Symmetry and dynamics of systems with toroidal moments

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The physical nature of the order parameter of a toroidal current state is analyzed. The toroidal order parameter can be identified with the toroidal moment density only in the static case. In the dynamic case, the toroidal order parameter also contributes to the electric polarization. This contribution does not reduce to the pattern of the electric field radiated by a toroidal dipole. The distinctive features of the dynamic effects in the toroidal state stem from the asymmetry of the electron momentum spectrum. The case can be made that the toroidal state is a universal model of orbital antiferromagnetism. An alternating electric field can induce a magnetic moment in a toroidal state. Its pronounced anisotropy permits acoustic oscillations to be excited by applying alternating electric and magnetic fields to a toroidal subsystem.

The model of an excitonic insulator[1] opens up broad opportunities for describing various types of charge and magnetic (current or spin) ordering in a system of collectivized electrons. A phase transition occurs in this model because of congruent regions in the electron spectrum, and the long-range order is described by the anomalous expectation value $\Delta \sim \langle \psi^{+}_{1\sigma} \psi_{2\sigma'} \rangle$, where the operator ψ^{+}_{σ} creates an electron with a spin σ, and the indices 1 and 2 specify different congruent regions of the spectrum. In general, the order parameter Δ has the complex structure

$$\Delta = \Delta_{\mathrm{Re}}{}^{s} + \sigma \Delta_{\mathrm{Re}}{}^{t} + i(\Delta_{\mathrm{Im}}{}^{s} + \sigma \Delta_{\mathrm{Im}}{}^{t}); \qquad (1)$$

here the σ are the Pauli matrices, and the $\Delta^{S,t}_{\mathrm{Re,Im}}$ are real quantities. The singlet real component (Δ^{S}_{Re}) and the triplet imaginary component (Δ^{t}_{Im}) of the order parameter are invariant under time reversal, while the singlet imaginary component (Δ^{S}_{Im}) and the real triplet component (Δ^{t}_{Re}) change sign upon time reversal. Consequently, the formation of Δ^{t}_{Re} or Δ^{S}_{Im} in a system gives rise to a magnetic order. The transition to a state of a spin antiferromagnetic is described by the real triplet component of the order parameter, Δ^{t}_{Re}. Chromium is a classical example of such an antiferromagnet with collectivized electrons.[2] Ferromagnetic order can be achieved as a result of the coexistence of Δ^{t}_{Re} and Δ^{S}_{Re} (Ref. 3).

The properties of the magnetic state which results from the formation of an imaginary singlet order parameter Δ^{S}_{Im} are extremely interesting.[4,5] A singlet spin structure of the orbit parameter implies that the magnetic ordering is of an orbital nature. A more detailed study of its properties requires incorporating the symmetry of the wave functions of the collectivized electrons in some way. This information is embodied in the interband momentum matrix element $\mathbf{P}_{12}$, which can be nonzero if the coordinates of congruent parts of the spectrum coincide in momentum space. If $\mathbf{P}_{12} \neq 0$, the formation of Δ^{S}_{Im} leads to a loss of symmetry by the system not only with respect to time reversal but also with respect to space inversion.[6] The macroscopic density of the magnetic momentum is zero; i.e., such a state is an orbital antiferromagnet according to the standard classification.[7] If, on the other hand, the order parameter is macroscopically inhomogeneous, a macroscopically nonuniform current arises in the system[8,5]:

$$\mathbf{j} = \frac{e}{c} \operatorname{rot} \operatorname{rot} (\gamma_T \mathbf{P}_{12} \Delta_{\mathrm{Im}}{}^{s}); \qquad (2)$$

where e is the electron charge, c is the velocity of light, and the coefficient γ_T is expressed in terms of the parameters of the microscopic model. Using the definition

$$\mathbf{M} = \frac{1}{2c} \int [\mathbf{rj}] dV$$

of the magnetic moment, we can easily verify that the total magnetic momentum of any sample of finite dimensions is again zero for an inhomogeneous Δ^{S}_{Im}, as it is in the case $\Delta^{S}_{\mathrm{Im}} = \mathrm{const}$.

In an external magnetic field, a Lorentz force acts on the currents in (2), deforming the system and raising its energy.[9,10] The response of a system with a spontaneous current as in (2) is thus dynamic in nature. The actual macroscopic currents in (2) should be distinguished from the Ampère currents $\mathbf{j}_s = \mu_0 \operatorname{rot} \mathbf{S}(\mathbf{r})$ [$\mathbf{S}(\mathbf{r})$ is the spin density, and μ_0 is the Bohr magneton], on which the Lorentz force does not act. The interaction with the magnetic field $\mathbf{H}(\mathbf{r})$ is described by the paramagnetic term $\mathbf{H}(\mathbf{r}) \cdot \mathbf{S}(\mathbf{R})$. This circumstance determines the different behavior in a magnetic field of an orbital magnetic material with $\mathbf{P}_{12}\Delta^{S}_{\mathrm{Im}} \neq 0$ and of spin magnetic materials with the same symmetry.

The order parameter Δ^{S}_{Im} describes microscopic correlations of the wave functions and in this sense is a microscopic parameter. The order parameter in a state with $\mathbf{P}_{12}\Delta^{S}_{\mathrm{Im}} \neq 0$ was given a macroscopic physical meaning in Ref. 11, where it was shown that the density of the toroidal momentum $\mathbf{T}$ serves as a macroscopic order parameter in this state, and the state which results from the transition can naturally be called a "toroidal current state." Analogously, in a ferroelectric the role of a microscopic order parameter is played by microscopic displacements, which unambiguously determine a macroscopic characteristic, the polarization. The toroidal moment is a fundamental characteristic of a medium

in the same way that the electric dipole moment (polarization) or magnetic moment is. The density of the toroidal dipole moment **T** is defined in terms of an integral over a unit cell[12]:

$$\mathbf{T}=\frac{1}{10c}\int [\mathbf{r}(\mathbf{r}\mathbf{j}(\mathbf{r}))-2r^2\mathbf{j}(\mathbf{r})]dV. \tag{3}$$

Electric, magnetic, and toroidal dipoles generate three independent families of electromagnetic multipoles.[12] The interaction of an arbitrary distribution of charges and currents with electric and magnetic fields can be described completely by simply specifying the multipole moments of all three types of the system.

In the microscopic model of Ref. 5 the density of the toroidal moment is introduced[11] in accordance with (2):

$$\mathbf{T}=\frac{e}{c}\gamma_T\mathbf{P}_{12}\Delta_{1m}{}^{S}. \tag{4}$$

The minimum length scale over which we are justified in introducing the toroidal moment density is determined by the correlation length of the order parameter. Geometrically, a toroidal dipole may be thought of as a current-carrying solenoid which has been bent into a torus; the current is flowing along the meridians of the torus. A microscopic object of this type was first studied by Zel'dovich.[13] A toroidal current configuration arose in Ref. 13 as a consequence of parity-breaking electroweak interactions. An important point is that the toroidal momentum in a toroidal current state[5,11] owes its existence to ordinary Coulomb interactions.

