Self-similar states in the dynamics of the *z*-pinch

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Self-similar solutions of the magnetohydrodynamic equations are obtained. They provide a detailed analytic description of the cylindrically-symmetric compression and expansion of a current-carrying plasma in the *z*-pinch. The solutions describe the convergence of shock waves to the pinch axis for a completely skinned current in the pinch, and the convergence to the axis of ionizing and MHD shock waves in a transverse magnetic field in the case where the current is partially skinned on the outer boundary of the pinch and partially trapped on the pinch axis. Solutions are obtained for the collapse of the current-carrying plasma liner toward an axial current. Solutions describing flows in the reverse *z*-pinch and the motion of the shock wave reflected from the pinch axis through the plasma converging on the axis are also presented. Self-similar solutions describing similar flows in finite plasma masses bounded by a current shell (tangential discontinuity) are considered.

1. INTRODUCTION

Gas-dynamic processes occurring at high energy densities have attracted the attention of numerous researchers since the time of Riemann and Rayleigh. The solution of the shock-wave cumulation problem obtained independently by Guderley[1] and by Landau and Stanyukovich[2] was followed in a very short time by solutions of the problems of strong explosion,[3] the motion of a gas under the influence of a short shock,[4,5] and cavity collapse[6] (see also Refs. 7–9). There has been a considerable recent increase in the interest in cumulation processes in connection with research into inertial thermonuclear fusion (see the review in Ref. 10).

An important and promising method of producing high-energy densities is to compress matter in cylindric geometry by high-intensity electromagnetic fields (magnetocumulative generators, pinches, and so on). Flows of this kind have been discussed by a number of authors.[11–20] In particular, MHD generalizations of the point explosion problem[13,17] solutions of self-similar problems in the dynamics of the reverse *z*-pinch,[14,16] and solutions for flows with uniform deformation[12,20] and with allowance for finite plasma conductivity[11,18,19] have been reported. On the other hand, the compression stage in a plasma carrying a high current, i.e., the pinch effect under the conditions of essentially inhomogeneous flow and growing total current in the plasma, has been examined only by numerical methods and in terms of simple models.

It is clear that solutions describing the compression or expansion of current-carrying plasma must be compatible with a realistic variation in the total current in the circuit consisting of the plasma and the external device. In this paper, we obtain general self-similar solutions corresponding to current growth $\dot{I} > 0$ (not necessarily in accordance with a power law) and describing the essentially inhomogeneous compression and expansion of cylindrically-symmetric plasma carrying a high current. The classes of self-similar solutions discussed here describe the convergence to the axis and the reflection from the axis of shock waves in a magnetic field, the collapse to the axis of plasma current-carrying shells, flows of the same form but including current shells (tangential discontinuities) on the outer boundary of the plasma, and focusing of the entire plasma mass on the axis.[1)]

The topicality of the self-similar solutions is dictated, above all, by the fact that they provide, in analytic form, the asymptotic picture of a complex nonstationary flow process for a wide range of initial conditions.[9] Another important point is that the self-similar solutions enable us to determine characteristic flow states from data provided by different numerical calculations, and open the way for an analytic study of the stability of compression dynamics and the realization of different possible equilibrium states of the *z*-pinch.

There are two main states that can occur, depending on the ratio of the current growth time $\tau_I = |I/\dot{I}|$ to the characteristic magnetic-field diffusion time $\tau_H = 4\pi\sigma R^2/c^2$, where R is the pinch radius and σ the plasma conductivity. When the current varies rapidly ($\tau_I \ll \tau_H$), the current in the pinch is completely skinned on the surface of the plasma pinch, and the magnetic field in the interior of the current channel is zero. The pinch compression dynamics is then described by the corresponding gas-dynamic flow, which includes shock waves converging to the axis, collapsing shells, and so on, with a magnetic piston in the form of a current shell on the outer boundary of the pinch. In the other limiting case ($\tau_I \gg \tau_H$), or in the typical experimental situation in which the current initially grows slowly and then much more rapidly, the flow takes the form of shock waves or plasma shells converging to the axis in the nonzero magnetic field produced by the current trapped in the plasma on the pinch axis.

For the fast-compression processes considered here, the plasma flow can be described by the nondissipative magnetohydrodynamic equations for which the validity criteria

$$\mathrm{Re} \gg 1, \quad \mathrm{Rm} \gg 1 \tag{1}$$

for hydrogen plasma can be written in the form

$$\begin{aligned} &10^3 I\,[\mathrm{kA}]\,T^{-5/2}\,[\mathrm{eV}]\,n^{1/2}\,[10^{14}\ \mathrm{cm}^{-3}] \gg 1, \\ &I\,[\mathrm{kA}]\,T^{1/2}\,[\mathrm{eV}]\,n^{-1/2}\,[10^{14}\ \mathrm{cm}^{-3}] \gg 1. \end{aligned} \tag{2}$$

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where Re and Rm are, respectively, the ordinary and magnetic Reynolds numbers. The last two conditions are satisfied in high-current discharges in practically the entire range of parameter values.

2. BASIC EQUATIONS. SELF-SIMILAR REPRESENTATION OF SOLUTIONS

We shall confine our attention to the cylindrically-symmetric problem and write down the basic equations in terms of cylindric coordinates. The equation of continuity, the equation for the magnetic field, the adiabatic equation, and the equations of motion are, respectively,

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(rnu)=0, \tag{3}$$

$$\frac{\partial H}{\partial t}+\frac{\partial}{\partial r}(Hu)=0, \tag{4}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+(\gamma-1)\frac{T}{r}\frac{\partial}{\partial r}(ru)=0, \tag{5}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{\partial P}{\partial r}+\frac{H}{4\pi r}\frac{\partial}{\partial r}(rH)=0, \tag{6}$$

where n is the particle-number density in the plasma ($n = n_e = n_i$), u is the radial component of the plasma velocity, H is the azimuthal component of the magnetic field, $\rho = m_i n$ is the mass density of the plasma, T is the plasma temperature in energy units ($T = T_e = T_i$), γ is the adiabatic exponent ($\gamma = \frac{5}{3}$ in all the calculations below), and $P = 2nT$ is the plasma pressure.

