Breakdown of the renormalization group in the theory of localization

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A supermatrix σ-model describing a two-dimensional system of disordered metallic granules is considered. A perturbation theory is developed in the limit of long-range interaction between the granules or large hopping amplitude. In contrast to the perturbation theory applied in other work, the dimensional-regularization procedure that serves to cut off the integrals at large momenta is not used here, since all the integrals converge. It is shown that the coefficients of the logarithms that arise in the two dimensional case depend on the structure of the lattice. This dependence suggests the absence of a one-parameter renormalization group in the theory of localization.

1. INTRODUCTION

The hypothesis, advanced in Ref. 1, of the existence of a renormalization group has had an enormous influence on the development of the theory of disordered metals. According to this hypothesis, the only quantity determining the properties of a disordered metal is the conductance of the system. Renormalization-group equations have been written down for this quantity. Using rather plausible assumptions about the form of the Gell-Mann–Low function, the authors of Ref. 1 predicted localization in an arbitrarily weak random potential in one and two dimensions. In a space with $d > 2$ dimensions a power-law behavior of the kinetic coefficients near the mobility edge was predicted. The hypothesis of the existence of a renormalization group agreed with the results of direct summation of the lowest orders of perturbation theory[2,3] and with the result of the analysis of σ-models[4–6] describing the kinetics of disordered metals. The existence of localization in wires in an arbitrarily weak random potential (first predicted by Thouless[7] and following also from the results of Ref. 1) has been confirmed by an exact microscopic calculation.[8,9]

Essentially, the scheme of the renormalization-group method in the theory of localization coincided fully with the corresponding scheme in the theory of phase transitions.[10,11] This similarity has been confirmed by analysis of perturbation-theory series[2,3] and by investigation of σ-models.[4–6]

However, it became clear rather quickly that there are certain points that distinguish localization theory from the theory of phase transitions. In the theory of phase transitions the main object is the order parameter, which is non zero below the transition point and vanishes above this point. The corresponding σ-models in the theory of phase transitions describe slow variations of this order parameter. In the theory of localization the σ-models describe variations of a certain matrix Q, which is in no way an order parameter for the metal-insulator transition. The eigenvalues of the matrix Q determine the density of states (see, e.g., Ref. 9). But the density of states is a smooth function of the concentration of impurities and does not vanish.[12] Therefore, the matrix Q is not connected directly with the Anderson transition.

It is not by chance that until now there have been debates as to what should be regarded as the upper critical dimensionality in localization theory. This dimensionality is easily determined in theories in which an order parameter exists. It is the absence of an order parameter in localization theory that causes the indicated difficulty. A recent investigation of the Anderson transition on a Cayley tree by the method of supersymmetry[13] led to the conclusion that there is a minimum metallic conductivity, as first predicted by Mott.[14] This result is in sharp contradiction with the predictions of the renormalization group. This makes it timely to carry out a further check on the basic propositions of the renormalization-group method.

At first glance, the σ-models written down for disordered metals by means of the replica method differ little from the corresponding models in the theory of phase transitions. But the limit $n \to 0$, where n is the number of replicas, is nontrivial. It was demonstrated recently[15] that the application of replica σ-models in a problem concerning the statistics of levels, for which exact results are available,[9] yields incorrect answers. Therefore, a proof of renormalizability carried out by the method of replicas cannot be regarded as reliable. The existence of the renormalization group for the supersymmetric σ-model[9] can also not be regarded as proven. This model differs fundamentally from the usual σ-models. The renormalizability of the usual σ-models is a consequence of the invariance of the Hamiltonian under rotations in the spaces of the spins or matrices. Because of this invariance the terms quadratic in the gradients can be written in the same way, whence follows the reproducibility of the Hamiltonian under renormalizations.

In the supersymmetric σ-model[9] there exists a special direction Λ in the space of the supermatrices Q. This is connected with the noncompactness of this model. The existence of a special direction is manifested most clearly in the calculation of the average value $\langle Q \rangle$ determining the density of states. It turns out that this average is always proportional to the supermatrix Λ.