Our basic purpose in the present study is to answer the following question: What are the similarities and differences between the properties of a toroidal current state and those of other magnetic systems having a toroidal moment?

The toroidal order parameter introduced in Ref. 11 is a new type of order parameter, not only because of its symmetry properties but also because of its physical content. There have been previous studies of t-odd polar vectors in the theory of phase transitions and magnetism. An antiferromagnetism vector, for example, may exhibit such transformation properties under certain conditions. In §1 below we show that an antiferromagnetic vector and the vector toroidal moment may be coupled but they will never coincide. In a broad class of spin magnets the toroidal moment density is nonzero, but only in the static limit are the properties of these systems analogous to those of a toroidal current state.

The use of an intuitive geometric model of a toroidal dipole leads to a simple qualitative explanation for essentially all the properties of a toroidal current state. However, according to the analysis in §2, it is correct to identify the singlet imaginary order parameter with the toroidal moment density **T** only in the static case. In the dynamic case, the toroidal order parameter defined as in (4) also contributes to the electric polarization; this contribution does not reduce to the pattern of the electric field radiated by a toroidal dipole. Dynamic effects are determined by the shape of the electron spectrum and may be used to experimentally distinguish a toroidal current state from spin magnets with a toroidal moment, since the latter make a relativistically small contribution to the dynamics. Here, in §2, we point out that there is a basis for regarding any antiferromagnetic ordering of an orbital nature as a toroidal current state.

In the last two sections of this paper we discuss some specific problems of the macroscopic dynamics of a toroidal current state. In §3 we show that in addition to the direct and inverse Faraday effects and birefringence, which are allowed by the symmetry of a toroidal current state, linearly polarized light may induce a magnetic moment in a toroidal current state. In §4 we study the relationship between toroidal and acoustic oscillations. Because of the pronounced anisotropy of a toroidal current state, it is possible to excite acoustic oscillations by applying electric and magnetic fields to the toroidal subsystem.

§1. SYMMETRY OF MAGNETIC MATERIALS; STATIC TOROIDAL POLARIZABILITY

The toroidal moment **T** is introduced as a quantity which describes the interaction of a system with a nonuniform vortical magnetic field. In the expression for the free energy density, the term

$$\delta\mathcal{F}_T=\mathbf{T}\,\mathrm{rot}\,\mathbf{B} \tag{5}$$

corresponds to the interaction of the toroidal moment with the electromagnetic field. Expression (5) arises in a multipole expansion of the interaction term **jA**, containing the total magnetic vector potential **A**, which determines the magnetic induction **B** (**B** = curl **A**). This is the structure of the term representing the interaction of the microscopic order parameter Δ_{1m}^{S} with the magnetic field. A variation of interaction term (5) over the vector potential gives rise to a dependence of the current on the order parameter like that in (2). To determine the equilibrium toroidal moment density in a magnetic field, we must add the term in (5) to the free energy density and vary the free-energy functional:

$$\delta F\{\mathbf{T}\}/\delta\mathbf{T}=0,\quad F\{\mathbf{T}\}=\int\{\mathcal{F}_0(\mathbf{T})+\delta\mathcal{F}_T\}dV. \tag{6}$$

Near the temperature of the toroidal transition, the free energy density has the usual form of a Landau expansion:

$$\mathcal{F}_0(\mathbf{T})=\alpha_T|\mathbf{T}|^2+\beta_T|\mathbf{T}|^4+g_{ij}\frac{\partial\mathbf{T}}{\partial x_i}\frac{\partial\mathbf{T}}{\partial x_j}; \tag{7}$$

here $\alpha_T=a(\Theta-\Theta_c)$, Θ_c is the toroidal transition temperature, and $a, \beta_T, g_{ij}>0$. The choice of the toroidal moment density as the order parameter in (7) is based on a calculation in a microscopic model. We obtain the functional (7) by replacing Δ_{1m}^{S} in the functional for the microscopic order parameter by T in accordance with (4). Here the coefficients in (7) are determined unambigously by the parameters of the microscopic model.

As was shown in Ref. 7, a magnetoelectric effect occurs in a toroidal current state by virtue of the following component of the energy:

$$\delta\mathcal{F}_{ME}=\lambda\mathbf{T}[\mathbf{EB}]=e_{ijk}\lambda_{ij}^{a}[\mathbf{EB}]_k; \tag{8}$$

where **E** is the electric field, e_{ijk} is the completely antisymmetric third-order tensor, λ_{ij}^{a} is an antisymmetric component of the magnetoelectric tensor, and the coefficient λ is

expressed in terms of the parameters of the microscopic model.

Expressions (5) and (8) and the condition for a minimum of the free energy, (6), determine the static toroidal polarizability, which is understood not only as the induction of a toroidal moment in external fields but also as a general change in the energy of the system due to the terms in (5) and (8). The physical picture which gives rise to the components in (5) and (8) of the energy of the interaction of the medium with the electromagnetic field may be either orbital (in a toroidal current state) or spin in nature. A toroidal moment has, in addition to an orbital component, a spin component $\mathbf{T}_S$ (Ref. 12):

$$\mathbf{T}_S = \frac{1}{2}\mu_0 \sum_a [\mathbf{r}_a \mathbf{S}_a]; \tag{9}$$

the summation is over the spins of all the charges in the system. An entity having the meaning of a spin toroidal moment also exists in a microscopic model.[6] It can be shown that if $\mathbf{P}_{12} \neq 0$ and $\Delta^t_{\mathrm{Re}} \neq 0$ there is a term in the free energy density which is analogous to (5) and which has a toroidal spin moment density

$$\mathbf{T}_S = \frac{e}{c}\gamma_S [\mathbf{P}_{12} \Delta_{\mathrm{Re}}{}^t]. \tag{10}$$

The existence of a magnetoelectric effect in a state with $\Delta^t_{\mathrm{Re}} \neq 0$ was observed in Ref. 14. The symmetry of the magnetoelectric tensor is given by the relative orientation of the vectors $\mathbf{P}_{12}$ and Δ^t_{Re}, i.e., by relativistic interactions. The magnetoelectric tensor is symmetric if $\mathbf{P}_{12}$ and Δ^t_{Re} are parallel, while it is antisymmetric if they are perpendicular.