We shall seek self-similar solutions of (3)–(6), for which all the variables can be written in the form of products of a function of time t and a function of the dimensionless self-similar coordinate

$$\xi=r/R(t), \tag{7}$$

where $R(t)$ is the time-dependent spatial scale of the problem. If we start with some arbitrary time $t = t_0$ and let $R_0 = R(t_0)$, and if we introduce the dimensionless compression

$$a(t)=R(t)/R_0, \tag{8}$$

we can separate the variables in (3)–(9) by writing the hydrodynamic variables in the form

$$u(r,\ t)=\dot{R}(t)\xi U(\xi), \tag{9}$$

$$n(r,\ t)=n_0 a(t)^{2\chi}N(\xi), \tag{10}$$

$$T(r,\ t)=T_0 a(t)^{-2\lambda}\Theta(\xi), \tag{11}$$

$$H(r,\ t)=H_0 a(t)^{\chi-\lambda}B(\xi), \tag{12}$$

where n_0, T_0, and H_0 are, respectively, constants with the dimensions of particle-number density, energy, and magnetic field. Without loss of generality, we may relate them as follows:

$$H_0^2/8\pi n_0 T_0=1. \tag{13}$$

The quantities U, N, Θ, and B introduced above, are dimensionless functions of ξ that describe the self-similar velocity, density, temperature, and magnetic-field profiles; χ and λ are certain exponents.

For inhomogeneous flows that include convergent shock waves, collapsing plasma liners, and so on, the separation of variables in the equation of motion is possible only if the function[8] $a(t)$ has a power-law dependence on time, and the exponent is uniquely related to λ:

$$a(t)=|t/t_0|^{1/(1+\lambda)}. \tag{14}$$

In the special case where $\lambda = -1$, Eq. (14) is replaced with the exponential function $a(t) = \exp(\pm t/t_0)$ (see Ref. 8). It is clear from (14) that states with $\lambda > -1$ describe compression for $t<0$ and expansion for $t>0$. The time $t = 0$ corresponds to collapse, i.e., the vanishing of the characteristic scale of the problem. When $\lambda < -1$, the characteristic scale vanishes for $|t| \to \infty$. Thus, times $t>0$ correspond to the compression of the plasma from a state of infinite radius at $t = 0$ to zero, which occurs asymptotically as $t \to \infty$.

For given γ, self-similar solutions belonging to this class form a two-parameter family in ideal magnetohydrodynamics.

3. AUTONOMOUS DYNAMIC SYSTEM

It is important to emphasize that we are dealing with a self-similar problem of the second kind, for which the self-similar exponent cannot be determined simply from considerations of dimensionality, or from conservative laws, but must be found from the solution of the problem itself either as an eigenvalue or a discrete or continuous series of eigenvalues on an interval.[2)] To solve this type of problem, the initial set of equations must be reduced to an autonomous dynamic system whose field of integral curves determines possible solutions and can be related to different physical problems.

To obtain the autonomous dynamic system from the original variables N, Θ, and B, we must transform to the variables

$$S=\gamma\Theta/\xi^2(1-U),\quad A=B^2/\xi^2N(1-U). \tag{15}$$

We shall assume henceforth that $U\leqslant 1$ (see below), so that S and A are nonnegative. The local values of the Mach number of a flow relative to the acoustic, Alfven, and fast magnetoacoustic velocities c_S, c_A and c_f (we recall that the last of these determines the rate of propagation of small perturbations across the magnetic field) can readily be expressed in terms of the variables U, S, and A, so that $S(1-U)$, $A(1-U)$, and $(S+A)(1-U)$ are self-similar representatives of c_s^2, c_A^2, and c_f^2, respectively. To be specific, we introduce one more relation between the normalizing constants:

$$T_0=m_i R_0^2/2(\lambda+1)^2 t_0^2, \tag{16}$$

and it is then readily shown that the Mach numbers are given by

$$M_{Sk}^2=u^2/c_S^2=U^2/(1-U)S,\quad M_{Ak}^2=U^2/(1-U)A,$$
$$M_{fk}^2=U^2/(1-U)(S+A). \tag{17}$$

The ratio of the kinetic to magnetic pressures is

$$\beta=\frac{8\pi P}{H^2}=\frac{2}{\gamma}\frac{S}{A}. \tag{18}$$

In terms of the variables U, S, and A, the equation of induction, the adiabatic equation, and the equation of motion reduce to the following forms, respectively:

$$(U-1)\frac{d\ln S}{d\ln\xi}+\gamma\left[\frac{dU}{d\ln\xi}+2\left(U-\frac{\lambda+1}{\gamma}\right)\right]=0, \tag{19}$$

$$(U-1)\frac{d\ln A}{d\ln\xi}+2\left[\frac{dU}{d\ln\xi}+U-\lambda-1\right]=0, \tag{20}$$

$$(U+S+A-1)\frac{dU}{d\ln\xi}+\left[U(U-\lambda-1)+2S\left(U+\frac{\chi-\lambda}{\gamma}\right)+A(\chi-\lambda+1)\right]=0. \tag{21}$$

The original nonautonomous set of equations (3)–(6) has thus been reduced to the three-dimensional autonomous dynamic system (19)–(21), whose trajectories completely define self-similar motion. The equation of continuity (3) can readily be integrated together with (19) if we take (10) into account and express N explicitly in terms of U, S, and ξ:

$$N=K_N S^{(\chi+1)/(\gamma-\lambda-1)}[\xi^2(1-U)]^{(\lambda+\gamma\chi+1)/(\gamma-\lambda-1)}, \tag{22}$$

where K_N is a positive constant of integration defining the scale N, which is arbitrary to the extent to which n_0 is arbitrary under the single restriction defined by (13). The profiles $N(\xi)$ and $B(\xi)$ are constructed with the aid of (22) from known profiles $U(\xi)$, $S(\xi)$, and $A(\xi)$.