In a situation when a rigorous proof of renormalizability is absent, summation of the perturbation-theory series can serve as a direct method of verifying the existence of the renormalization group. Such calculations have been performed in, e.g., Refs. 2 and 3. However, even direct summation of diagrams requires care. The point is that, in two-dimensional space, the integrals describing the contribution

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of the diffusion modes are formally logarithmically divergent at large and small momenta. The frequency can serve as a natural cutoff at small momenta. The cutoff at large momenta is more difficult. Naturally, in the one-loop approximation the method of cutoff is unimportant and difficulties arise only in the next orders. Of course, one could take into account in explicit form the region of momenta of the order of the inverse mean free path. But this would greatly complicate the calculation. Therefore, for the cutoff one usually applies the less cumbersome procedure of dimensional regularization. According to this scheme all the calculations are performed in a space with $d<2$ dimensions, in which there are no divergences at large momenta, after which one analytically continues to dimensions $d\geqslant 2$. Dimensional regularization makes it possible to eliminate short distances entirely from the analysis. This procedure is widely used in field theory[16] and the theory of phase transitions.[11] The basis for the correctness of the application of dimensional regularization has been our confidence in the universality of the quantities that have been calculated. But the universality, in its turn, is a consequence of the existence of the renormalization group.

If, however, it is not known in advance that the theory is renormalizable, it is impossible to assert that dimensional regularization will give a correct method of cutoff. Therefore, summation of diagrams by means of dimensional regularization can in no way prove the existence of the renormalization group. To verify renormalizability by summing the perturbation-theory series is possible only if we introduce short distances into the theory. For example, it is possible to consider models on a lattice, when the reciprocal lattice spacing serves as a natural cutoff. In this case, if the coefficients of the logarithms turn out to be dependent only on quantities governing the behavior at small momenta (such as the diffusion coefficient), one could suppose that the theory is renormalizable. But if the coefficients of the logarithms depend on the structure of the lattice, i.e., the contribution from short distances turns out to be important, one cannot speak of the existence of a one-parameter renormalization group.

Below we carry out a direct check on the renormalizability of a supersymmetric σ-model on a lattice[13] by means of summation of the perturbation-theory series. It is shown that the coefficients of the logarithms depend on the structure of the lattice, indicating the absence of renormalizability of the theory. We conclude that the hypothesis of Ref. 1 that a renormalization group exists is incorrect.

2. CHOICE OF MODEL

We shall investigate the model proposed in Refs. 9 and 13 for a disordered metal. In this model we consider a system of metallic granules. Electrons can tunnel from one granule to another. In each of the granules there are randomly distributed impurities. It is assumed that the mean free path in the granules is much greater than the interatomic distances.

The study of the kinetics of the electrons in such a system is conveniently carried out using the method of supersymmetry.[6,9] Performing the usual transformations, one can reduce the calculation of the density correlator $K(\mathbf{r},\mathbf{r}')$ that completely determines the kinetics of the noninteracting particles to the form

$$K(\mathbf{r},\mathbf{r}')=-2\pi^2\nu^2\int (Q_{13}^{12})_{rr'}(Q_{31}^{21})_{r'r}\exp(-F[Q])\prod_i dQ_i, \tag{1}$$

where

$$F(Q)=-\sum_{i,j}J_{ij}\,\mathrm{STr}\,Q_iQ_j+2i(\tilde{\omega}+i\delta)\sum\mathrm{STr}\,\Lambda Q_j,$$
$$\tilde{\omega}=\frac{1}{8}\omega\pi\nu V. \tag{2}$$

In formulas (1) and (2) the subscripts i,j,r, and r' label the granules. The constants J_{ij} describe the interaction between the granules. These constants can be expressed in terms of amplitudes T_{ij} for hopping from the ith to the jth granule:

$$J_{ij}=T_{ij}^2\nu^2V_iV_j\equiv T_{ij}^2/\Delta_i\Delta_j. \tag{3}$$

In (1)–(3) the symbol ν denotes the density of states at the Fermi surface, V_i is the volume of the ith granule, ω is the external frequency, and Δ_i is the spacing between the levels in the ith granule (if it is isolated from the other granules).

The symbol STr denotes the supertrace. The supermatrices Q and Λ have dimensions 8×8 and are equal to

$$Q=U\begin{pmatrix}\cos\hat{\theta} & i\sin\hat{\theta}\\ -i\sin\hat{\theta} & -\cos\hat{\theta}\end{pmatrix}\overline{U},$$
$$U=\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix},\quad \Lambda=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad U\overline{U}=1. \tag{4}$$

The form of the matrices u, v, and $\hat{\theta}$ depends on the presence of time-reversal symmetry or central symmetry. The bar above the supermatrix U denotes "charge" conjugation. The definition of this conjugation and the explicit form of the matrices u, v, and $\hat{\theta}$ can be found in the review in Ref. 9. The superscripts on the matrices Q in (1) label the blocks distinguished explicitly in (4), and the subscripts label the elements in these blocks.

