the observable quantities. We emphasize that this is the rule and not an exception. For example, in papers by Tolochko and the author[11,12] it is shown that the velocities of zero sound and ordinary in ^{3}He are very close for any value of the effective mass m^* of the quasiparticles. Thus, neither the atomic mass m nor the effective mass of an atom appear in the relations between observable quantities.

2. EXPERIMENT. EMPIRICAL RESULTS. THERMAL CONDUCTIVITY

The most precise data for the thermal conductivity $\varkappa$ of ^{3}He at constant volume of the liquid are given in the recent paper of Greywall,[13] which contains a comparison with the results of previous work. Greywall measured $\varkappa$ in a wide interval of V and T: $0.07<T<0.5$–1.0K, $26.24<V<36.84$ cm^3/mole, and determined in particular, the values of the first two coefficients in the expansion of $\varkappa$ for $T<T_F$:

$$\varkappa=\frac{C_\varkappa^0}{T}\left(1+\frac{T}{T_\varkappa^0}\right).\tag{4}$$

Kerrisk and Keller[14] measured the constant-pressure thermal conductivity $\varkappa_p$ of ^{3}He for $1.5<T<4.0$ K and $0<P<35$ bar. Using the equation of state obtained in Ref. 2 for ^{3}He at $T\gg T_F$:

$$P(T)=P(0)-P_0\ln(T/T_p)+\delta T^2,\tag{5}$$

we have determined on the basis of the data of Ref. 14 the value of $\varkappa$ at constant volume of the liquid, and have parametrized the temperature dependence of $\varkappa$ in the interval $1.5<T<2.5$ K by an expression of the form

$$\varkappa=C_\varkappa^\infty T(1+T_\varkappa^\infty/T).\tag{6}$$

We emphasize that the conversion $\varkappa_p\to\varkappa$ is necessary, since for $T>1.5$ K the density of ^{3}He at constant pressure depends strongly on T. The quantities P^0, T_p and δ in (5) are tabulated in Ref. 2 for several values of V. Below we have defined the parameters $\varkappa_0$ and $T_\varkappa$ by the relations

$$\varkappa_0^2=C_\varkappa^0C_\varkappa^\infty,\qquad T_\varkappa^2=C_\varkappa^0/C_\varkappa^\infty.$$

Then the dimensionless thermal conductivity $\varkappa/\varkappa_0$ is characterized by the asymptotic forms

$$\begin{gathered}\frac{\varkappa}{\varkappa_0}=\frac{T_\varkappa}{T}+\Delta_\varkappa^0\quad\text{for}\quad T\ll T_F;\\ \frac{\varkappa}{\varkappa_0}=\frac{T}{T_\varkappa}+\Delta_\varkappa^\infty\quad\text{for}\quad T\gg T_F,\\ \Delta_\varkappa^0=T_\varkappa/T_\varkappa^0,\qquad \Delta_\varkappa^\infty=T_\varkappa^\infty/T_\varkappa.\end{gathered}\tag{7}$$

The parameters $\varkappa_0$, $T_\varkappa$, and $\Delta_\varkappa^\infty$ for five values of the molar volume V are given in Table I. For comparison we also give in the Table the values of the parameters Δ_S^0, Δ_S^∞, and T_S which characterize the asymptotic forms of the spin entropy per particle and which we have obtained in Ref. 2 from an analysis of the data of Greywall[1] (in the notation of Ref. 2, $\gamma^\sigma=S_0/T_S$):

$$\begin{gathered}\frac{S^\sigma}{RS_0}=\frac{T}{T_S}-\Delta_S^0\frac{T^3}{T_S^3}\ln\frac{\theta_S}{T}\quad\text{for}\quad T\ll T_F,\\ \frac{S^\sigma}{RS_0}=1-\Delta_S^\infty\frac{T_S}{T}\quad\text{for}\quad T\gg T_F.\end{gathered}\tag{8}$$

TABLE I.

$P(T=0)$, bar	V, cm^3/mole	T_F, K	T_S, K	$T_\varkappa$, K	T_η, K	Δ_S^0	Δ_S^∞	$\Delta_\varkappa^0$	$\Delta_\varkappa^\infty$	$\varkappa_0$, erg/sec · cm · K	η_0, μP
0	36,84	0,359	0,403	0,251	0,382	0,832	0,639	3,1	2,8	115,6	17,5
5	32,59	0,286	0,349	0,201	0,309	0,970	0,645	3,2	2,8	104,0	21,3
10	30,39	0,250	0,318	0,173	0,265	1.030	0,625	3,2	2,9	96,4	24,0
20	27,70	0,207	0,270	0,136	0,210	1,050	0,602	3,2	2,9	84,6	28,5
30	26,14	0,185	0,239	0,111	0,180	0,970	0,581	3,3	2,1	76,2	32,6

Note. The parameters T_S, Δ_S^0, and Δ_S^∞ characterizing the spin entropy are determined by the expressions (8), the parameters $T_\varkappa$, $\Delta_\varkappa^0$, $\Delta_\varkappa^\infty$, and $\varkappa_0$ characterizing the thermal conductivity are determined by (7), and the parameters T_η and η_0 characterizing the viscosity are determined by (11).

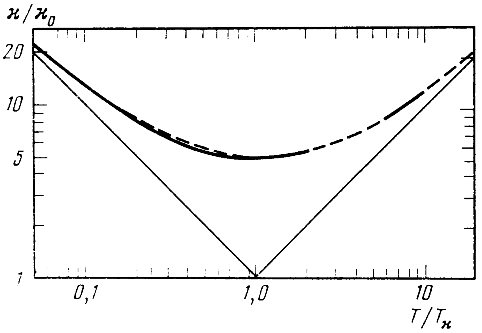

FIG. 2. Dependence of $\log(\varkappa/\varkappa_0)$ on $\log(T/T_\varkappa)$ for two values of the molar volume. Solid curve: $V = 36.84$ cm^3/mole. Dashed curve: $V = 27.7$ cm^3/mole. The data are those of Greywall[13] and Kerrisk and Keller.[14] The straight lines are the asymptotes: $\varkappa/\varkappa_0 = T_\varkappa/T$ for $T \ll T$ and $\varkappa/\varkappa_0 = T/T_\varkappa$ for $T \gg T_\varkappa$.