It is readily shown that the singularities of the set (19)–(21), which are due to the vanishing of the coefficients in front of the derivatives of U, S, and A in these equations when

$$U=1 \tag{23}$$

$$U+S+A=1, \tag{24}$$

occur when the $\xi=$ const lines coincide (for particular values of ξ) with the characteristics of the original set of hyperbolic equations (3)–(6) on the r, t plane. Characteristics coinciding with particle trajectories (here, they correspond to all three types of perturbation, namely, perturbations of entropy, Alfven perturbations, and slow magnetoacoustic perturbations) are described by (23), whilst characteristics corresponding to fast magnetoacoustic waves are described by (24) (c_{f-} or c_{f+}, depending on the direction of propagation, i.e., toward or away from the axis). The simplicity of (24) explains the particular choice of variables in (15).

When $U\leqslant 1$ during the compression stage, condition (24) can be satisfied only on a certain c_{f-} characteristic, called the limiting characteristic. For $\lambda>-1$, this characteristic divides the flow pattern in the r, t plane into a region with $U+S+A>1$, which is causally related to points on the line $r=0$, $t<0$, and a region with $U+S+A<1$, which is not causally related to the point $r=0$, $t=0$ that corresponds to collapse. We can therefore produce a change in the flow pattern in a particular region, i.e., destroy the self-similarity of the flow as a whole by introducing time-dependent boundary conditions and, if this is not accompanied by shock waves converging to the axis, the flow pattern near $r=0$, $t=0$ will not change under these conditions.

To describe the motion of the plasma particles, we renumber them so that the Lagrange coordinate q of each particle is the radius at which a given particle is located at the time when the corresponding self-similar coordinate assumes a particular value $\xi=\xi_0$ (we recall that, in self-similar solutions with inhomogeneous deformation, a plasma particle with a fixed Lagrange coordinate corresponds not to a constant but to a variable $\xi=\xi_q(t)$; the argument t of ξ_q will hereafter be omitted). Let us introduce the function $R_q(\xi)$ that implicitly describes the motion of this particle. By definition,

$$R_q(\xi_0)=q, \tag{25}$$

and, by virtue of (14),

$$|t/t_0|=(R_q(\xi_q)/\xi_q R_0)^{\lambda+1}. \tag{26}$$

Using (3) and integrating the equation of motion of the particle together with (19), we obtain

$$R_q(\xi)=qK_R[S(1-U)^\gamma\xi^{2(\lambda+1)}]^{-1/2(\gamma-\lambda-1)}, \tag{27}$$

where the positive normalizing constant K_R is determined from (25) and is independent of q. Substituting $\xi=\xi_q$ in (27), we obtain the parametric representation of the functions $\xi_q(t)$ and $R_q(t)$ from (26) and (27). If we know the density, temperature, velocity, and magnetic field profiles in terms of ξ, we can determine, for a given particle, the dependence of these variables on time, which will not, of course, be of the power type.

4. THE PHASE SPACE OF THE PROBLEM

Equations (19)–(21) are more conveniently examined by introducing the new independent variable τ, defined by

$$d\ln\xi=(1-U)(1-U-S-A)\,d\tau. \tag{28}$$

We thus remove singularities due to transitions across the characteristic (see Section 2), and obtain the dynamic system

$$dU/d\tau=(1-U)\,[U(U-\lambda-1)+2S(U+(\chi-\lambda)/\gamma)+A(\chi-\lambda+1)], \tag{29}$$

$$dS/d\tau=S\{-\gamma U^2-2\gamma UA+U[(2-\gamma)\lambda+\gamma+2]+A[(2-\gamma)\lambda+\gamma(\chi+1)+2]+2(\chi+1)S-2\lambda-2\}, \tag{30}$$

$$dA/d\tau=2A\{U-\lambda-1+S[U+\lambda+1+2(\chi-\lambda)/\gamma]+A(\chi+2-U)\}. \tag{31}$$

The self-similar solutions in which we are interested are described by the trajectories of (29)–(31) in the phase space (U, S, A). The asymptotic trajectories corresponding to $\tau\to\pm\infty$ are determined by the singular points of this system (this dynamic system does not have more complicated attractors). The type of the singularity and the nature of the approach to it correspond physically to specific boundary conditions (for $r=0$, $r=\infty$, on the free surface of the plasma, or on the limiting characteristic), which the solution represented by the given trajectory must satisfy. Singular points are thus seen to determine possible states of self-similar motion.