The classification of magnetic materials which is ordinarily used is based on the classes of magnetic or Shubnikov (or black-white) symmetry. The symmetry of a structure is described by specifying the set of transformations which leave the given structure invariant. The transformations of magnetic symmetry incorporate all the crystallographic symmetry operations, which now also act on the axial vectors of the magnetic moments, supplemented by the operation of time reversal, which reverses the directions of all the magnetic moments (and currents). In this sense, orbital and spin magnets are completely equivalent and are described by the same magnetic classes. In addition to the magnetic (black-white) symmetry, a so-called color symmetry has been developed for describing spin systems. Color symmetry is based on a separation of the operations which act on structural (crystallographic) variables and which act on spin variables.[15] Color symmetry makes it possible to describe in detail the spin configurations in magnetic materials. Substantial progress was made in this direction after Andreev and Marchenko[16,17] pointed out a way to construct all magnetic structures of an exchange nature (when the exchange interactions responsible for the formation of the magnetic order are far stronger than relativistic interactions). According to Ref. 16, for a given space group of crystal symmetry there is an isomorphism between possible types of exchange structures and irreducible representations of the space group with an overall dimensionality no greater than three. In this manner, an analog of the Nigli-Indenbom theorem in magnetic symmetry is established for exchange classes.

Let us examine the toroidal spin structures which are allowed by the exchange symmetry. This classification of exchange structures is based on an expansion of the spin density $\mathbf{S}(\mathbf{r})$ in the functions $\varphi^{(\alpha)}_{in}(\mathbf{r})$, which implement an irreducible representation of the given symmetry space group of the crystal:

$$\mathbf{S}(\mathbf{r}) = \sum_{i,n,\alpha} \mathbf{S}^{(\alpha)}_{in} \varphi^{(\alpha)}_{in}(\mathbf{r}), \tag{11}$$

where α is the index of the irreducible representation, i specifies the row of the representation, and the index $n = 1, 2, \ldots$ ensures the completeness of the set of functions $\varphi^{(\alpha)}_{in}(\mathbf{r})$ for describing an arbitrary distribution of the unlocalized spin density.[18] Using (11), we find the following expression for the density of the spin toroidal moment in (9):

$$\mathbf{T}_S = \sum_{i,n,\alpha} [\mathbf{S}^{(\alpha)}_{in} \mathbf{d}^{(\alpha)}_i]. \tag{12}$$

Here $\mathbf{d}^{(\alpha)}_i$ is the polar vector of the expectation value of the coordinate operator over the function $\varphi^{(\alpha)}_{in}(\mathbf{r})$, which falls off quite rapidly at infinity:

$$\mathbf{d}^{(\alpha)}_i = \langle \mathbf{r} \varphi^{(\alpha)}_{in}(\mathbf{r}) \rangle.$$

It is easy to see that the only structures which can make a nonvanishing contribution to the toroidal moment density are those which correspond to irreducible representations which are part of the vector representation of the given group. The direction of $\mathbf{d}^{(\alpha)}_i$ is determined by the symmetry of the crystal without consideration of the magnetic structure, but the density of the toroidal moment will be nonzero provided that the magnetic vectors $\mathbf{S}^{(\alpha)}_{in}$ describing the given structure contain components perpendicular to $\mathbf{d}^{(\alpha)}_i$. If $\mathbf{d}^{(\alpha)}_i$ and $\mathbf{S}^{(\alpha)}_{in}$ are parallel, the system is characterized by a magnetic quadrupole moment

$$\mu_{ik}(\mathbf{r}) = \frac{\mu_0}{c}(r_i S_k(\mathbf{r}) + r_k S_i(\mathbf{r})). \tag{13}$$

We can illustrate these arguments with the simple example of crystallographic class $\mathbf{C}_S$. The class $\mathbf{C}_S$ contains two one-dimensional irreducible representations, A' and A'' (A' is a unit representation), and it allows the existence of nine exchange structures.[18] The toroidal moment can be nonzero only in six of the structures, corresponding to the representation A'':

$$(\mathbf{C}_s|A'')[R\sigma_z, RU_2, U_\infty''], \quad (\mathbf{C}_s|A'', A'')[R\sigma_z, RU_2],$$
$$(\mathbf{C}_s|A'', A'', A'')[R\sigma_z], \quad (\mathbf{C}_s|A', A'')[U_2'\sigma_z, RU_2],$$
$$(\mathbf{C}_s|A', A', A')[RU_2''\sigma_z], \quad (\mathbf{C}_s|A', A'', A'')[U_2'\sigma_z].$$

The magnetic structure is given in parentheses; the exchange class (the set of elements which leave the exchange structure invariant) is given in square brackets; U' is a rotation around the axis specified by the direction of the ferrimagnetism vectors $\mathbf{S}'_n$, which transform in accordance with the unitary representation A'; U'' is a rotation around the vectors $\mathbf{S}''_n$, which correspond to representation A''; and U_2 is a (180°) rotation

around the axis perpendicular to $\mathbf{S}_n'$ and $\mathbf{S}_n''$. In all the structures the vector toroidal moment density lies in the symmetry plane. The last three structures, which contain components of the spin density which transform in accordance with the unit representation, are ferrimagnetic. To determine the toroidal moment in these structures unambiguously, we need to subtract the constant component from the magnetization, since the magnitude of the toroidal moment depends on the choice of coordinate system if there is a nonzero resultant magnetic moment.

More complex symmetry classes can be treated in a similar way. The spin structure of Cr_2O_3, for example, a standard collinear antiferromagnet of crystal class $\mathbf{D}_{3d}$, transforms in accordance with the representation A_{2u} (Ref. 17). The Z component of a polar vector transforms in accordance with the same representation. The vector $\mathbf{d}^{2u}$ in (12) is directed parallel to the Z symmetry axis. The vector $\mathbf{S}^{2u}$ has the same direction in Cr_2O_3. Consequently, Cr_2O_3 is a magnetoelectric material, with a nonzero quadrupole moment in (13). In the antiferromagnet Fe_2O_3, the crystal class is the same, $\mathbf{D}_{3d}$, but the magnetic structure is given by the representation A_{2g} (Ref. 17). The vector $\mathbf{d}$ is zero in this representation, and there is no magnetoelectric effect in Fe_2O_3. The toroidal moment, on the other hand, and the corresponding antisymmetric component of the magnetoelectric tensor (8) are nonzero in many magnetic ferroelectrics, e.g., boracites.[19,10]

According to (12), the antiferromagnetic vector $\mathbf{L}$ is always coupled with the toroidal moment $\mathbf{T}$ in a spin magnetic. Furthermore, in certain cases (e.g., if the symmetry transformations which send structurally equivalent atoms of different sublattices into each other do not contain inversion or reflections) an antiferromagnetic vector may transform in a given group as a polar t-odd vector, i.e., as the vector $\mathbf{T}$. However, the magnitudes, directions, and physical meanings of these vectors are different, and there is no polar t-odd vector which can be identified with the toroidal moment. In addition, the transformation properties of the toroidal moment vector and of the antiferromagnetic vector are generally different. The difference can be illustrated by the invariants $\mathbf{M}\cdot\mathrm{curl}\,\mathbf{L}$ and $\mathbf{M}\cdot\mathrm{curl}\,\mathbf{T}$, which have similar structures. The first of these invariants stems from the relativistic spin-spin interaction, and its contribution to the free energy is small. The second invariant stems from the definition of the toroidal moment in (5), and it corresponds to any ordering with a nonzero toroidal moment. This invariant is thus of exchange nature and can be written in the form $\mathbf{M}\frac{\partial}{\partial x_i}\mathbf{I}$, as can be verified easily by expressing the toroidal moment in terms of the antiferromagnetic vectors in (12). If there is a polar direction in a magnetic system, one always finds relativistic interactions which shape the toroidal moment in this direction. This circumstance means that it is possible to single out magnetoelectrics with an antisymmetric magnetoelectric tensor as an independent class. Magnetoelectrics have been classified[20] on the basis of the magnitude of the spin-orbit interaction with respect to the crystal field and with respect to the nature of the concomitant effects: magnetoelectrics, ferroelectrics, ferromagnets,and antiferromagnets. If, on the other hand, we take into account the symmetric component of the magnetoelectric tensor, which transforms under symmetry operations in the same way as the tensor magnetic quadrupole moment (13), while the toroidal and magnetic quadrupole moments given an exhaustive description of the magnetic characteristics of any system in second order of a multipole expansion, we find a new classification of magnetoelectrics:

1) Quadrupole magnetoelectrics. This category includes substances with a symmetric magnetoelectric tensor which belong to the following 27 magnetic symmetry classes:

$$\mathbf{D}_2,\ \mathbf{C}_{2v}(\mathbf{C}_2),\ \mathbf{D}_{2h}(\mathbf{D}_2),\ \mathbf{S}_4,\ \mathbf{C}_4(\mathbf{C}_2),\ \mathbf{C}_{4h}(\mathbf{S}_4),\ \mathbf{D}_4,\ \mathbf{D}_{2d}(\mathbf{D}_2),$$
$$\mathbf{C}_{4v}(\mathbf{C}_4),\ \mathbf{D}_{4h}(\mathbf{D}_4),\ \mathbf{D}_3,\ \mathbf{C}_{3v}(\mathbf{C}_3),\ \mathbf{D}_6,\ \mathbf{D}_{3d}(\mathbf{D}_3),\ \mathbf{D}_{3h}(\mathbf{D}_3),$$
$$\mathbf{C}_{6v}(\mathbf{C}_6),\ \mathbf{D}_{6h}(\mathbf{D}_6),\ \mathbf{D}_{2d},\ \mathbf{D}_4(\mathbf{D}_2),\ \mathbf{D}_{2d}(\mathbf{S}_4),\ \mathbf{C}_{4v}(\mathbf{C}_{2v}),$$
$$\mathbf{D}_{4h}(\mathbf{D}_{2d}),\ \mathbf{T},\ \mathbf{T}_h(\mathbf{T}),\ \mathbf{T}_d(\mathbf{T}),\ \mathbf{O},\ \mathbf{O}_h(\mathbf{O}).$$

2) Toroidal magnetoelectrics. These materials have an antisymmetric magnetoelectric tensor and belong to the 11 magnetic classes

$$\mathbf{C}_{3v},\ \mathbf{C}_{4v},\ \mathbf{C}_{6v},\ \mathbf{D}_{4h}(\mathbf{C}_{4v}),\ \mathbf{D}_{2d}(\mathbf{C}_{2v}),\ \mathbf{D}_3(\mathbf{C}_3),$$
$$\mathbf{D}_{3d}(\mathbf{C}_{3v}),\ \mathbf{D}_{3h}(\mathbf{C}_{3v}),\ \mathbf{D}_4(\mathbf{C}_4),\ \mathbf{D}_6(\mathbf{C}_6),\ \mathbf{D}_{6h}(\mathbf{C}_{6v}).$$

3) Mixed quadrupole-toroidal magnetoelectrics. These materials have a magnetoelectric tensor of general form and belong to the 20 magnetic classes

$$\mathbf{C}_1,\ \mathbf{C}_2,\ \mathbf{C}_S,\ \mathbf{C}_{2v},\ \mathbf{C}_2(\mathbf{C}_1),\ \mathbf{C}_S(\mathbf{C}_1),\ \mathbf{C}_{2v}(\mathbf{C}_S),\ \mathbf{C}_3,\ \mathbf{C}_4,\ \mathbf{C}_6,$$
$$\mathbf{C}_i(\mathbf{C}_1),\ \mathbf{C}_{2h}(\mathbf{C}_2),\ \mathbf{C}_{2h}(\mathbf{C}_S),\ \mathbf{C}_{4h}(\mathbf{C}_4),\ \mathbf{S}_4(\mathbf{C}_2),\ \mathbf{C}_{3h}(\mathbf{C}_3),$$
$$\mathbf{C}_{3i}(\mathbf{C}_3),\ \mathbf{C}_{6h}(\mathbf{C}_6),\ \mathbf{D}_2(\mathbf{C}_2),\ \mathbf{D}_{2h}(\mathbf{C}_{2v}).$$

It was shown above that the toroidal moment density is nonzero in a broad class of spin magnets.[1)] The antiferromagnetic vectors $\mathbf{S}_{in}^{(\alpha)}$ in (11), which carry information on not only the toroidal moment but also other multipole moments, serve as order parameters in these systems. The response of such systems to the static fields described by the interaction terms in (5) and (8), i.e., their static toroidal polarizability, is indistinguishable from the static polarizability of a toroidal current state (it should be recalled that a description of diamagnetism goes beyond the scope of the multipole expansions which we are considering in the present paper). The situation is different with regard to the dynamic polarizability.

§2. DYNAMIC TOROIDAL POLARIZABILITY

The low-frequency dynamics of an ordered system near a phase transition, where the characteristic frequencies are low, is conveniently described by a Lagrangian formalism. As the generalized coordinate and velocity here we should choose the order parameter and its time derivative. In this approach, the relationship between the coefficients of the Lagrangian and the characteristics of the microscopic model is established from the equation of motion, whose role is played by the self-consistency equation for the time-dependent order parameter. At low characteristic frequencies, one can expand the self-consistency equation in a series in the

frequency and thereby reconstruct the form of the kinetic part of the Lagrangian. The potential part, $\mathscr{F}$, on the other hand, of the Lagrangian density $\mathscr{L}$ has exactly the same form as the free energy density in (7).

Kopaev and Tugushev[22] have studied the behavior of a toroidal current state in an alternating electric field. They showed that a term

$$\delta\mathscr{L}_E=\eta\mathbf{TE} \tag{14}$$

in the kinetic part of the Lagrangian [the toroidal order parameter T here is defined in accordance with (4)] gives rise to structural features in the absorption of an electromagnetic field near the toroidal transition. A term analogous to that in (14) arises if we use Maxwell's equation

$$\operatorname{rot}\mathbf{H}=-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \tag{15}$$

in (5). The electromagnetic interaction of the toroidal moment with the electromagnetic field which is described by this substitution contains a relativistically small factor according to (15). This interaction corresponds to the pattern of electromagnetic waves radiated by a toroidal dipole with a time-varying current.[23] The coefficient η found from the self-consistency equation for the microscopic model, in contrast, is not relativistically small; it is determined by microscopic parameters of a Coulomb nature. What is the physical meaning of the interaction (14)?