One's attention is drawn to the closeness of the values of $\Delta_\varkappa^0$ and $\Delta_\varkappa^\infty$, which makes natural an empirical dependence of the form

$$\varkappa/\varkappa_0 = T_\varkappa/T + T/T_\varkappa + \Delta_\varkappa, \tag{9}$$

where $\Delta_\varkappa$ is a very weak function of V and T. In Fig. 2 we give the dependence of $\log(\varkappa_0/\varkappa)$ on $\log(T/T_\varkappa)$ for two values of V: 36.84 and 27.7 cm^3/mole. It can be seen that $\varkappa/\varkappa_0$ is indeed a universal (i.e., almost independent of the density of the liquid) function of $T/T_\varkappa$. For molar volumes in the range 36.84–27.7 cm^3/mole scaling for the dependence of $\varkappa$ on T holds with approximately the accuracy with which one can determine values of $\varkappa$ from the data of Refs. 13 and 14, viz., 1–3%. The point is that, unfortunately, the latter papers give not tabulated values of $\varkappa$ and $\varkappa_p$ but the coefficients of the empirical expansions of $\varkappa$ in powers of T and V (Ref. 13) and in powers of P and T (Ref. 14). The accuracy of these expansions is estimated by the authors of Ref. 14 and 15 to be 1% and 3%, respectively. But near the melting curve for $V \approx 26.14$ cm^3/mole the universality of the dependence of $\varkappa$ on T holds only with an accuracy of 10%. In Fig. 3 we show the dependence of the parameter $\Delta_\varkappa$ on $\log(T/T_\varkappa)$ for four values of V. The average value $\overline{\Delta}_\varkappa$ is approximately equal to 3, while near the minimum of $\varkappa$ at $T \approx T_\varkappa$ the optimum value of $\Delta_\varkappa = 3.15$. On the V,T phase diagram there is a region in which there are no experimental data for $\varkappa(T)$. For $V = 36.84$ cm^3/mole this is a wide interval $0.5 < T < 1.5$ K. Since in experiment one measures essentially one universal function $A_\varkappa$ it is possible to determine $\varkappa$ for such V and T on the basis of the empirical formula (9).

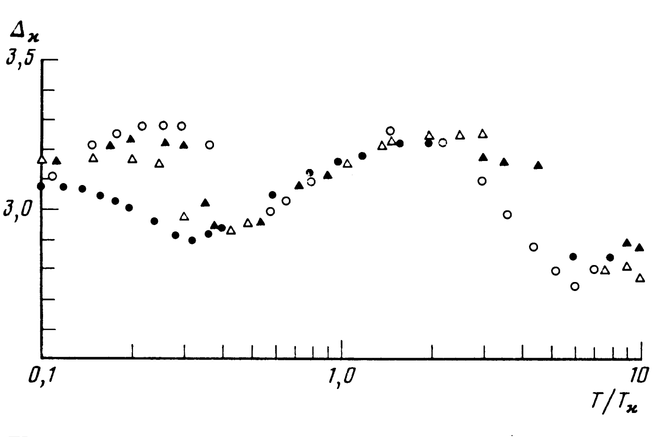

FIG. 3. Dependence on $\log(T/T_\varkappa)$ of the parameter $\Delta_\varkappa$ defined by the relation (9); $V = 36.84$ (●), $V = 32.59$ (△), $V = 30.39$ (▲), $V = 27.70$ (○) cm^3/mole. The data are from Refs. 13 and 14.

The viscosity

For the viscosity of ^{3}He experimental data obtained with the same apparatus for a wide range of V and T are lacking. Even in the region $0.1 < T < 1.0$ K, which is easily accessible to the modern experimenter, the data from different papers are in poor agreement with each other. Therefore, difficulty has risen in choosing an experimental dependence of η on T for ^{3}He at the saturated-vapor pressure that could be adopted as a standard and used to determine the dependence of η on the density of the liquid. Since we expect that for $0.05 < T < 2.5$ K the dependence of η on T, like that of $\varkappa$ on T, is universal, and the asymptotic forms of η for $T \ll T_F$ and $T \gg T_F$ are known, a natural empirical formula for η has the form

$$\eta/\eta_0 = T_\eta^2/T^2 + 1 + \Delta_\eta T_\eta/T, \tag{10}$$

where $\Delta\eta$ is a weak function of the molar volume, and the parameters η_0 and T_η are determined from the asymptotic forms of η for $T \gg T_F$ and $T \ll T_F$:

$$\eta = C_\eta{}^\infty\left(1 + \frac{T_\eta{}^\infty}{T}\right), \quad \frac{\eta}{\eta_0} = 1 + \Delta_\eta{}^\infty \frac{T_\eta}{T} \quad \text{for} \quad T > T_F,$$

$$\eta = \frac{C_\eta{}^0}{T^2}\left[1 + \left(\frac{T}{T_\eta{}^0}\right)^\nu\right], \tag{11}$$

$$\frac{\eta}{\eta_0} = \frac{T_\eta{}^2}{T^2} + \Delta_\eta{}^0\left(\frac{T_\eta}{T}\right)^{2-\nu} \quad \text{for} \quad T < T_F,$$

where

$$T_\eta{}^2 = C_\eta{}^0/C_\eta{}^\infty, \quad \eta_0 = C_\eta{}^\infty, \quad \Delta_\eta{}^\infty = T_\eta{}^\infty/T_\eta, \; \Delta_\eta{}^0 = (T_\eta/T_\eta{}^0)^\nu.$$

The expressions (11) are written by analogy with (6). Among experimenters there is no agreed opinion as to the value of the index ν in (11) that determines the correction to the leading ($\propto T^{-2}$) term in the expansion of η for $T \ll T_F$. The values $\nu = 1, 2, 3$ have been considered. In a paper by Pethick[15] it is shown that in the approximation of an almost ferromagnetic liquid $\varkappa$ and D for ^{3}He contain a correction linear in T, while for η this correction is equal to zero. Since there is no strong exclusion of such a correction, this means only that it is small, i.e., $\nu = 1$ but $\Delta_\eta^0 < 1$. Therefore, we expect that η will be characterized by a scaling dependence (10) on T, with an unknown function Δ_η of T/T_η that has the limits Δ_η^0 for $T \ll T_F$ and Δ_η^∞ for $T \gg T_F$, with $\Delta_\eta^\infty > \Delta_\eta^0$. The dependence of the parameter C_η^0 on V is determined in the paper by Parpia *et al.*[16] For the molar volume $V_0 = 36.84$ cm^3/mole the value $C_\eta^0 = 2.55\,\mu$P $\cdot$ K^2 from Ref. 16 agrees with the results of Abel *et al.*,[17] who measured the attenuation of sound in ^{3}He, this measurement being equivalent at low T to a measurement of the viscosity. For the parameter C_η^∞, however, we can give an estimate only for $V = V_0$: $C_\eta^\infty = 17.5\ \mu\text{P}$. This value was obtained on the basis of a parametrization of the dependence of η on T for $0.14 < T < 1.4$ K, as measured by Betts *et al.*[18] In this region of T the density of ^{3}He depends weakly on T and one may treat η and η_p as the same:

$$\eta=[2.55/T^2+17.5+9.4/T]\ \mu\text{P}\ . \tag{12}$$

We note that the expression (12) has the correct low-temperature limit, since $C_\eta^0 = 2.55$. There is a later and frequently cited paper by Black *et al.*[19] which gives an empirical dependence of η on T in the wider range $0.05 < T < 3.0$ K:

$$\eta=2.21/T^2+26.3/T^{1/3}.$$

We decided against parametrizing an already empirical dependence, and chose as our basis the tabulated data of Betts *et al.*[18] The dependence of the parameter C_η^∞ on V can be found by requiring that the ratio T_η/T_F, like $T_\varkappa/T_F$, depend weakly on V. In the next section of the paper we shall give physical but not rigorous arguments in favor of the following relation between C_η^∞ and $C_\varkappa^\infty$:

$$\frac{C_\eta^\infty(V)}{C_\eta^\infty(V_0)}=\left(\frac{V_0}{V}\right)^{2/3}\frac{C_\varkappa^\infty(V)}{C_\varkappa^\infty(V_0)}. \tag{13}$$

Values of the parameters $\eta_0 = C_\eta^\infty$ and $T_\eta = (C_\eta^0/C_\eta^\infty)^{1/2}$, determined on the basis of the data from Refs. 14 and 16 and the relation (13), are given in Table I. In Fig. 1 it can be seen that the relation (13) does indeed give a weak dependence of the parameter T_η/T_F on V_0/V. When V varies from 36.84 to 26.14 cm^3/mole this parameter varies within the range 1.06–0.98, while the parameter $T_\varkappa/T_F$ varies within the range 0.7–0.6. The qualitative dependence of η on V and T for $T > 0.14$ K can now be determined on the basis of formula (10) with a constant value $\Delta_\eta^\infty = 1.41$. For this value of Δ_η^∞ for $V = V_0$ the expressions (10) and (12) for $\eta(V_0)$ coincide. In order to reduce as much as possible the possible inaccuracy in the value of η measured in Ref. 18, we have calculated the dependence of the ratio $\eta(V)/\eta(V_0)$ on n/n_0. For comparison, in Fig. 4 we also show the dependence of $\varkappa(V)/\varkappa(V_0)$ on n/n_0, calculated on the basis of data from Refs. 13 and 14. In Fig. 4 it can be seen that for $T > 0.4$ K the viscosity increases with increase of the density of the liquid. But at low T, as is known from experiment, η is a decreasing function of n. This result could have been foreseen: At high T quantum liquids should be similar in their properties to normal liquids, for which $\varkappa$ and η increase with increase of the density. Therefore, on the V,T phase diagram there should be lines on which $\partial\varkappa/\partial V$ and $\partial\eta/\partial V$ vanish. We are fortunate in that these lines lie in the region $T < E_k$, i.e., in the region where $\varkappa$ and η have a scaling dependence on T. We note that the dependence (10) of η on T agrees qualitatively with the results of Betts *et al.*,[20] who measured η for $P = 0$ and $P = 13.7$ atm in the region of low T. According to their data, for $T \approx 0.3$ K the viscosity is almost the same for $P = 0$ and $P = 13.7$ atm.

Our results are also in qualitative agreement with the data from the papers of Dyumin, Esel'son, and Rudavskiĭ,[21,22] who, on the basis of measurements of the sound attenuation, calculated the dependence of η on T and n, assuming that the second viscosity of ^{3}He is negligibly small. But according to the data of Refs. 22 the change of sign of the derivative of η with respect to the density of the liquid occurs at $T \approx 0.8$ K, while our value is half this. In addition, at $T \approx 0.5$ K, the value of η determined in Ref. 22 differs by 20% from the data of Ref. 18, which we have taken as our standard. Direct measurement of η in the wide interval $36.8 > V > 20$ cm^3/mole and $T < 10$ K would make it possible not only to determine to what accuracy the dependence of the viscosity on T is universal but also to what ascertain whether there is on the V,T phase diagram a region in which the model of an amorphous solid (glass) is applicable for ^{3}He. The point is that the laws $S \propto T$ and $\varkappa \propto T$ obtained in the framework of this model by Andreev and Kosevich[3,23] can also be explained without assuming that there is short-range order in the liquid. For the viscosity, however, the authors of Ref. 23 obtained the law $\eta \propto 1/T$, which is not fulfilled for ^{3}He or ^{4}He at densities close to the equilibrium density n_0. For such n and for ^{4}He the dependence (11) holds, while the parameter T_η^∞ is so small that the term T_η^∞/T is only a correction. At large densities, however, as

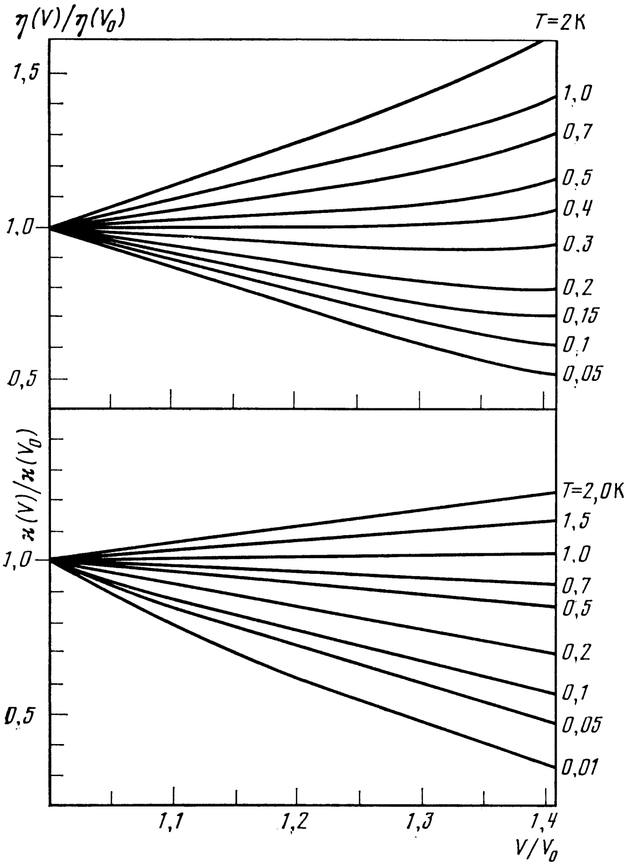

FIG. 4. Dependence of the ratios $\eta(V)/\eta(V_0)$ and $\varkappa(V)/\varkappa(V_0)$ on the parameter $n/n_0 = V_0/V$, where n_0 is the equilibrium density of ^{3}He. The data for $\varkappa$ are from Refs. 13 and 14, and the values of η are calculated using the expression (10).

follows from the analysis of the experimental data performed by Andreev and Kosevich for ^{4}He and hydrogen, there is a region of T in which the term T_η^∞/T is of the order of unity, but there is no region in which the strong inequality $T_\eta/T \gg 1$ is fulfilled. It is possible that this is connected with the fact that for ^{4}He the semiquantum window $T_\lambda < T < E_k$ is narrower than the region $T_F < T < E_k$ for ^{3}He. Therefore, it is useful to trace the dependence of the parameter T_η^∞ on n experimentally. According to the ideas that we have developed, the parameters $T_\varkappa^\infty$ and T_η^∞ decrease with increase of n, and their dependence on n is the same as that of the parameter T_F. The glass model, however, predicts and increase of T_η^∞ with increase of n. Therefore, the change of sign of $\partial T_\eta^\infty/\partial n$ would be an indication of a crossover of the dependence of η on T to the glass regime. It is also of interest to investigate the region of T in which $\eta(T)$ has a minimum, in order to ascertain whether or not the gas dependence $\eta \propto T^{1/2}$ is fulfilled for $T > E_k$.