To ensure that the trajectory as a whole can be given a particular meaning, it must not cross the plane (24) in phase space (we recall that we have assumed that $U \leqslant 1$) since, otherwise, (28) shows that $d\xi/d\tau$ will change sign and the dependence of the hydrodynamic variables on the self-similar coordinate will not be single-valued, which is physically unsatisfactory. A phase curve crossing the plane (24) is permissible only if it is made up of three trajectories of (29)–(31), one of which is a singular point of (29)–(31) on the plane (24) and the other two tend to this point from either side of the plane for $\tau \rightarrow \pm\infty$ which, according to (28), corresponds to a continuously increasing ξ. On the plane (24), the points form a line whose equation is

$$S^2 - \frac{1}{\gamma}[\gamma+2\chi-(2-\gamma)\lambda]S+\lambda=A^2+\chi A, \quad U=1-S-A. \quad (32)$$

It can be shown that the hyperbola (32) is a nondegenerate singular line upon which one eigenvalue is zero at almost every point, and two others are nonzero. The possibility of continuing the trajectory of (29)–(31) through a given point P' on the hyperbola (32) into the other side of the plane (24), i.e., the possibility of constructing a phase curve representing the self-similar solution from the three trajectories of (29)–(31), is determined by the nonzero eigenvalues ω_1 and ω_2 at this singular point. When ω_1 and ω_2 are real but have different signs, only one phase curve can be drawn through the point P' in the direction of increasing ξ from each side of the plane (24), and both these curves are analytic at P'. When ω_1 and ω_2 are real and have the same sign, the phase curves continued through the point P' tend to it from one side of the plane (24) and form a two-dimensional manifold, but only two of these curves continue through P' smoothly (analytically) whilst all others are self-similar solutions with a weak discontinuity on the limiting characteristic (see Refs. 6, 21, and 22). When ω_1 and ω_2 are complex, the trajectories near the point P' cross the plane (24) an infinite number of times, and are therefore physically meaningless.

It is clear from (19)–(21) that, apart from the hyperbola (32), the system (29)–(31) will have no singular points for $\lambda \neq -1$ outside the invariant planes of the system: $U=1, S=0$, and $A=0$ [Fig. 1 illustrates the trajectories of (19)–(21) on the invariant planes]. The trajectories on the invariant planes do not in themselves have a physical meaning within the framework of our problem of z-pinch dynamics. Actually, motions with $U=1$ should be examined within the framework of the homogeneous deformation model;[12,20] the plane $A=0$ corresponds to the pure gas-dynamic problem (see Refs. 1, 6–8, 10), and the plane $S=0$ corresponds to self-similar motion of the plasma with $\beta \equiv 0$, which is unrealistic under the conditions of the pinch. From our point of view, the most interesting trajectories are those running outside the invariant planes in phase space, and the singular points to which these trajectories tend for $\tau \rightarrow \pm\infty$. It is clear from Fig. 1 that this condition is not satisfied by singular saddle points lying on a finite part of the $U=1$ plane: P_4 $(S=A=0)$, P_5 $(S=\gamma\lambda/2(\chi+1)$, $A=0)$, P_7 $(S=0, A=\lambda/(\chi+1))$. This also applies to the

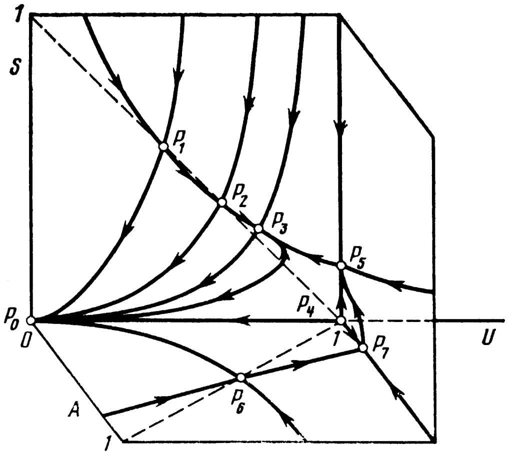

FIG. 1. Trajectories of the dynamic system (19)–(21) on the invariant planes ($\lambda = 0.22$, $\chi = 0$). Arrows show the direction of increasing ξ. The plane (24) cuts the invariant planes along the dashed lines.

singular points P_1 and P_2 on the $A=0$ plane and P_6 on the $S=0$ plane, which are intersections of the singular hyperbola (32) with the invariant planes.

The singular point P_0 $(U=S=A=0)$ is an attracting node for $\xi \rightarrow \infty$ when $\lambda > -1$ and for $\xi \rightarrow 0$ when $\lambda < -1$. It can correspond both to the exact solution of the equations of self-similar motion, describing stationary cold plasma in the absence of the magnetic field and density profile $n \sim r^{2\mu}$, and the asymptotic state for $\xi \rightarrow 0$ or $\xi \rightarrow \infty$. When $\lambda > -1$ and the trajectory tends to P_0 for $\xi \rightarrow \infty$, we obtain the following asymptotic behavior from (29)–(31):

$$U \propto \xi^{-\lambda-1}, \quad S \propto \xi^{-2(\lambda+1)}, \quad A \propto \xi^{-2(\lambda+1)}. \quad (33)$$

This means that the trajectory entering P_0 for $\xi \rightarrow \infty$ is characterized by definite limiting values of the local Mach number ($M_{S\infty}$, $M_{A\infty}$, $M_{f\infty}$) and the parameter β_∞ [see (17)]. At time $t=0$, $\xi=\infty$ will correspond to any finite value of r, i.e.,

$$u(r,\,0)=\text{const}\cdot r^{-\lambda}, \quad n(r,\,0)=\text{const}\cdot r^{2\chi},$$
$$T(r,\,0)=\text{const}\cdot r^{-2\lambda}, \quad H(r,\,0)=\text{const}\cdot r^{\chi-\lambda}. \quad (34)$$