The model described by formulas (1) and (2) is a σ-model on a lattice. Of course, if Q varies slowly from granule to granule this model goes over into the continuum supersymmetric σ-model that was considered in Refs. 6 and 9.

Below we study the case when the granules form a two-dimensional regular lattice. Here it is assumed that the interaction can be an arbitrary function of the distance between the ith and jth granules ($J_{ii}=0$). The volumes V_i of all the granules are assumed to be the same. For the continuum model, perturbation theory is well developed in the limit of large diffusion coefficients D. The terms of the corresponding perturbation-theory series are powers of logarithms of the frequency. The coefficients of the logarithms depend only on the bare diffusion coefficient. However, calculations carried out until now have used the dimensional-regularization procedure, which, generally speaking, does not follow from any physical properties of the system. On the other hand, the calculation of the perturbation-theory series for the model defined by formulas (1) and (2) requires no assumptions about the cutoff at large momenta, since the momenta are bounded by the reciprocal-lattice constants.

Therefore, the calculations for the proposed model on a lattice can serve as a check on the basic ideas of the renormalization group. If the coefficients of the logarithms are found to depend only on the diffusion coefficient, the hypothesis of the existence of a renormalization group will be confirmed. But if these coefficients turn out to depend on the structure of the lattice, this fact will disprove the existence of the renormalization group.

For spin models on a lattice there exists a regular method[17] of constructing an expansion applicable for large coupling constants or long ranges of the interaction. In this method one separates out the mean spin in the mean field approximation, after which one performs an expansion in the deviations from this mean spin. An analogous expansion can also be performed for the lattice σ-model (1), (2). The mean value of the supermatrix Q_i in this model is exactly equal to the supermatrix Λ for all coupling constants J_{ij}. Separating out this mean value, we reduce the functional (2) to the form

$$F[Q]=-\sum_{i,j} J_{ij}\,\mathrm{STr}\,(Q_i-\Lambda)\,(Q_j-\Lambda)-\alpha\sum_j \mathrm{STr}\,\Lambda Q_j, \qquad (5)$$

where

$$\alpha=2(J-i(\tilde{\omega}+i\delta)) \qquad J=\sum_j J_{ij}.$$

As in Ref. 17, the subsequent calculation of the integral (1) can be carried out by expanding in the interaction of the deviations from the mean spin (the first term in (5)). It turns out to be somewhat more convenient to do the calculation using decoupling of the second interaction by integration over auxiliary fields. The details of the calculations are presented in the following Sections.

3. REDUCTION TO AN INTEGRAL OVER AUXILIARY FIELDS

We shall calculate the density correlator $K(r, r')$ (1), (5) in the limit of large J_{ij} or long range r_0 of the interaction between the granules. Decoupling the interaction between the granules in (1), (5) by integration over an auxiliary field Z, we reduce the density correlator to the form

$$K(r,r')=-\frac{1}{8}\pi^2\nu^2\int \exp\Big[-\sum_{i,j}(J^{-1})_{ij}\,\mathrm{STr}\,Z_iZ_j\Big]$$
$$\times\frac{\partial}{\partial(A_{31}{}^{21})_r}\frac{\partial}{\partial(A_{13}{}^{12})_{r'}}\exp\Big[\sum_i M(Z_i)\Big]\prod_i dZ_i, \qquad (6)$$

where

$$M(Z)=\ln\int\exp\left[2\,\mathrm{STr}\,Z(Q-\Lambda)+\alpha\,\mathrm{STr}\,\Lambda Q\right]dQ. \qquad (7)$$

In formula (6) the integration is over the supermatrices Z, which have dimensions 8×8 and are self-adjoint ($Z=\overline{Z}$). The supermatrix A_i in (6) is the transverse part of the supermatrix Z_i:

$$A={}^1/_2(Z-\Lambda Z\Lambda). \qquad (8)$$

Correspondingly, the longitudinal part of the supermatrix Z is

$$B={}^1/_2(Z+\Lambda Z\Lambda). \qquad (9)$$

The subsequent calculations will be performed in the following order: We calculate the function $M(Z)$ (7), expanding the exponential in the integral in powers of Z; we substitute $M(Z)$ into (6) and integrate over B; we integrate in (6) over A. All the calculations will be performed with the accuracy needed to obtain the two-loop approximation.