We write an expression for the energy of the interaction of an arbitrary system with electric and magnetic fields:

$$\hat{H}_{int}=-\hat{\mathbf{d}}\mathbf{E}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}\hat{\mathbf{T}}-\hat{\mathbf{M}}\mathbf{H}; \tag{16}$$

here the operator $\hat{\mathbf{d}}=e\hat{\mathbf{r}}$ represents the electric dipole moment, the operator $\hat{\mathbf{M}}=\mu_0(\hat{\mathbf{L}}+\hat{\mathbf{S}})$ represents the magnetic dipole moment, and the operator

$$\hat{\mathbf{T}}=\frac{-i\mu_0}{5}r^2\nabla+\frac{1}{5}\left[\hat{\mathbf{r}}\left(\mu_0\hat{\mathbf{L}}+\frac{5}{2}\mu_0\hat{\mathbf{S}}\right)\right]$$

represents the toroidal dipole moment.[12] When we transform to the interaction energy in the multipole approximation (16), a term which is a total time derivative drops out of the Lagrangian, and we lose information about the diamagnetic moment. It follows from a comparison of (14) and (16) that the term in (14) describes not the interaction of the toroidal moment with the electric field but a specific dynamic electric polarization. This conclusion can also be derived from the microscopic model. In the microscopic model, the static polarization is determined by the real singlet order parameter[4,5]

$$\mathbf{P}=\gamma_P\mathbf{P}_{12}\Delta_{\mathrm{Re}}{}^S. \tag{17}$$

A direct microscopic calculation by the method of Ref. 5 shows that in the dynamic case the expression for the polarization in (17) can be redefined as

$$\mathbf{P}=\gamma_P\mathbf{P}_{12}\Delta_{\mathrm{Re}}{}^S+\tilde{\gamma}_P\frac{\partial}{\partial t}\mathbf{P}_{12}\Delta_{\mathrm{Im}}{}^S. \tag{18}$$

The quantity in (4) may thus be identified with the toroidal moment density only in the static case. The second term in (18) describes the electron contribution to the polarization; in the static case, the electrons create microscopic currents which form the toroidal moment density of the system. It can thus be said that the toroidal order parameter defined by (4) contributes to the electric polarization. The toroidal polarization is caused not by a change in the current density in a cell [according to (16), this component is relativistically small] but by a shift of the electron density participating in the formation of the toroidal moment.

Analogously, using the microscopic definition of the current density,[5] we can show that the toroidal moment density should also be redefined, and in the dynamic case it has the structure

$$\mathbf{T}=\gamma_T\mathbf{P}_{12}\Delta_{\mathrm{Im}}{}^S+\tilde{\gamma}_T\frac{\partial}{\partial t}\mathbf{P}_{12}\Delta_{\mathrm{Re}}{}^S. \tag{19}$$

Relations (18), (19), and (14) (at $\eta\gg 1/c$) were derived from a completely definite microscopic model of a toroidal current state.[4,5] Certain general questions, however, e.g., regarding the change in the toroidal moment density in an alternating electromagnetic field, can be studied without resorting to a model.

We consider an arbitrary system describable by a Hamiltonian $\hat{H}=\hat{H}_0+\hat{H}_{\mathrm{int}}$, here $\hat{H}_{\mathrm{int}}$ is the Hamiltonian of the interaction with the external field in (16). In first-order perturbation theory in the external field, we find the following expression for the expectation value of the operator $(\hat{\mathbf{q}})$ representing some arbitrary physical quantity in the absence of attenuation, in the long-wave limit:

$$\langle\hat{q}_i\rangle=\frac{1}{\hbar}\sum_{n,k}\left[\frac{(\hat{q}_i)_{0n}(\hat{H}_{int})_{n0}}{\omega_{n0}-\omega-i0}+\frac{(\hat{q}_i)_{n0}(\hat{H}_{int})_{0n}}{\omega_{n0}+\omega+i0}\right][f_0^{(0)}-f_n^{(0)}]; \tag{20}$$

here the index n specifies the set of quantum numbers which describe the stationary excited states of the system (we are assuming that the expectation value of the operator $\mathbf{q}$ over the ground state is zero); k specifies the one-electron wave functions of the ground state, $|0,k\rangle$; $\hbar\omega_{nn'}=E_n-E_{n'}$; E_n is an eigenvalue of the unperturbed Hamiltonian ($H_0'\psi_n=E_n\psi_n$; and $f_n^{(0)}$ is the equilibrium distribution function. Expression (20) makes it possible to introduce, along with the electric dipole polarizability $\alpha_{ij}(\omega)$ and the magnetoelectric polarizability $\beta_{ij}(\omega)$, which are ordinarily used (Ref. 24, for example), as in

$$P_i=\alpha_{ij}(\omega)E_j,\quad M_i=\omega\beta_{ij}(\omega)E_j, \tag{21}$$

a toroidal polarizability $t_{ij}(\omega)$, as in

$$T_i=\omega t_{ij}(\omega)E_j, \tag{22}$$

where E_j is the jth component of the electric field. The polarizability $\alpha_{ij}(\omega)$ is directly related to the dielectric constant (in the nonrelativistic limit) by $\varepsilon=1+4\pi\alpha$, $\beta_{ij}(\omega)$ describes the dynamic magnetoelectric effect, and $t_{ij}(\omega)$ describes the relativistic component of ε due to oscillations of the toroidal moment.

In accordance with the structure of interaction operator (16), we can distinguish in $t_{ij}(\omega)$ in (22) an intrinsic

toroid-toroid polarizability $t_{ij}^{T}(\omega)$,

$$t_{ij}^{T}=\frac{2}{\hbar c}\sum_{k,n}\frac{\langle 0,k|\hat{T}_i|n\rangle\langle n|\hat{T}_j|0,k\rangle}{\omega_{n0}^2-(\omega+i0)^2}\omega_{n0}(f_0^{(0)}-f_n^{(0)}), \quad (23)$$

and a toroid-dipole cross polarizability $t_{ij}^{d}(\omega)$,

$$t_{ij}^{d}=\frac{2}{\hbar}\sum_{k,n}\frac{\langle 0,k|\hat{T}_i|n\rangle\langle n|\hat{d}_j|0,k\rangle}{\omega_{n0}^2-(\omega+i0)^2}(f_0^{(0)}-f_n^{(0)}). \quad (24)$$

It follows from (23) and (24) that the toroid-toroid polarizability t_{ij}^{T} is relativistically small with repect to the toroid-toroid dipole polarizability t_{ij}^{d}. The reason is the form of the operator representing the interaction of the toroidal moment with the electric field in (16). The contribution of an alternating electric field to the toroidal moment density due to the toroid-dipole polarizability (24) and (22) is of the same order of magnitude as the contribution of interaction (14) to the toroidal order parameter in (4). However, these contributions are of different origin, and in the Lagrangian density the toroid-dipole polarizability corresponds to the independent term

$$\delta\mathscr{L}_{PT}=J(\dot{\mathbf{P}}\mathbf{T}-\mathbf{P}\dot{\mathbf{T}}). \quad (25)$$

A nonrelativistic component of the toroidal moment density arises in an alternating field only in systems in which the toroidal moment is of an orbital nature. For spin magnets we can first calculate the expectation value of $\mathbf{S}(\mathbf{r})$ in (9) with the help of (20); after this calculation, we carry out the spatial integration in the definition of the toroidal moment density. The magnetic elements of the operator $\hat{\mathbf{L}}=\mathbf{S}(\mathbf{r})$ which act only on the spin variables in (20) vanish by virtue of the orthogonality of the coordinate (orbital) components of the wave functions. They can differ from zero only to the extent that there is a spin-orbit interaction. In the effective Lagrangian, this circumstance is reflected by the relativistically small value of the coefficient J in (25) for spin magnets.