From (27), we find that $R_q(\xi \rightarrow \infty) \rightarrow \text{const}$ as $t \rightarrow 0$. This means that solutions represented by trajectories of this type are similar to those known in gas dynamics for shock-wave cumulation or collapsing cavities, for which plasma particles are at finite distances from the axis at the instant of collapse. From (34), we conclude that these solutions are meaningful only for combinations of indices λ and χ for which the mass, energy, and current densities at the time of collapse do not have nonintegrable singularities on the axis, i.e., for

$$\chi+1>1, \quad \chi-\lambda+1>0. \quad (35)$$

The singular point P_3 on the $A=0$ plane is defined by

$$U=(\lambda+1)/\gamma, \quad S=(\lambda+1)^2(\gamma-1)/2\gamma(\chi+1). \quad (36)$$

In Fig. 1, it is a node on the $A=0$ plane, which is attracting for $\xi \to \infty$ but, for other values of λ and χ, it may lie on the other side of the plane (24) and constitute a saddle. For $\lambda > -1$, the trajectories tend to the singular point P_3 from space ($A \neq 0$) when $\xi \to \infty$. Near this singular point, $R_q(\xi) \to 0$ and $\Theta(\xi) \to \infty$ for $\xi \to \infty$, i.e., the singular point P_3 represents the asymptotic self-similar flow corresponding to the focusing of the entire plasma mass in the pinch into a single line along the axis.

Of all the singular points at infinity, the following two are of particular interest for us. First, the point P_8, which is a node when (35) is satisfied and is attractive for $\tau \to \infty$:

$$U \to 1, \quad A \to \infty, \quad S/A \to 0. \tag{37}$$

It corresponds to the free surface of the plasma in a finite magnetic field, since $N\Theta \to 0$ and $\xi \to$ const as the point is approached. Second, the point P_9, which exists when both (35) and the condition $\lambda > \chi$ are satisfied, is defined by

$$U \to 0, \quad A \to \infty, \quad S \to \infty, \quad S/A \to \gamma(\chi-\lambda+1)/2(\lambda-\chi). \tag{38}$$

This singular point corresponds to a two-dimensional attracting manifold for $\tau \to \infty$ (i.e., for $\xi \to 0$). In other words, it corresponds to a trajectory which tends to it for $\xi \to 0$ and forms a two-dimensional surface. The point P_9 describes the state of the plasma on the axis of the pinch after the collapse, when a reflectd shock wave propagates away from the axis.

Apart from the smooth-flow regions described by the trajectories of (29)–(31), the self-similar solutions that we are considering may include either type of discontinuity allowed in magnetohydrodynamics for a transverse magnetic field, namely, shock waves and tangential discontinuities.[3)] Since the shock wave front corresponds to a fixed value $\xi = \xi_1$, and if we label states ahead of and behind the shock-wave front with indices 1 and 2, we can use (9) and (17) to deduce that the Mach numbers of the plasma flowing onto and away from the shock-wave front, in the frame in which this front is at rest, are respectively given by

$$M_{Sk}^2 = (1-U_k)/S_k, \quad M_{Ak}^2 = (1-U_k)/A_k, \tag{39}$$

where $k = 1, 2$. We can use (39) together with the well-known expressions for shock adiabats of transverse ionizing and MHD shock waves in terms of Mach numbers (see, for example, Refs. 23 and 24) to determine state 2 from state 1 in terms of our variables, and vice versa. Thus, for the MHD shock wave, we can readily show that possible states 1 lie below the plane (24) in phase space (for $U + S + A < 1$), whereas states 2 (behind the shock-wave front) fill the space between the plane (24) and the conical surface with apex at $U = 1, S = 0, A = 0$, whose parametric equation is

$$S = \frac{\gamma(\gamma-1)(1-v)^3}{2v^2(\gamma v+2-\gamma)},$$

$$A = \frac{(\gamma+1)v+2-\gamma}{v(\gamma v+2-\gamma)}, \quad \frac{\gamma-1}{\gamma+2} \leqslant v \leqslant 1. \tag{40}$$

The tangential discontinuity in the above solutions can take the form of a current shell (magnetic piston) separating the plasma from the vacuum. The discontinuity moves together with the plasma particles, i.e., it is characterized by a fixed Lagrange coordinate q and a time dependent self-similar coordinate $\xi_q(t)$ (see Section 2). The continuity of the resultant pressure across the discontinuity[25] can be written in terms of our variables in the form

$$B^2(\xi_q) + 2N(\xi_q)\Theta(\xi_q) = B_p^2. \tag{41}$$

In the quasistationary approximation (i.e., to within small terms of the order of $\dot{R}^2/c^2$), the magnetic field in the vacuum region has the profile

$$H(r, t) = H_0 a(t)^{\chi-\lambda+1} B_p(\xi_q)\xi_q(R_0/r) \tag{42}$$

[see (12)], where H_p is determined from (41) and the time dependence of ξ_q (which is not of the power type) is given by (26) and (27). This means that, when the tangential discontinuity is present, the time dependence of the parameters of the flow as a whole may differ from the power-type dependence (see Section 5 below).

5. SELF-SIMILAR STATES OF MOTION

Collapse of the plasma liner

We shall now consider self-similar solutions for the range of parameters (35) that was not previously examined (other types of solution can be found in the literature cited in Section 1). The phase-space trajectories, which correspond for $\lambda > -1$ to physically meaningful solutions prior to collapse, tend to the singular point P_0 or P_3 as $\xi \to \infty$.

By continuing the trajectory in the direction of decreasing ξ, we arrive at the point P_8 for a finite value of ξ (we can choose $\xi = 1$). The motion over the phase curve between P_8 and P_0 includes transitions through one of the points of the singular hyperbola (32) that represents the limiting characteristic. This type of solution describes the motion of a current-carrying plasma shell converging (if $t < 0$) to an axial current surrounded by an evacuated region, or to a shell diverging from it (if $t > 0$). The plasma density vanishes on the inner boundary of the shell (free surface, on which the magnetic field is continuous) and at infinity.