The calculation of the integrals that appear in (7) upon expansion of the exponential in powers of Z is performed by changing to the variables u, v, and $\hat{\theta}$ using the expressions (4). In the present case of broken time-reversal symmetry these variables have the form

$$\hat{\theta}=\begin{pmatrix}\theta & 0\\ 0 & i\theta_1\end{pmatrix},\quad 0<\theta<\pi,\quad \theta_1>0,\quad u=u_1u_2, \qquad (10)$$

$$u_1=\begin{pmatrix}1-2\eta\overline{\eta} & 2\eta\\ -2\overline{\eta} & 1-2\overline{\eta}\eta\end{pmatrix},\quad u_2=\begin{pmatrix}e^{i\varphi\tau_3} & 0\\ 0 & e^{i\chi\tau_3}\end{pmatrix},$$

$$v=\begin{pmatrix}1+2\varkappa\overline{\varkappa} & 2i\varkappa\\ -2i\overline{\varkappa} & 1+2\overline{\varkappa}\varkappa\end{pmatrix}$$

where, in turn,

$$\eta=\begin{pmatrix}\eta_1 & 0\\ 0 & -\eta_1{}^*\end{pmatrix},\quad \overline{\eta}=\begin{pmatrix}\eta_1{}^* & 0\\ 0 & \eta_1\end{pmatrix} \qquad (11)$$

$$\varkappa=\begin{pmatrix}\varkappa_1 & 0\\ 0 & -\varkappa_1{}^*\end{pmatrix},\quad \overline{\varkappa}=\begin{pmatrix}\varkappa_1{}^* & 0\\ 0 & \varkappa_1\end{pmatrix},\quad \tau_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$0\leqslant\varphi\leqslant 2\pi,\quad 0\leqslant\chi\leqslant 2\pi.$$

In (10) and (11) η_1 and $\varkappa_1$ are Grassmann variables, and η_1^* and $\varkappa_1^*$ are the elements conjugate to them.

Performing the integration first over η, η^*, $\varkappa$, $\varkappa^*$, Φ, χ, and then over θ and θ_1, we can obtain the first terms of the expansion of $M(Z)$ in Z. The necessary integrals are given in the Appendix. In principle, the formulas (A.2)–(A.10) do not contain all the integrals that arise when $M(Z)$ is expanded in powers of Z to sixth order. This is connected with the fact that in all cases it will be necessary to integrate over B in order to obtain an effective free-energy functional depending only on A. In the calculation of certain terms it is more convenient to integrate first over B, and next over Q. Accordingly, not all the possible integrals are needed. For example, the Appendix does not contain the integral

$$\langle(\mathrm{STr}\,AQ)^2(\mathrm{STr}\,B(Q-\Lambda))^3\rangle_Q.$$

To write out the terms of the expansion of $M(Z)$ in powers of Z to sixth order is fairly simple. For this it is necessary to make use of the formulas (A.2)–(A.10), which are in fact terms of the cumulant expansion. Because one obtains very cumbersome expressions and, in addition, it will still be necessary to integrate over B, we do not give these expressions here. In the next two Sections the density correlator will be calculated in the one-loop and two-loop approximations.

4. THE ONE-LOOP APPROXIMATION

Before discussing the interaction of the fields A and B, we write the Lagrangian $L_0(Z)$ of these fields without the interaction (the terms quadratic in a and B):

$$L_0[Z]=\sum_{i,j}(J^{-1})_{ij}\,\mathrm{STr}\,Z_iZ_j-\sum M_2(Z_i), \tag{12}$$

where $M_2(Z)$ contains only terms quadratic in Z. Using formulas (A.2)–(A.4) and going over to the Fourier representation in the coordinates, we obtain

$$L_0[A,B]=L_0'[A]+L_0''[B],$$

$$L_0'[A]=\sum_{\mathbf{k}}[(1/J(\mathbf{k})-2/\alpha)\,\mathrm{STr}\,A_{\mathbf{k}}A_{-\mathbf{k}}], \tag{13}$$