3. RELATIONS BETWEEN THE KINETIC COEFFICIENTS

1. We now give the arguments that have helped us to discover such simple dependence of $\varkappa$ and η on T. We shall also determine the dependence of the kinetic coefficients of the molar volume and find relations between them. We recall first of all the generally accepted expressions for $\varkappa$, η, and D, applicable at low T:

$$\varkappa = \frac{1}{3}\frac{C_V}{V} v_F^2 \tau_\varkappa, \qquad \eta = \frac{1}{5} n v_F^2 m^* \tau_\eta,$$
$$D = \frac{1}{3} v_F^2 (1+Z_0)\tau_D, \tag{14}$$

and on the basis of the Fermi-liquid relations

$$T_F = \frac{P_F^2(1+Z_0)}{3m^* k_B}, \qquad v_F = \frac{P_F}{m^*},$$
$$\frac{C_V}{V} v_F = \frac{\pi^{4/3} T k_B}{3^{1/2} a^2 \hbar}, \qquad n = \frac{1}{a^3} = \frac{P_F^3}{3\pi^2\hbar^3}$$

we eliminate from the expressions (14) the unobservable parameters Z_0, m^*, v_F, and the times $\tau_\varkappa$, τ_η, and τ_D by expressing the kinetic coefficients in terms of geometrical quantities, viz., the mean free path $l_\varkappa = v_F\tau_\varkappa$, $l_\eta = v_F\tau_\eta$, $l_D = v_F\tau_D$, and the interatomic spacing a:

$$\varkappa = \left(\frac{\pi^2}{9}\right)^{2/3} \frac{T k_B^2 l_\varkappa(T)}{a^2\hbar}, \qquad \eta = \frac{(3\pi^2)^{1/3}\hbar l_\eta(T)}{5a^4},$$
$$D = \frac{k_B a l_D(T) T}{\hbar (3\pi^2)^{1/3}\Phi(T)}. \tag{15}$$

The expressions (15) correspond to the dimensions adopted for the kinetic coefficients in the literature:

$[\varkappa]$ = erg/sec · cm · K, $[\eta]$ = g/cm · sec = P,

$[D]$ = cm^2/sec, $[T]$ = K.

Since all the Fermi-liquid parameters have dropped out of the expressions for $\varkappa$, η, and D, it should be expected that the relations (15) have a wider region of applicability that does Fermi-liquid theory itself. As can be seen from (15), in an experiment one is essentially measuring the mean free paths l_η, $l_\varkappa$, and l_D. The expression for D contains the function $\Phi = \chi T/C_k$, so that for $T \ll T_F$ the combination T/Φ is equal to the parameter T_F. It is convenient to normalize the kinetic coefficients to their values for $V_0 = 36.84$ cm^3/mole:

$$\varkappa = \left[\frac{V_0}{V}\right]^{1/3} T\lambda_\varkappa \frac{l_\varkappa(T)}{a}, \qquad \eta = \frac{V_0}{V}\lambda_\eta \frac{l_\eta(T)}{a},$$
$$D = \left[\frac{V}{V_0}\right]^{2/3} \frac{T l_D(T)\lambda_D}{a\Phi(T)}. \tag{16}$$

The parameters λ in (16) are determined by the expressions

$$\lambda_\varkappa = \left[\frac{\pi^2}{9}\right]^{2/3} \frac{k_B^2}{\hbar a_0} = 487.9\ \frac{\text{erg}}{\text{sec}\cdot\text{cm}\cdot\text{K}^2},$$
$$\lambda_\eta = \frac{(3\pi^2)^{1/3}\hbar}{5a_0^3} = 10.67\cdot 10^{-6}\ \text{P},$$
$$\lambda_D = \frac{k_B a_0^2}{\hbar(3\pi^2)^{1/3}} = 65.68\cdot 10^{-6}\ \frac{\text{cm}^2}{\text{sec}\cdot\text{K}};$$

a_0 is the equilibrium interatomic spacing: $a_0 = 3.94$ Å.

Since we have verified that the characteristic scale for ^{3}He at temperature $T_c < T < E_k$ is the magnetic temperature T_F, and since it is known that for $T \ll T_F$ the mean free path increases like T^{-2}, it is convenient to define the lengths l^0 by the expressions

$$l_\varkappa(T) = l_\varkappa^0 \frac{T_F^2}{T^2}, \quad l_\eta(T) = l_\eta^0 \frac{T_F^2}{T^2}, \quad l_D(T) = l_D^0 \frac{T_F^2}{T^2}. \tag{17}$$

On the basis of the experimental data and the relations (16) and (17) it is possible to find the dependence of l^0 on the molar volume. In Fig. 5 we show the dependence of the parameters $l_\varkappa^0/a$ and l_η^0/a on $n/n_0 = V_0/V$. It is seen that this dependence is weak, and the interatomic spacing a is the only geometrical scale for the liquid at $T \ll T_F$. The parameter $l_\varkappa^0/a$ varies in the range 0.45–0.52, while l_η^0/a varies in the range 1.85–2.13. The dependence of the parameter C_D^0 in the law $D = C_D^0/T^2$ for $T \ll T_F$ on V has not been determined experimentally, but can be predicted by assuming that the parameter l_D^0/a, like $l_\varkappa^0/a$ and l_η^0/a, depends weakly on V. Neglecting the dependence of all the parameters l^0/a on V, from (16) and (17) we obtain

$$\frac{C_\varkappa^0(V)}{C_\varkappa^0(V_0)} = \left[\frac{V_0}{V}\right]^{1/3} \frac{T_F^2(V)}{T_F^2(V_0)}, \qquad \frac{C_\eta^0(V)}{C_\eta^0(V_0)} = \frac{V_0}{V}\frac{T_F^2(V)}{T_F^2(V_0)},$$
$$\frac{C_D^0(V)}{C_D^0(V_0)} = \left[\frac{V}{V_0}\right]^{2/3} \frac{T_F^3(V)}{T_F^3(V_0)}. \tag{18}$$