The behavior of the solution for $\xi \to \infty$ is given by (33) and (34) and, as $\xi \to 1$, i.e., on the inner boundary of the plasma liner, we have

$$U \propto 1 - (\chi-\lambda+1)(\xi-1),$$

$$\Theta \propto (\xi-1)^{[(\gamma-1)(1-\chi)+\lambda(\gamma-3)]/(\chi-\lambda+1)},$$

$$N \propto (\xi-1)^{(\chi+\lambda+1)/(\chi-\lambda+1)}.$$

This flow can be given a number of physical interpretations. For example, it describes the collapse, under the influence of a strong current, of the liner formed by parallel wires, one of which lies along the pinch axis and the other is the generator of a cylinder drawn around this axis. Another possibility is the motion of plasma around a current filament, whose compression is accompanied by the concentration of the plasma in the neighborhood of the filament, so that it is surrounded by a vacuous region.

For an arbitrary choice of λ and χ in the range (35), the solution can be constructed as follows. We first choose an

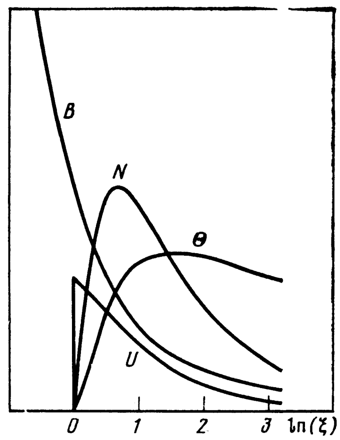

FIG. 2. Current-carrying plasma shell collapsing onto a concentrated axial current ($\lambda = 0.1, \chi = -0.4$).

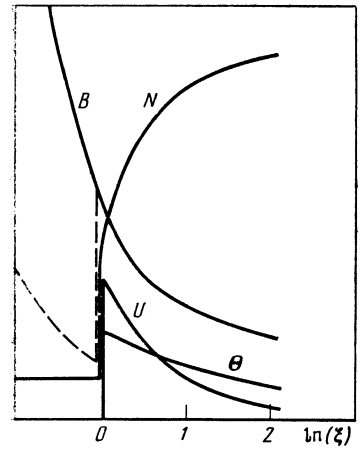

FIG. 3. Convergence of an ionizing shock wave through a stationary nonconducting gas onto a current along the axis. Solid curve—gas-dynamic motion of ionizing shock waves in a magnetic field; dashed curve—MHD state ($\lambda = 0.25, \chi = 0$). The gas-dynamic analog of the flow corresponds to the Guderley problem.[1]

arbitrary point P' on the singular hyperbola (32), whose neighborhood for $U+S+A>1$ lies in the region of attraction of the node P_8 as $\tau \to \infty$. Next, we choose an intrinsic direction corresponding to the departure from P' as τ increases. Trajectories leaving P' in this direction with $U+S+A$ decreasing and increasing with increasing τ are attracted to nodes P_0 and P_8, respectively. Simultaneous numerical integration of (28)–(31) by this method, augmented by the well-known asymptotic behavior near singular points, enables us to construct the required self-similar solution. Solutions differing by the choice of the initial point P' should be similar provided analyticity is preserved at P'. Quantitatively, each such trajectory corresponds uniquely to a particular combination of limiting mass numbers $M_{S\infty}$ and $M_{A\infty}$ that describe the approach to the singular point P_0. Figure 2 shows the profiles of the self-similar variables for a solution of this kind that is analytic at P'. In this figure, $M_{S\infty} = 1.53$ and $M_{A\infty} = 0.99$.

We emphasize that the values of the exponents λ and χ are not determined in this case by some additional considerations, such as, for example, those in the gas-dynamic problem of self-similar collapse of a shell (see Ref. 6). In this case, there are physically sensible solutions on a two-dimensional region of the λ, χ plane. Different combinations of the indices determine the time-dependence of the pinch radius and the current through it, as well as the asymptotic profiles of self-similar variables near the axis and at infinity.

Shock waves converging on an axis with a localized current

Consider a phase curve which describes the collapse of a shell and passes through some non-singular point Q' whose projection onto the $A=0$ plane coincides with the point Q ($U = 2/(\gamma+1)$, $S = 2\gamma/(\gamma+1)$), joined to the singular point P_0 by a shock transition. It is readily shown that, at Q', the self-similar solution allows contact through the ionizing shock wave with the trivial solution described by the singular point P_0, i.e., with the cold nonconducting gas in which the density has the profile $n \sim r^{2\chi}$, and the magnetic field is $H \sim 1/r$. On the shock-wave front, the gas becomes conducting and thus conveys the interaction between the flow and the magnetic field.[23,24] The portion of the phase curve between Q' and the singular point P_0 (which now represents the state of the plasma at infinity) describes the self-similar solution of the problem of ionizing shock waves converging toward the current-carrying axis, and is meaningful only for $t<0$. The solution constructed in this way can describe the initial stage of development of the pinch if the breakdown within the pinch occurs not only on the periphery but also on its axis, so that the current shell travels through a neutral gas which it traps and ionizes until it reaches the axial current filament. This type of flow can also occur in cylindrically symmetric systems with a central electrode along the axis if the breakdown is first initiated on the periphery and this is followed by compression toward the axis. The corresponding profiles are shown in Fig. 3. The solid and dashed distributions $B(\xi)$ correspond, respectively, to the gas-dynamic (magnetic field continuous across the shock wave) and MHD [magnetic field compressed by the shock wave by the factor $(\gamma+1)/(\delta-1)$] limiting states of propagation of the ionizing shock wave.[24]