The term in (14) gives rise to an unusual dependence of the polarizability current on the toroidal order parameter:

$$\mathbf{j}=\gamma_E\frac{\partial^2}{\partial t^2}\mathbf{T}. \quad (26)$$

More detailed information on the time evolution of the toroidal polarization current in (26) can be found by substituting into (26) an expression for the toroidal order parameter induced by the external field, $\mathbf{T}_{\text{ind}}$. At low frequencies, $\mathbf{T}_{\text{ind}}$ is determined from a variation of the Lagrangian. The contribution of first order in the field, $T_{\text{ind}}^{(1)}\sim\dot{E}$, describes structural features in the dielectric constant,[22] but it carries no information about the breaking of t-invariance in a toroidal current state and in this sense is deficient in content. The term of lowest order in the electric field which incorporates the breaking of t-invariance in a toroidal current state has the structure $T_{\text{ind}}^{(2)}\sim T_0(\dot{E})^2$. Terms which are even in the frequency of the external field thus appear in expression (26) for the polarization current. The condition for such contributions to the polarization current to arise can be studied in its general form, without resorting to a multipole expansion, by using a term $\mathbf{A}\tilde{\mathbf{p}}$ ($\mathbf{A}$ is the vector potential) as a perturbation. Analysis of the expression for the nonlinear conductivity of second order in the field[25] shows that terms which are even in the frequency can appear in the polarization current only as the result of an asymmetry of the spectrum of one-electron excitations with respect to the momentum:

$$E_n(\mathbf{k})\neq E_{n'}(-\mathbf{k}).$$

A breaking of t-invariance, i.e., the establishment of magnetic order in the system, has an asymmetry of this type. However, not all arbitrary magnetic orders can give rise to an asymmetry of the spectrum.

In a toroidal current state, an asymmetry of the spectrum arises from the Coulomb interaction.[1] In spin magnets, on the other hand, such an asymmetry is possible only to the extent that there are spin-orbit interactions. In the latter case, we can speak in terms of an induced (or extrinsic) toroidal order parameter in describing the physical picture of the interaction of the system with an alternating electric field, and, as before, we can use a phenomenological description based on relations (14), (26), and (27). The relativistic nature of the dynamic polarizability of spin magnets allows us to make use of dynamic effects for a reliable identification of toroidal current states where these effects are large. A question which arises in this connection is how a toroidal current state is related to other possible models of orbital antiferromagnetism which lead to an asymmetry of the spectrum; in other words, there is the question of how general the model of a toroidal current state is. In spin magnets the definition of an antiferromagnet includes the requirement that the magnetic atoms of the different sublattices be structurally equivalent (otherwise, the system would be ferrimagnetic). In orbital magnets this requirement is naturally replaced by the requirement of a topological equivalence of the contours with a current which make contributions of opposite signs to the magnetization. The simplest configuration on which the vector field of a current is specified and which satisfies this requirement is a torus. (The presence or absence of a center of inversion in such a system is determined by the relative arrangement of the toroidal configurations.) A torus is a "flux guide"[23]: a system which channels the magnetic induction flux (in the absence of an external field) and which allows arbitrary deformations without a disruption of screening. Configurations in which deformations do disrupt the screening correspond to an orbital ferrimagnetism. There is thus a basis for regarding the toroidal current state as a universal model of orbital antiferromagnetism.

As was shown above, the order parameter of a toroidal current state, (4), can be identified in the static case with the density of the toroidal moment of the system. It is pertinent to note that in the microscopic model[4,5] which has already been studied the order parameter is a single-component parameter, (4). The microscopic model of Refs. 4 and 5 is thus not applicable to transitions accompanied by the formation of an orbital toroidal moment in all magnetic classes which allow the existence of a polar t-odd vector (there are a total of 31 such classes; see §1); it is applicable only in those classes where this vector and the corresponding antisymmetric component of the magnetoelectric tensor have one inde-

pendent component. There are 26 such classes. For five classes,

$$C_1,\ C_s,\ C_2(C_1),\ C_i(C_1),\ C_{2h}(C_s),$$

with a two-component and a three-component antisymmetric magnetoelectric tensor, a microscopic theory for a toroidal current state has yet to be constructed.

§3. OPTICAL ACTIVITY AND PHOTOMAGNETISM OF A TOROIDAL CURRENT STATE

The interaction discussed in the preceding section, (14), gives rise to an induced component of the toroidal order parameter which varies as a function of time at the frequency of the external field. In addition to the variable component, the electromagnetic field may induce a static component, bilinear in the vectors **E** and **E***, in the toroidal order parameter. Such an interaction is responsible for the optical activity of a toroidal current state. To find its structure, we use the Onsager relation for the dielectric constant:

$$\varepsilon_{ij}(\omega,\ \{\tau_\alpha\},\ \mathbf{k})=\varepsilon_{ji}(\omega,\ \{-\tau_\alpha\},\ -\mathbf{k}); \tag{27}$$

here $\{\tau_\alpha\}$ is the set of parameters which change sign upon time reversal. In the absence of dissipation, the following relations should also hold[7]:

$$\varepsilon_{ij}(\omega,\ \{\tau_\alpha\},\ \mathbf{k})=\varepsilon_{ji}^*(\omega,\ \{\tau_\alpha\},\ \mathbf{k}). \tag{28}$$

It follows from (27) and (28) that the imaginary antisymmetric component of the dielectric constant, which describes the Faraday effect, is proportional to odd powers of the toroidal order parameter. The corresponding term in the Lagrangian is

$$\delta\mathcal{L}_\Phi\sim -i\lambda_{ij}^a[\mathbf{E}\mathbf{E}^*]_i T_j. \tag{29}$$