$$L_0''[B]=\sum_{\mathbf{k}}[(1/J(\mathbf{k}))\,\mathrm{STr}\,B_{\mathbf{k}}B_{-\mathbf{k}}$$
$$-2c(\mathrm{STr}\,B_{\mathbf{k}}\Lambda\cdot\mathrm{STr}\,B_{-\mathbf{k}}\Lambda-\mathrm{STr}\,B_{\mathbf{k}}\cdot\mathrm{STr}\,B_{-\mathbf{k}})],$$

$$J(\mathbf{k})=\sum_{r'}J_{rr'}e^{i\mathbf{k}(r-r')},$$

where the coefficient c is defined in (A.1).

Using the definition (5) of α, we can convince ourselves that the coefficient of A^2 in (13) vanishes in the limit $\mathbf{k}\to 0$, $\omega\to 0$. The fields A describe the diffusion modes. The fields b are not gapless in the limit $\mathbf{k}\to 0$, $\omega\to 0$. The existence of the fields B leads to an additional effective interaction between the fields A.

Substituting (13) into (6) and calculating the Gaussian integrals over the elements of the matrices A (the integral over B is equal to unity), we obtain an expression for the density correlator in the Fourier representation:

$$K(\mathbf{k})={}^1/_4(\pi\nu)^2(J-J(\mathbf{k})-i\tilde{\omega})^{-1},$$
$$K(\mathbf{k})=V\sum_{r'}K(r,r')e^{i\mathbf{k}(r-r')}. \tag{14}$$

At small momenta it reduces to the usual diffusion pole:

$$K(\mathbf{k})={}^1/_4(\pi\nu)^2(D_0\mathbf{k}^2-i\tilde{\omega})^{-1},\quad D_0=-{}^1/_2J''(0). \tag{15}$$

The coefficient D_0 in (15) plays the role of the effective diffusion coefficient. Formula (15) can be obtained easily from (2) if in (2) we expand the first term with respect to the gradients and choose a definite parametrization of the supermatrix Q.

For example, this supermatrix can be presented in the form[6,9]

$$Q=W+\Lambda(1-W^2)^{1/2}, \tag{16}$$

where W is the "transverse" supermatrix ($W\Lambda+\Lambda W=0$). Substituting (16) into (1) and (2) and calculating the Gaussian integrals, we arrive at (15).

The difference between this approach and those applied previously consists in the fact that now the diffusion modes are known even for large momenta (formula (14)). This makes it possible to include short distances in the theory and to dispense with dimensional regularization.

We now take into account the interaction of the field A, which describes the interaction of the diffusion modes. This interaction is made up of terms in $M(Z)$ that are not quadratic in A, and interactions through the fields B. To find the total effective interaction it is necessary to integrate in (6) over the fields B. This can be done by perturbation theory, by taking as the zeroth approximation the free B-field Lagrangian in (13). The Gaussian integrals that arise are easily calculated if we make use of the following formulas, which can be verified by direct calculation:

$$\langle B_{\mathbf{k}}MB_{-\mathbf{k}}\rangle_B=-2cJ^2(\mathbf{k})M, \tag{17}$$

$$\langle B_{\mathbf{k}}NB_{-\mathbf{k}}\rangle_B=\frac{1}{16}J(\mathbf{k})(\mathrm{STr}\,N+\Lambda\,\mathrm{STr}\,\Lambda N+\tau_3\,\mathrm{STr}\,\tau_3N$$
$$+\Lambda\tau_3\,\mathrm{STr}\,\tau_3\Lambda N),$$

$$\langle\mathrm{STr}\,B_{\mathbf{k}}N_1\cdot\mathrm{STr}\,B_{-\mathbf{k}}N_2\rangle_B={}^1/_2J(\mathbf{k})\,\mathrm{STr}\,N_1N_2$$
$$+cJ^2(\mathbf{k})(\mathrm{STr}\,\Lambda P_1\cdot\mathrm{STr}\,\Lambda P_2-\mathrm{STr}\,P_1\cdot\mathrm{STr}\,P_2),$$

where

$$\langle\ldots\ldots\rangle_B=\int(\ldots\ldots)e^{-L_0''[B]}dB,$$

M is a transverse supermatrix ($M\Lambda+\Lambda M=0$), and N_1, N_2 and N are longitudinal supermatrices ($N\Lambda-\Lambda N=0$). Integrating over B, we arrive at an effective Lagrangian $\tilde{L}[A]$ that now contains only the fields A. Collecting all powers of A up to the sixth, we obtain