Reverse *z*-pinch

The segment of the trajectory between P' and the singular point P_8 is the self-similar solution with the shock wave propagating through a cold neutral gas in the direction away from the pinch axis. It is followed by moving plasma bounded at the rear by the free surface which is separated by a vacuous region from the current-carrying axis (here, we must have $t>0$). This flow configuration corresponds to experiments with the reverse *z*-pinch, where the azimuthal magnetic field repels the shock wave away from the axis.[26] Figure 4 shows the corresponding profiles obtained for the same exponent values as in Fig. 3. We emphasize once again that Figs. 3 and 4 are portions of the same phase trajectory in the U, S, A space. The parameter χ has a direct physical meaning for these curves: it determines the density profile in

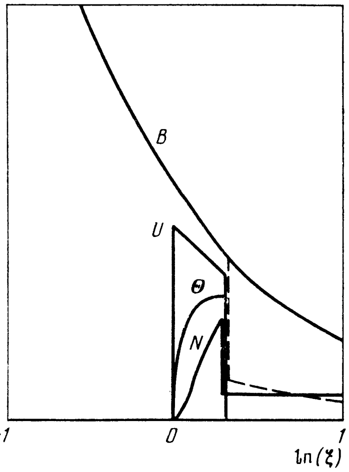

FIG. 4. Motion in the reverse z-pinch: the current shell pushes the shock wave away from the pinch axis. The values of λ, χ and the significance of the dashed lines are the same as in Fig. 3.

the undisturbed gas. In the case of uniform density, $\chi = 0$, as in Figs. 3 and 4.

Shock waves reflected from the axis

After the collapse at $t = 0$, the reflected shock wave begins to propagate through the medium away from the axis, whilst the medium continues to converge on the axis. The corresponding profiles can be constructed in the same way as in the analogous gas-dynamic problem.[1,10] The solution has the following qualitative form in phase space: (1) segment of the trajectory entering the singular point P_9 for $\xi \to 0$ (it describes the motion of the plasma between the axis and the reflected shock wave), (2) the MHD shock wave, and (3) segment of the trajectory tending to the singular point P_0 as $\xi \to \infty$ from the side of negative U (it describes the motion of the plasma ahead of the reflected shock-wave front, and is characterized by the same limiting values of the Mach numbers $M_{S\infty}$, $M_{A\infty}$, $M_{f\infty}$ and the parameter β_∞ as the corresponding solution prior to collapse). Figure 5 shows the profiles of self-similar variables near the reflected shock wave for the same self-similar solution as in Fig. 2 (the same λ and χ) but with $t > 0$, i.e., after collapse. As in the analogous gas-dynamic problem,[1,10] the reflected shock wave has a finite intensity (although the pressure on the axis at the time it is formed becomes infinite), since the plasma ahead of the shock wave front is also highly compressed and hot. In the special case of Fig. 5, the shock wave reflected from the axis is characterized by Mach numbers $M_{S1} = 2.7$ and $M_{A1} = 3.0$, plasma and magnetic-field compression $N_2/N_1 = B_2/B_1 = 2.2$, and heating $\Theta_2/\Theta_1 = 2.3$.

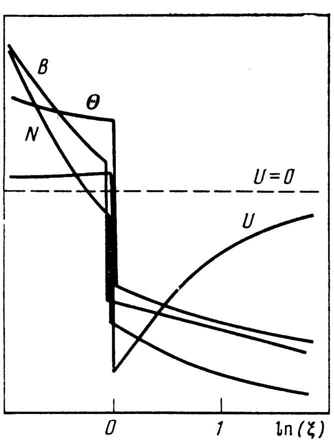

FIG. 5. Plasma flow in the neighborhood of a shock wave reflected from the axis under the same conditions as in Fig. 2.

Solutions of this type are of independent physical interest, since the flow with the shock wave reflected from the axis and moving through the converging plasma is characteristic of the early stage of development of the pinch, i.e., immediately after the convergence of the shock wave on the axis.

6. SELF-SIMILAR MOTION OF FINITE PLASMA MASSES

The interpretation of self-similar solutions is complicated by the fact that, according to (34) and when (35) is satisfied, the plasma mass, energy, and current densities diverge as $r \to \infty$. The most natural way of avoiding this divergence is to consider a self-similar flow limited by a magnetic piston, i.e., a tangential discontinuity with a given Lagrange coordinate q. Physically, this corresponds to the skinning of part of the current flowing through the pinch on its outer surface. This approach retains only the trivial divergence due to the assumed cylindrical symmetry of the problem, i.e., the logarithmic divergence of the magnetic-field energy in the vacuum part, due to the fact that the field decreases as $1/r$. In the real geometry, in which the plasma column has a finite length L, the assumption of cylindrical symmetry is justified for $r \ll L$, and this is one further argument in favor of introducing the pinch boundary in an explicit form, since self-similar profiles have no meaning for $r \gtrsim L$. In this formulation, pure gas-dynamic problems appear to be the most natural, as well. Thus, the Guderley problem[1] corresponds to the self-similar solution of the MHD problem that describes the compression of a pinch in which the current rises so rapidly that it is completely skinned on its outer surface (see Section 1).