Since the vector product in (29) transforms as a t-odd axial vector, the tensor λ_{ij}^a is nonzero in those magnetic classes of toroidal current states which also allow the existence of a magnetic moment. Among the 31 toroidal magnetic symmetry classes, there are 13 classes which meet this condition:

$$C_1,\ C_2,\ C_s,\ C_2(C_1),\ C_s(C_1),\ D_2(C_2),\ C_{2v}(C_s),\ C_4,\ D_4(C_4),\ C_3,\ C_6,\ D_3(C_3),\ D_6(C_6).$$

The gyration vector g in these classes in defined, according to (29), by

$$g_i=\lambda_{ij}^a T_j.$$

The real component of the dielectric constant, which is responsible for birefringence, is a symmetric tensor, bilinear in the parameters τ_α or in the wave vector **k** and one of the parameters τ_α. The contribution to the dielectric constant which is bilinear in the magnetic field and in the toroidal order parameter describes the induction of a magnetic moment by an electromagnetic field in a toroidal current state,

$$M_k=\lambda_{ijkl}^s(E_iE_j^*+E_i^*E_j)T_l. \tag{30}$$

A pseudotensor which is symmetric with respect to the first pair of indices may be nonzero in all toroidal classes. In contrast with the inverse Faraday effect, the magnetic moment in (30) may be induced not only by circularly polarized light but also by linearly polarized light. Expression (30) also holds in the case of dissipation. Interband absorption may promote the occurrence of a toroidal current state both by introducing excess charge carriers and by suppressing processes which fix the phase of the order parameter. [The toroidal order parameter in (30) is proportional to the intensity of the electromagnetic field here.] Accordingly, a toroidal photomagnetism may be observed in materials which in equilibrium are stable with respect to a transition to a toroidal current state but in which there is a large Coulomb constant corresponding to a transition to a toroidal current state. Foremost among such materials are band antiferromagnets and ferroelectrics of the electron type (the vibron model as a particular case). The appearance of a magnetic moment in an electromagnetic field has been observed in GeTe at a temperature below the ferroelectric transition.[26] The light in the experiment of Ref. 26 was unpolarized, however, and nothing was said about a dependence of the magnetic moment on the light intensity; further study is thus required to evaluate the possibility of interpreting the experiment of Ref. 26 on the basis of a toroidal photomagnetism.

§4. ELASTIC OSCILLATIONS IN A TOROIDAL CURRENT STATE

In the effects discussed above, the distinctive features of a toroidal current state are manifested in the magnitude of the interaction of the toroidal order parameter with an external field. The interaction of the toroidal subsystem with the crystal lattice also has some important distinctive features.

The invariance of the Hamiltonian of an ordinary spin ferromagnet or antiferromagnet under spin rotations is known to give rise to a precession Landau-Lifshitz equation in a description of the magnetization oscillations:

$$\dot{\mathbf{M}}=g\left[\mathbf{M}\frac{\delta\mathcal{H}}{\delta\mathbf{M}}\right] \tag{31}$$

(g is the gyromagneic ratio, and $\mathcal{H}$ is the effective Hamiltonian of the magnetic material) and to the existence in the magnon spectrum of modes which soften in the long-wave limit ($\mathbf{q}\to 0$). The crystal anisotropy and the interaction with the acoustic subsystem (Ref. 27, for example) give rise to a gap in the magnon spectrum. The magnetization precession in (31) results from the appearance of a mechanical moment in an external field (the term $-\mathbf{MH}$ in the Hamiltonian). In a toroidal current state in an external field, the mechanical moment associated with the terms in (8) and (14) also acts on the vector **T**. The polar vector **T** in a toroidal current state, however, is rigidly tied to the lattice, since its direction is specified by the direction of the vector matrix element of the momentum, $\mathbf{P}_{12}$; i.e., it is determined by the crystal potential. As a result, the dynamics of the toroidal order parameter **T** is described by the nonprecession equation of motion.

$$\ddot{\mathbf{T}}=\lambda\delta\mathcal{H}/\delta\mathbf{T}. \tag{32}$$

The components of **T** in (32) which are transverse with respect to the initial anisotropy axis of $\mathbf{P}_{12}$ arise only when the interaction of the toroidal and acoustic subsystems is taken into account. Deformations of the lattice in an acous-

tic wave cause local changes in the direction of the given initial symmetry of the vector $\mathbf{P}_{12}$.

Let us consider a medium which is isotropic in terms of its elastic properties in a toroidal current state with a uniaxial symmetry of the toroidal subsystem (e.g., $\mathbf{C}_{6v}$). We direct the intensity of the spontaneous toroidal moment along the sixfold axis (the Z axis).

The free energy functional of the system is

$$F=\int dV\{\mathcal{F}_t+\mathcal{F}_{el}+\mathcal{F}_{t\text{-}el}\}, \tag{33}$$

where the contribution of the toroidal subsystem to the free energy, $\mathcal{F}_t$, is found in accordance with (7) with the additional term $(1/2)\alpha_T'(T_x^2+T_y^2)$. We write the elastic energy as

$$\mathcal{F}_{el}=\mu\left(u_{ij}-\frac{1}{3}\delta_{ij}u_{kk}\right)^2+\frac{K}{2}u_{kk}^2 \tag{34}$$

(μ is the shear modulus, K is the bulk modulus, and u_{ij} is the strain tensor). The energy of the interaction of the toroidal and elastic subsystems can in general be written as

$$\mathcal{F}_{t\text{-}el}={}^1/_2\varkappa_{ijnl}T_iT_ju_{nl}+{}^1/_2\varkappa'_{ijnl}\dot{u}_iT_ju_{nl}. \tag{35}$$

The first term here is responsible for the appearance of transverse components of the interband momentum matrix element $\mathbf{P}_{12}$ (and thus of the vector $\mathbf{T}$) in the static case. The second term is analogous to the term $\mathbf{T\dot{P}}$, where $\mathbf{P}$ is the polarization [cf. (25)]. The number of independent components of the tensors $\hat{\varkappa}$ and $\hat{\varkappa}'$ in the $\mathbf{C}_{6v}$ class is six. $\mathcal{F}_{t\text{-}el}$ can be written in terms of its components as follows:

$$\begin{aligned}\mathcal{F}_{t\text{-}el}&={}^1/_2\varkappa_1T_z^2u_{zz}+2\varkappa_2(T_x^2+T_y^2)(u_{xx}+u_{yy})\\&+\varkappa_3[(T_x^2-T_y^2)(u_{xx}-u_{yy})+4T_xT_yu_{xy}]+\varkappa_4(T_x^2+T_y^2)u_{zz}\\&+\varkappa_6T_z^2(u_{xx}+u_{yy})+4\varkappa_5(T_zT_xu_{xz}+T_zT_yu_{yz})\\&+{}^1/_2\varkappa_1'T_z\dot{u}_zu_{zz}+2\varkappa_2'(T_x\dot{u}_x+T_y\dot{u}_y)(u_{xx}+u_{yy})\\&+\varkappa_3'[(T_x\dot{u}_x-T_y\dot{u}_y)(u_{xx}-u_{yy})+2(T_x\dot{u}_yu_{xy}+T_y\dot{u}_xu_{xy})]\\&+\varkappa_4'(T_x\dot{u}_x+T_y\dot{u}_y)u_{zz}+\varkappa_6'T_z\dot{u}_z(u_{xx}+u_{yy})\\&+4\varkappa_5'(T_z\dot{u}_xu_{xz}+T_x\dot{u}_zu_{xz}+T_y\dot{u}_zu_{yz}+T_z\dot{u}_yu_{yz}).\end{aligned} \tag{35a}$$