As shown in Ref. 2, the parameter T_F measured by Thompson *et al.*[24] is characterized by a power dependence on V. Therefore, the kinetic coefficients, as can be seen from (18), also have a power dependence on V:

$$\frac{T_F(V)}{T_F(V_0)} = \left[\frac{V}{V_0}\right]^{\nu_F}, \qquad \frac{C_\varkappa^0(V)}{C_\varkappa^0(V_0)} = \left[\frac{V}{V_0}\right]^{\nu_\varkappa},$$
$$\frac{C_\eta^0(V)}{C_\eta^0(V_0)} = \left[\frac{V}{V_0}\right]^{\nu_\eta}, \qquad \frac{C_D^0(V)}{C_D^0(V_0)} = \left[\frac{V}{V_0}\right]^{\nu_D}. \tag{19}$$

The indices ν in (19) can be expressed in terms of ν_F: $\nu_F = 1.93$, $\nu_\varkappa = 2\nu_F - 1/3 = 3.53$, $\nu_\eta = 2\nu_F - 1 = 2.86$,

TABLE II.

V, cm^3/mole	T_F, K	$C_\varkappa{}^0$, erg/sec · cm	$C_\eta{}^0$, P · (mK)^2 (Ref. 16)	$C_\eta{}^0$, P · (mK)^2	$C_D{}^0$, cm^2 · (mK)^2/sec
36,84	0.359	29.08	2.55	2,63	1,44
33,87	0.305	23,36	2,23	2,23	0.904
32.07	0.277	19.89	1.97	1.97	0,662
30.76	0.256	17.37	1,75	1,77	0,512
29,71	0.238	15.35	1.63	1.60	0.406
28,86	0,224	13.71	1.46	1,46	0,332
28.13	0.212	12,30	1,33	1.33	0,274
27,56	0.205	11,20	1.22	1,22	0.237
27,06	0.198	10,24	1.16	1,14	0,205
26.58	0,191	9.32	1.10	1.04	0.177
26,14	0.185	8,47	1,06	0.96	0.153

Note. The dependences of the pressure P, chemical potential $\underline{\mu}$, sound velocity u, average energy E^+ per atom, average potential energy $\overline{V}$, and average kinetic energy E_k on the density are calculated from the expressions (24), (29), (31), and (32); $n_0 = 0.01635$ Å^{-3} is the equilibrium density of ^{3}He.

$\nu_D = 3\nu_F + 2/3 = 6.46$. For $C^0_\varkappa$ and C^0_η the relations (19) agree with the data of Refs. 13 and 16 to within 15%. In deriving them we have relied only on the assumption that at low T there are, for ^{3}He, no other scales besides T_F and a. Since the inaccuracy of the expressions (19) may be partly due to the inaccuracy of the measurement in Ref. 25 of the dependence of T_F on V, we have also determined the relative quantities, in order to reduce the power of the parameter T_F by as much as possible:

$$\frac{C_\eta{}^0(V)}{C_\eta{}^0(V_0)} = \frac{C_\varkappa{}^0(V)}{C_\varkappa{}^0(V_0)}\left[\frac{V_0}{V}\right]^{1/3},$$
$$\frac{C_D{}^0(V)}{C_D{}^0(V_0)} = \frac{C_\varkappa{}^0(V)\,VT_F(V)}{C_\varkappa{}^0(V_0)\,V_0T_F(V_0)}. \tag{20}$$

In the interval $36.8 > V > 27.0$ cm^3/mole, the relationship (20) of $C^0_\varkappa$ tp C^0_η agrees with the data of Greywall[13] and Parpia *et al.*[16] to within 3%, and it was this that determined our choice of these data as the basis for the analysis of experiment for $\varkappa$ and η. We emphasize that these are the only two papers in which the data are in such exact agreement. In Table II we give experimental values of $C^0_\varkappa$, C^0_η, and C^0_D, and the corresponding values calculated on the basis of (20). The data of Greywall are adopted as the standard. For the parameter C^0_η for $V = V_0$ the optimal value is 2.63 P · (μK)^2, which gives the best agreement between the relation (20) and the data of Refs. 13 and 16 in a wide range of V. For C^0_D at $V = V_0$ we have adopted the value $1.44 \cdot 10^{-6}$ cm^2 · K^2/ sec determined by Wheatley.[25] The dependence of C^0_D on V has not been investigated in detail, but the value $C^0_D = 0.177 \cdot 10^{-6}$ cm^2 · K^2/ sec at $V = 26.6$ cm^3/mole (Table II) agrees well with the value $C^0_D = 0.167 \cdot 10^{-6}$ cm^2 · K^2/ sec also determined by Wheatley.[25] The relations (20) are based only on the scaling hypothesis, i.e., on the assumption that at low T all the mean free paths have the same dependence on V:

$$l_\varkappa(V)/l_\varkappa(V_0) = l_\eta(V)/l_\eta(V_0) = l_D(V)/l_D(V_0).$$

The expressions (18), on the other hand, are more specific and less exact. They are based on the assumption that the dependence of l on a and T_F can be factorized: $l \approx aT^2_F/T^2$.

2. For higher $T > T_F$ the question arises of the limits of applicability of the expressions (16). Since they do not contain model, unobservable quantities, the only limitation lies in the requirement that the mean free paths be greater than a. For $l < a$ the concept of a mean free path loses its meaning. For $T_F < T < E_k$ we shall define lengths l^∞ by the expressions

$$l_\varkappa(T) = l_\varkappa{}^\infty\left(1 + \frac{T_\varkappa{}^\infty}{T}\right), \quad l_\eta(T) = l_\eta{}^\infty\left(1 + \frac{T_\eta{}^\infty}{T}\right),$$
$$l_D(T) = l_D{}^\infty\left(1 + \frac{T_D{}^\infty}{T}\right). \tag{21}$$

In the region $T_F < T < E_k$, besides corrections $\propto T_F/T$ there are also corrections $\propto T^2/E^2_k$, which may be disregarded for $T^3 < E^2_k T_F$. In Fig. 5 we show the dependence of the parameter $l^\infty_\varkappa/a$ on n, determined on the basis of the data from Refs. 13 and 14, and the expressions (16) and (21); this parameter varies in the range 0.95–1.25. Since the dependence of the parameter C^∞_η on n has still not been determined experimentally, it is reasonable to find it on the basis