The introduction of the current shell has an obvious effect on the solutions of Section 4, which describe the collapse of the plasma shell and the convergence of shock waves (the reverse z-pinch configuration involves a finite mass per unit length, by definition). For the first of these, Fig. 6 shows the time dependence of the concentrated current I_0 flowing along the axis (power-type dependence) and the current I_p flowing through the plasma:

$$I_p(t) = \frac{cH_0R_0}{2}\left|\frac{t}{t_0}\right|^{(\chi-\lambda+1)/(\lambda+1)}\{\xi_q B_p(\xi_q) - B_1\}, \qquad (43)$$

where the time dependence of the self-similar coordinate ξ_q of the current shell is given by (26) and (27), and B_p is obtained from (41). $I_0 \to 0$ as $t \to 0$ and I_p remains finite, i.e., close to the time of collapse, most of the current flows

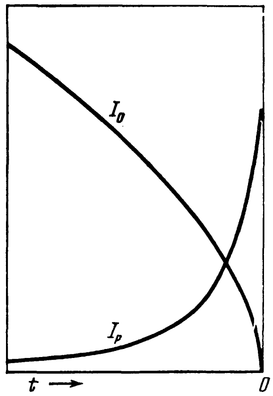

FIG. 6. Time dependence of the axial current I_0 and the current I_p flowing through the plasma (different scales) under the conditions of Fig. 2 when the current is partially skinned on the pinch boundary.

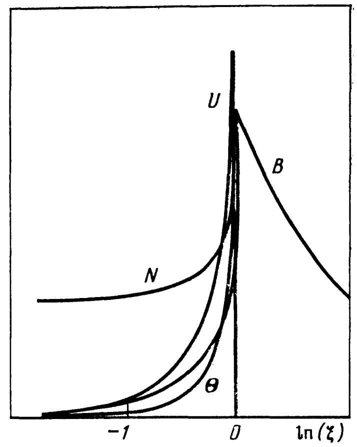

FIG. 8. Motion in the z-pinch for $\lambda = -2, \chi = 0$ well before the appearance of the shock wave on the surface of the pinch on which the current is partially skinned.

through the plasma of the collapsing shell and not along the axis.

The introduction of tangential discontinuities will also enable us to interpret self-similar solutions for which the corresponding trajectories cannot be satisfied by any reasonable boundary conditions at $r = 0$ and $r = \infty$. An example is provided by any of the trajectories joining the nodal points P_8 and P_3 under the conditions of Fig. 2, for which the pressure diverges as $\xi \to \infty$. However, since we then have $R_q(\xi) \to 0$, this type of solution describes the collapse of a plasma shell onto the axis or the expansion away from the axis for $t < 0$ and $t > 0$, respectively. This state differs from the situation described in Section 4 by the fact that, at the time of collapse, all the plasma particles are collected on the axis, and this is responsible for the divergence in pressure at this time. Figure 7 shows the profiles for this particular flow at any finite time t. They contain no divergences. We emphasize that Figs. 2 and 7 are constructed for the same set of indices λ, χ, but the solutions are quite different. In particular, the radius of the pinch and the current have a different time dependence for the same law of variation of the scale in (14). Thus, I_p remains finite as $t \to 0$ under the conditions of Fig. 6 but, in Fig. 7, this current diverges. This means that the state that we are considering can be realized under the same physical conditions as were examined in Section 4, but for a more rapid rise in the current with time.

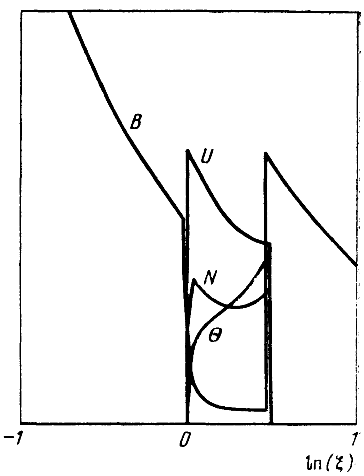

FIG. 7. Collapse of a current-carrying plasma shell onto an axial current, with the entire plasma mass concentrated on the axis at time $t = 0$ (the values of λ and χ are the same as in Fig. 2). The current is partially skinned on the pinch surface.

Another example is the solution for $\lambda = -2, \chi = 0$, for which the singular point P_0 is an attracting node as $\xi \to 0$, and the asymptotic expressions (33) describe the situation near the axis. Here, there is no reasonable way of passing through the plane (24) and continuing the solution on the other side of it. The self-similar solutions corresponding to the segment of the trajectory between P_0 and some point Q' at which the plane (24) is crossed for $\xi = \xi'$ is valid up to the time t' at which the self-similar coordinate of the tangential discontinuity, ξ_q, becomes equal to ξ'. As this time is approached, the profiles of self-similar variables near the pinch boundary continue to sharpen up until their derivatives with respect to ξ become infinite at time $t = t'$, which corresponds to the formation of a strong discontinuity (shock wave) on the outer boundary of the pinch, after which the solution becomes meaningless. The corresponding profiles are shown in Fig. 8.

7. CONCLUSIONS

The above solutions enable us to describe the dynamics of a current channel under compression, as follows. As the electric current in the pinch circuit increases, the plasma in the current channel is concentrated on the axis by the converging shock wave and the collapsing plasma shells. Since only part of the plasma mass is concentrated in this way, the compression process takes the form of several cycles in which shock waves reflected from the axis are replaced by converging shocks. Different stages of this process are described by the above self-similar solutions. Eventually, the process of the essentially inhomogeneous pulsing compression relaxes to homogeneous deformation for which the current in the external circuit remains constant (near the maximum) and each plasma particle is associated with a particular phase of the profile representing the pinch structure. Dissipative processes may turn out to be important during the subsequent stages. The flow structure in the latter case will be examined in another paper.

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[1)]For a concrete investigation of the z-pinch dynamics in terms of the self-similar solutions, see Ref. 27.

[2)]A similar situation was encountered for the first time in physics in the theory of thermal propagation of a flame and in detonation theory (see Ya. B. Zel'dovich, Collected Papers, Vol. 1, Nauka, Moscow, 1984).

[3)]The tangential discontinuity across the plasma-vacuum boundary is a special case.[25,27]

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