$$\tilde{L}[A]=L_0'[A]+L_1[A]+L_2[A], \tag{18}$$

where $L_1[A]$ and $L_2[A]$ contain the terms needed for the calculation of the one-loop and two-loop approximations, respectively. In this Section we consider the one-loop approximation; here, therefore, we write out only $L_1[A]$, in the form

$$L_1[A]=L_{11}[A]+L_{12}[A],$$

$$L_{11}[A]=-\frac{4}{\alpha^4}\sum_{i,j}\left(J_{ij}-\frac{\alpha}{2}\delta_{ij}\right)\mathrm{STr}\,A_i^2A_j^2, \tag{19}$$

$$L_{12}[A]=-\frac{8c}{\alpha^4}\Big[\sum_{i,j}\Big((J^2)_{ij}-\alpha J_{ij}+\frac{\alpha^2}{4}\delta_{ij}\Big)$$
$$\times\mathrm{STr}\,A_i^2\cdot\mathrm{STr}\,A_j^2+\alpha\sum_i(J^2)_{ii}\,\mathrm{STr}\,A_i^2\Big],$$

where

$$(J^2)_{ij}=\sum_k J_{ik}J_{kj}.$$

The term $L_{11}[A]$ in the limit of slow variations of A in space can also be obtained directly from the continuum σ-model of Refs. 6 and 9, which arises after expansion of the finite differences in (2) in powers of the gradients. By substituting (16) into the expression for the free energy of the continuum model, expanding in powers of W, and making the replacement $W=2A/\alpha$, we obtain, in the lowest orders, the sum of $L_0'[A]$ and $L_{11}[A]$ in the limit of long waves.

The term $L_{12}[A]$ in the limit of long waves is small, since it is proportional to the fourth power of the gradients (as $\tilde{\omega}\to 0$). This term has an anomalous structure, since it is a product of traces. Such a term cannot be obtained from the expansion in (2) in powers of the gradients and powers of W. If we had wished to write a continuum model in which the second term in (19) would appear, we should have had to add to the ordinary continuum model[6,9] a term of the type

$$(\mathrm{STr}\,\Lambda\nabla^2Q)^2. \tag{20}$$

Of course, this term contains higher-than-usual powers of the gradients. But it explicitly contains the matrix Λ, which breaks the symmetry. Therefore, in the model under consideration it is not possible to discard the contribution of short distances by means of dimensional regularization, since short distances break the symmetry. It is in this aspect that the supersymmetric σ-model describing localization differs fundamentally from the usual compact σ-models in which the contribution of short distances could not break the global symmetry.

To corroborate this qualitative argument we shall perform the calculation of the density correlator in explicit form. Allowance for the interaction between the A-fields leads to the result that in place of (14) one should write

$$K(\mathbf{k})={}^{1}/_{4}(\pi\nu)^{2}[\tilde{\alpha}(\mathbf{k})/2-J(\mathbf{k})]^{-1},$$
$$2/\tilde{\alpha}(\mathbf{k})=2/\alpha+\Sigma(\mathbf{k}), \tag{21}$$

where $\Sigma(\mathbf{k})$ is the self-energy part.

The Gaussian integrals arising in the perturbation theory are easily calculated if one makes use of the formulas

$$\langle A_{\mathbf{k}}MA_{-\mathbf{k}}\rangle_{A}=0,$$

$$\langle A_{\mathbf{k}}NA_{-\mathbf{k}}\rangle_{A}=\frac{D(\mathbf{k})}{16}(\mathrm{STr}\,N-\Lambda\,\mathrm{STr}\,\Lambda N+\tau_3\,\mathrm{STr}\,\tau_3N-\Lambda\tau_3\,\mathrm{STr}\,\Lambda\tau_3N),$$

$$\langle \mathrm{STr}\,A_{\mathbf{k}}M_1\cdot\mathrm{STr}\,A_{-\mathbf{k}}M_2\rangle_{A}=\frac{D(\mathbf{k})}{2}\,\mathrm{STr}\,M_1M_2,$$

where

$$\langle\ldots\ldots\rangle_{A}=\int(\ldots\ldots)e^{-L_0'[A]}dA,$$

M, M_1, and M_2 are transverse supermatrices, N is a longitudinal supermatrix, and