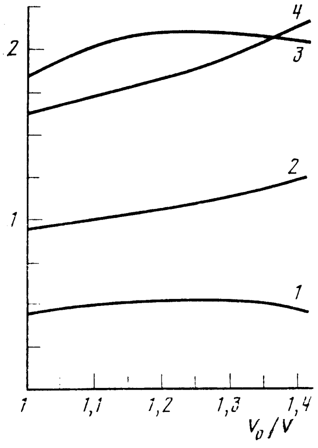

FIG. 5. Dependence of the ratios $l^0_\varkappa/a$ (1), $l^\infty_\varkappa/a$ (2), l^0_η/a (3), and l^∞_η/a (4) on the parameter V_0/V; a is the interatomic spacing. The mean free paths $l^0_\varkappa$, $l^\infty_\varkappa$, l^0_η, and l^∞_η are determined from the relations (16), (17), (21), and (22) and the data of Refs. 13, 14, and 16.

of the scaling hypothesis for high $T \ll T_F$:

$$l_\eta^\infty(V)/l_\eta^\infty(V_0)=l_\varkappa^\infty(V)/l_\varkappa^\infty(V_0). \tag{22}$$

It was the relation (22) which, with allowance for (16), led us to the relation (13) between $C_\varkappa^\infty$ and C_η^∞ that enabled us to define the scale T_η. Since it has turned out that the parameter T_η/T_F depends weakly on V, the scaling hypothesis for $T > T_F$ agrees with the universality of the dependence of η on T. The parameter l_η^∞/a, determined from the relation (22), varies in the range 1.64–2.17 upon variation of n (Fig. 5).

3. The expression (16) gives a qualitatively correct description of the dependence of the spin-diffusion coefficient D on T as measured by Hart and Wheatley.[26] For $T \gg T_F$ the parameter Φ in (16) can be replaced by 1, which corresponds to a Curie law for χ; in this case $D \propto T(1 + T_D^\infty/T)$. But since for low T we have $D \propto 1/T^2$, the dependence of D on T is characterized by a minimum at $T \sim T_F$. However, there is no quantitative agreement: The parameter T_D^∞ is too large, so that up to $T \sim 3$ K we cannot neglect the correction $\propto T_D^\infty/T$. This is not surprising, since already at $T \approx 0.5$ K the mean free path $l_D(T)$, determined on the basis of the data of Ref. 26 and the relation (16), is comparable to the interatomic distance, and this makes the very definition (21) of the length l_D^∞ meaningless. Thus, the region of applicability of the expression (16) for D is considerably narrower than the regions of applicability of the expressions (16) for $\varkappa$ and η. In our opinion, this is due to the fact that for $T > T_F$ the transport of spin is no longer determined by single-particle excitations and has a collective character, since it is associated with the establishment of equilibrium in a dense gas of spin excitations. For phenomena associated with the transport of energy and momentum, on the other hand, the system of spin excitations is passive, and plays the role of an external field for the single-particle excitations. It is for this reason that the elementary gas-kinetic relations (16) for $\varkappa$ and η are in such good agreement with experiment in a wide range of T and n, and the lengths $l_\varkappa^\infty$ and l_η^∞ are greater than a.

4. ENERGY OF THE GROUND STATE OF ^{3}He

1. The average kinetic energy E_k of one ^{3}He atom at $T = 0$ is the most important quantum scale of the liquid. For $T < E_k$ the rough, global properties of the liquid depend weakly on T. It turns out that this important characteristic of the substance can be determined with good accuracy from a comparison of the properties of liquid ^{3}He and ^{4}He. We give the definition of the quantities E_k and E_{k4} for ^{3}He and ^{4}He:

$$E_k=2\int n_p \frac{p^2}{2m}\frac{d^3p}{(2\pi)^3}\Big/ n,$$
$$E_{k4}=\int n_{p4} \frac{p^2}{2m_4}\frac{d^3p}{(2\pi)^3}\Big/ n_4. \tag{23}$$

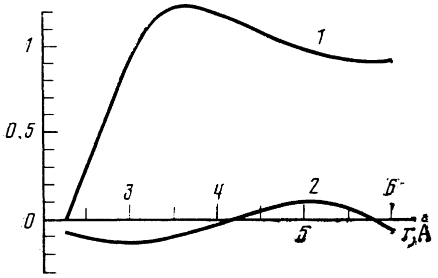

FIG. 6. Dependence on r of 1) the correlation function g of ^{3}He, and 2) the difference of the correlation functions g of ^{3}He and the correlation function g_4 of ^{4}He. The data are those of Achter and Meyer.[28]

In (23) n and n_4 are the densities of ^{3}He and ^{4}He, and n_p and n_{p4} are the momentum-distribution functions of the particles. The function n_{p4} has been measured in experiments on neutron scattering in ^{4}He, and this enabled Woods and Sears[27] to find the values $E_{k4} = 13.5 \pm 1.2$ K at $T = 1.1$ K, and $E_{k4} = 14.1 \pm 1.0$ K at $T = 4.2$ K. For ^{4}He the quantities E_{k4} and n_{p4} depend weakly on T for $T < E_{k4}$. Since ^{3}He absorbs neutrons strongly, the experimental determination of the function n_p is difficult. But if we assume that the atoms of the liquid interact only through a paired nonretarded potential $v(r)$ that is the same for ^{3}He and ^{4}He, we can determine E_k from the expressions for the total average energies E^+ and E_4^+ per atom:

$$E^+=E_k+\overline{V}, \quad E_4^+=E_{k4}+\overline{V}_4. \tag{24}$$

The average potential energies $\overline{V}$ and $\overline{V}_4$ can be expressed in terms of the potential v and the pair correlation functions g and g_4:

$$\overline{V}=\frac{n}{2}\int v(r)g(r)\,d^3r,$$
$$\overline{V}_4=\frac{n_4}{2}\int v(r)g_4(r)\,d^3r. \tag{25}$$

We note that the energies E^+, E_k, and $\overline{V}$ depend on T implicitly, through the functions g and n_p. The functions g and g_4, unlike n_p, have been investigated in detail in experiments on x-ray scattering both in ^{4}He and in ^{3}He. In Fig. 6 we give the dependences of g and $g - g_4$ on r in the interval 2–6 Å, as constructed from the data of Achter and Meyer.[28] It can be seen that g and g_4 differ little, and therefore it is reasonable on the basis of the expressions (24) and (25) to separate out in the energy E_k the small contribution $g - g_4$:

$$E_k=E^+ + \frac{n}{n_4}(E_{k4}-E_4^+)-\delta E_k, \quad \delta E_k=\frac{n}{2}\int (g-g_4)\,v\,d^3r. \tag{26}$$

In the zeroth approximation ($\delta E_k = 0$) and with the values $E_4^+ = -7.14$, $E^+ = -2.52$ K given in Ref. 28, and the value $n_0/n_4 = 0.75$ for the ratio of the equilibrium densities of ^{3}He and ^{4}He, we arrive on the basis of (26) at the estimate $E_k = 13 \pm 0.9$ K. An exact calculation of δE_k is impossible, since at short distances, where the potential v is large, the functions g and g_4 are poorly determined from the experimental x-ray scattering data. But since it is known that small r the potential v corresponds to repulsion of the atoms, and $g < g_4$, we can give an upper bound for δE_k:

$$\delta E_k < \frac{n}{2}\int_{r_0} v(g-g_4)\,d^3r, \tag{27}$$

where r_0 is the value of r at which v vanishes.