Equating the variational derivatives of the functional to zero, we find the following system of equations to describe the ground state:

$$\begin{aligned}&T_{0z}^2=\frac{1}{\beta_T}(\tau_T-\varkappa_1u_{zz}^0-2\varkappa_6(u_{xx}^0+u_{yy}^0)),\\&u_{zz}^0=-\frac{1}{b_1}\left(\frac{1}{2}\varkappa_1T_{0z}^2+b_2(u_{xx}^0+u_{yy}^0)\right),\\&b_1u_{xx}^0+b_2(u_{yy}^0+u_{zz}^0)+\varkappa_6T_{0z}^2=0,\\&b_1u_{yy}^0+b_2(u_{xx}^0+u_{zz}^0)+\varkappa_6T_{0z}^2=0,\\&b_1=K+{}^4/_3\mu,\quad b_2=K-{}^2/_3\mu,\end{aligned} \tag{36}$$

from which we determine $T_0\|z$ and the spontaneous strains

$$u_{xx}^0=u_{yy}^0\neq u_{zz}^0.$$

Linearizing the system of equations of motion in (32) and

$$\rho\dot{u}_i=\frac{\partial}{\partial x_j}\frac{\delta\mathcal{H}}{\delta u_{ij}}\frac{1+\delta_{ij}}{2},$$

we find the spectrum of homogeneous oscillations (with $\mathbf{q}=0$):

1) $\omega=0$; this is a quasiphonon branch.

2) $\omega^2=\lambda(-\alpha_T+3\beta_TT_{0z}^2+\varkappa_1u_{zz}^0+4\varkappa_6u_{xx}^0)$; this is a gap corresponding to a branch of coupled longitudinal toroidal oscillations (longitudinal with respect to the anisotropy axis) and longitudinal acoustic oscillations.

3) $\omega^2=\lambda(\alpha_T'+8\varkappa_2u_{xx}^0+2\varkappa_4u_{zz}^0)$; this is a gap corresponding to the branches of coupled transverse acoustic and toroidal oscillations.

An interesting effect is the excitation of transverse inhomogeneous elastic oscillations in an alternating electric field crossed with an alternating magnetic field, e.g., in the field of an electromagnetic wave. This effect stems from the presence of the $T_zT_xu_{xz}$ term in the free energy, which contributes to the equation of motion of the acoustic subsystem:

$$\rho\ddot{u}_z=\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial u_{xz}}\sim\frac{i}{2}q_xT_{0z}T_x$$

(ρ is the density of the medium).

Since $T_x\sim[\mathbf{EH}]_x$, an external electromagnetic wave incident parallel to the x axis excites transverse acoustic oscillations with a wave vector $\mathbf{q}\|x$ (the displacements of the medium occur along the z axis). In general, one of the fields $\mathbf{E}$ or $\mathbf{H}$ may be constant, and the effect should disappear when it is turned off. The magnitude of the effect is determined by the coefficient $\varkappa_5$ in (35a), which is of a Coulomb nature. An alternating electromagnetic field can also excite elastic oscillations in spin magnets, but there the effect will be determined by the small parameter of the spin-lattice interaction.

An important point is the actual crystal anisotropy of the system. If we replace the crystal class $\mathbf{C}_{6v}$ by a class with a more symmetric basis plane (e.g., $\mathbf{C}_{12v}$), the invariants similar to $T_zT_xu_{xz}$ drop out of the functional. In this case we are left with only the excitation of longitudinal acoustic oscillations, which is associated with invariants of the type $T_z^2u_{zz}$.

In several materials, e.g., magnetic ferroelectrics, we would expect a toroidal current state to coexist with a ferromagnetic order.[10] In this case the free energy functional has terms describing the interaction of the toroidal and magnetic subsystems:

$$\mathcal{F}_{TM}=\pi_{ijkl}u_{ij}M_kT_l-\tau\mathbf{T}\,\mathrm{rot}\,\mathbf{M}-\nu T_z^2M_z^2. \tag{37}$$

The first term in (37) corresponds to a piezomagnetic effect in the toroidal current state. In terms of its transformation properties the tensor π_{ijkl} is analogous to the toroidal photomagnetism tensor λ^S_{ijkl} in (30). Because of the invariant $\mathbf{T}\cdot\mathrm{curl}\,\mathbf{M}$, a minimum arises in the spectrum of one of the quasimagnon branches at finite values of $\mathbf{q}$; as a result, an incommensurate magnetic structure may form.

In addition to interacting directly with the toroidal order parameter in (37), the magnetic moment $\mathbf{M}$ contributes to the magnetic induction. It follows from the results of §§1 and 2 that the total interaction of the toroidal order parameter with the field is of the form

$$\mathcal{F}_{int}=-\mathbf{T}\,\mathrm{rot}\,\mathbf{B}'-\eta\mathbf{T\dot{E}}; \tag{38}$$

here η is a coefficient of Coulomb nature [see (14)], $\mathbf{B}'$ is the

magnetic induction which is external with respect to the toroidal subsystem,

$$\operatorname{rot}\mathbf{B}'=\operatorname{rot}(\mathbf{H}+4\pi\mathbf{M})=\frac{4\pi}{c}\mathbf{j}_{ext}+4\pi\operatorname{rot}\mathbf{M},$$

and $\mathbf{j}_{ext}$ is the external current.

The contribution to the induction from the self-field of the toroidal currents in (2) is of the form $\mathbf{T}\cdot\operatorname{curl}\operatorname{curl}\mathbf{T}$. The contributions of this structure, however, are already embodied in the last term in (7). Their order of magnitude with respect to the terms of the electromagnetic self-effect is $(c/v)^2$ (v is a scale velocity of the material), and it would exceed the accuracy of this treatment to incorporate the electromagnetic self-effect. Analogously, if a magnetic order $\mathbf{M}$ is orbital, and the coefficient τ in (37) does not contain a relativistically small term, the contribution of $\mathbf{M}$ to the induction $\mathbf{B}'$ in (38) can be ignored. In the opposite case, it should be taken into account.

In (38), we should understand $\mathbf{B}'$ as the static component of the electromagnetic field. The interaction of the toroidal order parameter with the alternating component is taken into account in the second term in (38), which does not contain a relativistically small factor, in contrast with the corresponding term found by means of Maxwell's equation from the first term in (38).

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[1)]One should not confuse spin magnets having a nonzero toroidal moment with the toroidal spin state,[21] where there is a toroidal ordering of the spin current.

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