For $r > r_0$ the interaction of two inert-gas atoms is well described by the Lennard-Jones potential

$$v(r) = 4\varepsilon^0\{(r_0/r)^{12} - (r_0/r)^6\}. \tag{28}$$

For ^{3}He and ^{4}He the values $\varepsilon^0 = 10.22$ K and $r_0 = 2.556$ Å are adopted. A rough estimate for δE_k can now be obtained from (27), (28) if we replace the function $g - g_4$ in (27) by its minimum value, equal to -0.157; $\delta E_k < 1.6$ K. But numerical integration on the basis of Achter's and Meyer's tabulated values of the functions g and g_4 gives the bound $\delta E_k < 1$ K. Thus, we have a reliable lower bound for the average kinetic energy of a ^{3}He atom: $E_k > 12 \pm 0.9$ K.

2. The pair correlation functions g and g_4 carry the maximum possible information about the local properties of the liquids. These functions have the meaning of the probability of finding an atom at the point r if another is situated at the point $r = 0$. If there are no correlations in the positions of the particles, then $g = g_4 = 1$. But if the correlations are so strong that the Frenkel picture of the liquid state[29] is applicable and the substance is an amorphous solid, then g has a gigantic peak at $r \approx a$. In addition, g falls off extremely fast with decrease of r, since the atoms are well localized and separated by a barrier. According to the data of Ref. 28, the maximum value of g_4 is 1.33, and maximum value of g is 1.24. In our opinion, the numbers 0.33 and 0.24 are an objective measure of the short-range order in liquid ^{3}He and ^{4}He. These substances are the most non-Frenkel of all known liquids. It seems that the concepts of "short-range order" and "low temperatures" for quantum liquids are incompatible. From the analysis performed in Ref. 2 and the present paper of the experimental data for the magnetic susceptibility, specific heat, thermal conductivity, and viscosity it also follows that for molar volumes $36.8 > V > 26.2$ cm^3/mole and temperatures $T < 2.5$ K for ^{3}He the physical laws are the same. No indications of the appearance or development of any sharply expressed short-range order have been discovered. Of course, this does not rule out the applicability of the Frenkel picture for high $T > 3$ K and low molar volumes $V < 20$ cm^3/mole. Since the Frenkel correlations are small, we shall neglect them and determined the quantities $\overline{V}$ and $\overline{V}_4$ (25) using the function $g = g_4 = 1$ for $r > r_0$ and $g = g_4 = 0$ for $r < r_0$. The calculation with the potential (28) leads to the value $\overline{V}_4 = -20.8$ K and $E_{k4} = 13.7$ K. Comparison with the empirical value $E_{k4} = 13.5$ K convinces us that the choice of g and g_4 is reasonable, and makes it possible to determine the dependence of $\overline{V}$ on the density n from (25):

$$\overline{V} = \frac{n}{2}\int_{r_0} v(r)\,d^3r = \overline{V}(n_0)\frac{n}{n_0},$$

$$\overline{V}(n_0) = \frac{16\pi r_0^3 n_0 \varepsilon^0}{9} = 15.58 \text{ K}. \tag{29}$$

To find the average kinetic energy E_k on the basis of this dependence and the relation (24) we need the values of the total average energy E^+. The dependence of E^+ on n at $T = 0$ is found from the thermodynamic identities that relate the sound velocity u, chemical potential μ, pressure P, and energy E^+:

$$u^2 = \frac{\partial\mu}{\partial n}\frac{n}{m} = \frac{\partial P}{\partial n}\frac{1}{m}, \qquad \mu = E^+ + \frac{\partial E^+}{\partial n} n. \tag{30}$$

The most precise measurements of the velocity of sound in ^{3}He have been carried out by Roach *et al.*[30] The data of Ref. 30 agree to within 0.5% with the linear dependence

$$u = -u_1 + u_2\frac{n}{n_0}, \quad u_1 = 359.084 \text{m/sec}\ u_2 = 540.75 \text{ m/sec} \tag{31}$$

We note that, as shown by Dyumin and Essel'son,[22] a linear dependence of u on n holds for all $T < 1.4$ K. Integrating the identities (30) with allowance for (31), we find the dependences of P, μ, and E^+ on the parameter $x = n/n_0 - 1$:

$$\frac{P}{mn_0} = u_0^2 x + u_0 u_2 x^2 + \frac{u_2^2}{3}x^3,$$

$$\frac{\mu(n) - \mu(n_0)}{m} = u_0^2 x + \frac{u_2^2 x^2}{2} + u_1^2[\ln(1+x) - x], \tag{32}$$

$$\frac{E^+(n) - E^+(n_0)}{m} = \frac{u_0^2}{2}x^2 - \left[u_1^2 - u_1 u_2 + \frac{1}{3}u_2^2\right]\frac{x^3}{1+x} + u_1^2\left[\ln(1+x) - x + \frac{x^2}{2}\right].$$

At the equilibrium density, $\partial E^+/\partial n = 0$, and, according to (31), $\mu(n_0) = E^+(n_0)$. Since the equilibrium value u_0 of the

TABLE III.

n/n_0	P, bar	u, m/sec	μ, K	E^+, K/atom	E_k, K/atom	$\overline{V}$, K/atom
1,00	0	181,7	−2,520	−2,520	13,06	−15,58
1,05	1,562	208,7	−1,845	−2,504	13,86	−16,36
1,10	3,587	235,7	−1,011	−2,456	14,68	−17,14
1,15	6,134	262,8	−0,009	−2,372	15,55	−17,92
1,20	9,262	289,8	1,169	−2,250	16,45	−18,70
1,25	13,03	316,8	2,532	−2,086	17,38	−19,47
1,30	17,50	343,9	4,085	−1,880	18,37	−20,25
1,35	22,74	370,9	5,834	−1,627	19,40	−21,03
1,40	28,79	398,0	7,784	−1,326	20,48	−21,81

Note. The parameters $C_\eta^0 = \eta T^2$ and $C_D^0 = DT^2$ are calculated using the relation (20). The values of $C_\varkappa^0 = \varkappa T$ are the data of Greywall,[13] the values of C_η^0 are the data of Parpia *et al.*,[16] and the values of T_F are the data of Thompson *et al.*[24]

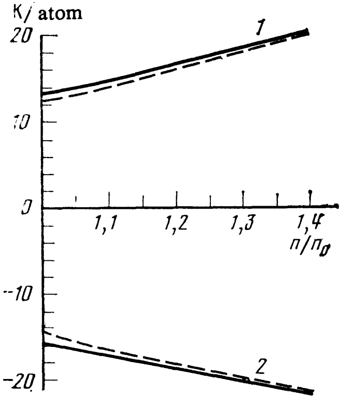

FIG. 7. Dependence on the parameter n/n_0 of 1) the average kinetic energy E_k, and 2) the average potential energy $\overline{V}$. The solid curves are the result of calculations using the expressions (29) and (32), and the dashed curves are the results of Manousakis *et al.*[32]

sound velocity is related to the velocities u_1 and u_2 (30) by $u_0 = u_2 - u_1$, it follows that, by determining the four parameters n_0, u_0, u_2, and $\mu(n_0)$ experimentally, it is possible to find from (32), (31) the dependences of P, u, μ, and E^+ on n, while the relations (29), (24) determined E_k as a function on n. In Table III we give the calculated values of the parameters P, μ, u, E^+, E_k, and $\overline{V}$. For n_0 and $\mu(n_0)$ we take the values $n_0 = 0.01635$ Å^{-3} and $\mu(n_0) = -2.52$ K. To within 25% the calculated values of P and u agree with the data of Abraham and Osborne.[31] The values that we have determined for $\overline{V}$ and E_k also agree with the results of a microscopic calculation of these quantities by Manousakis *et al.*,[32] as can be seen from Fig. 7. This agreement is not accidental. At $T = 0$ ^{3}He is a simple liquid: The correlation function g depends weakly on the density n and on the atomic mass, as can be seen by comparing g and g_4. The average potential energy $\overline{V}$ depends chiefly on the parameters of the potential v (29). We note that in Ref. 32 the calculation of the ground-state energy was performed with a potential with similar values of the parameters ε^0 and r_0 in (28). It is of interest to trace the dependence of $\overline{V}$ on the atomic mass m on the basis of a microscopic calculation. In our opinion, the quantities $\overline{V}$ and $\overline{V}_4$ as functions of m and m_4 for real ^{3}He and ^{4}He are close to their limiting values for $m = m_4 = 0$. This is the limit of a dense quantum gas. An amorphous solid, however, is close to the limit $m = m_4 = \infty$, which corresponds to an ideal classical crystal.

3. Having determined the energy E_k, it is possible to find the value of the mean square momentum $\overline{p^2}$ of the particles. For an ideal gas $\overline{p^2} = 3p_F^2/5$, while for real ^{3}He, as can be seen from (23), $\overline{p^2}/p_F^2 = E_k/\varepsilon_F$. Upon change of the density n the parameter $\overline{p^2}/p_F^2$ changes from 2.6 to 3.3. Thus, for ^{3}He the Fermi momentum is not the characteristic momentum, while the Fermi energy is the characteristic energy. Even at $T = 0$ the occupation numbers n_p are small, and this weakens the effects associated with the Pauli principle. The Landau quasiparticles have a small statistical weight against the background of the total number of particles, this being due to the very small value of the Migdal discontinuity a (Ref. 33) of the function n_p. We shall show that the function n_p has not only this discontinuity but also a logarithmic branch point at $p = p_F$, and shall give an estimate the parameter a. For this, using the results of Galitskiĭ,[34] we relate the function n_p to the imaginary part of the one-particle Green function G:

$$n_p = \frac{1}{\pi} \int_{-\infty}^{0} \operatorname{Im} G(p, \varepsilon)\, d\varepsilon. \tag{33}$$

For momenta p close to p_F and for small ε the function G has the form

$$G = \frac{a}{\varepsilon - \xi + i|\varepsilon|\varepsilon/\varepsilon_0}, \quad \xi = (p - p_F)v_F. \tag{34}$$

We divide the region of integration in (33) into two parts:

$$n_p = \frac{1}{\pi} \int_{-\omega_0}^{0} \operatorname{Im} G(p, \varepsilon)\, d\varepsilon + \frac{1}{\pi} \int_{-\infty}^{-\omega_0} \operatorname{Im} G(p, \varepsilon)\, d\varepsilon = n_p^F + n_p^R$$

and choose the parameter ω_0 to be smaller than ε_0, so that it becomes possible to determine the function n_p^F from (34). With logarithmic accuracy,

$$n_p^F = a\theta(p) - a\frac{\xi}{\pi\varepsilon_0} \ln \frac{\varepsilon_0^2}{\xi^2}, \qquad \theta = 1 \text{ for } p < p_F, \quad \theta = 0 \text{ for } p > p_F. \tag{35}$$

Since the function n_p can be measured in experiments on the scattering of fast neutrons in ^{3}He, we have the theoretical possibility of finding all three parameters a, p_F, and ε_0 characterizing the function G (34) for $p \to p_F$ and $\varepsilon \to 0$. The difficulties encountered by experiments here are the same as in the analogous problem of the determination of the density of condensate particles in liquid ^{4}He: The discontinuity a is very small. On the basis of the results of Refs. 11 and 12 we can give an upper bound for the parameter a: $a < m_0^*/m^*$ where m^* is the effective mass of the heavy quasiparticles and m_0^* is the effective mass of the light quasiparticles. The mass m^* is determined from the experimental data for the specific heat for $T < T_F$, while the mass m_0^* is determined from the corresponding data for $T > T_F$ (Ref. 2). This estimate is based on the relation $a/a_0 = m_0^*/m^*$ obtained in Ref. 11, where a_0 is the "bare" discontinuity of the function n_p—the discontinuity determined without allowance for the effect of the spin excitations on the one-particle excitations. The parameter a_0 is difficult to calculate, but it is known that $a_0 < 1$, since allowance for the contribution of other, nonspin excitations can only decrease the statistical weight of the one-particle states. The same bound exists on the magnitude of the discontinuity of the electron function n_p in metals. Having determined the parameter a with allowance for only the electron-phonon interaction, we can be sure that allowance for the electron-electron interaction will only decrease the quantity a. According to the data of Refs. 1 and 2, the parameter m_0^*/m^* varies for ^{3}He in the range 0.127–0.05 when the density is varied from n_0 to $n = 1.4n_0$. On the basis of (23) and (25) we can give an estimate for the contribution of the Landau quasiparticles to the energy E_k. This contribution is equal to $3a\varepsilon_F/5$, and is small than $3\varepsilon_F m_0^*/5m^*$, which amounts to 3% of the value of E_k when $n = n_0$, and 1% when $n = 1.4n_0$. Thus, in its global properties, liquid ^{3}He is very far removed both from an amorphous solid and from an ideal Fermi gas.

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