$$D(k)=[J^{-1}(k)-2/\alpha]^{-1}. \tag{23}$$

The first term in (19) in the present model with broken time-reversal symmetry makes no contribution in the first approximation (cooperons are absent). However, the second and third terms in (19) make a nonzero contribution in first order. After straightforward calculations we write the corresponding contribution $\Sigma_1(\mathbf{k})$ to the self-energy part in the form

$$\Sigma_1(\mathbf{k})=\frac{4c}{\alpha^2}\int\frac{[J(\mathbf{p}-\mathbf{k})-J(\mathbf{p})]^2}{\alpha/2-J(\mathbf{p})}\frac{d^2\mathbf{p}}{(2\pi)^2}. \tag{24}$$

It can be seen from the expression (24) that $\Sigma_1(0)=0$. The last term in (19) just serves to cancel the terms arising from the preceding term in (19) at zero momenta. Diagrammatically, the contribution of the one-loop approximation is depicted by the sum of diagrams in Fig. 1. In this Figure a solid line corresponds to the propagator $D(\mathbf{k})$ (23), and a dashed line depicts the propagator corresponding to the Lagrangian L_0' (13).

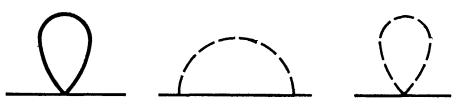

FIG. 1.

Expanding the integrand in (24) in powers of $\mathbf{k}$, we obtain in the region of small momenta

$$\Sigma_1(\mathbf{k})=\frac{4c\mathbf{k}^2}{\alpha^2}\int\frac{[J_x'(\mathbf{p})]^2}{\alpha/2-J(\mathbf{p})}\frac{d^2\mathbf{p}}{(2\pi)^2}. \tag{25}$$

The coefficient of $\mathbf{k}^2$ in (25) determines the correction to the diffusion coefficient D_0 and, as we should expect, is not logarithmic at small ω. Therefore, in this order the breaking of the global symmetry at short distances discussed above does not lead to violation of the predictions of the renormalization group. In the next, two-loop approximation, however, the anomalous terms do lead to the appearance of additional logarithms, the coefficients of which depend on the properties of the model at short distances.

The diffusion-coefficient correction determined by the expression (25) is small if the inequality

$$D_0\gg 1 \tag{26}$$

is fulfilled.

Here, if the range r_0 of the interaction is large, the quantity J can be much smaller than or of the order of unity, since $D_0\sim Jr_0^2$. The quantity $1/D_0$ serves as the parameter in the expansion performed.

5. THE TWO-LOOP APPROXIMATION

In the preceding Section we calculated the contribution of the first approximation. We now consider the next approximation. To calculate the contribution of this approximation it is necessary first of all to obtain the corresponding term $L_2[A]$ in the effective Lagrangian $\tilde{L}[A]$ (18). By rather cumbersome computations, we find

$$L_2[A]=L_{21}[A]+L_{22}[A]+L_{23}[A],$$

$$L_{21}[A]=-\frac{16}{\alpha^7}\sum_{i,j,k}\Big[\mathrm{STr}\,A_i^2A_j^2A_k^2\Big(J_{ik}J_{jk}-\frac{3}{2}\alpha\delta_{ik}J_{ij}+\frac{\alpha^2}{4}\delta_{ij}\delta_{ik}\Big)+\mathrm{STr}\,A_i^2A_kA_j^2A_kJ_{ik}J_{jk}\Big]-\frac{1}{4\alpha^6}\sum_{i,j}\Big[(J_{ij})^2\times(\mathrm{STr}\,A_i^2\cdot\mathrm{STr}\,A_j^2+2(\mathrm{STr}\,A_iA_j)^2-2(\mathrm{STr}\,\Lambda\tau_3A_iA_j)^2)\Big], \tag{27}$$

$$L_{22}[A]=-\frac{64}{\alpha^7}\sum_{i,j,k}\mathrm{STr}\,A_i^2\cdot\mathrm{STr}\,A_j^2\cdot\mathrm{STr}\,A_k^2\cdot\Big\{2c^2(J^2)_{ik}(J^2)_{jk}-c(5c\alpha-8b)(J^2)_{jk}J_{ik}+\Big(2c\alpha(\alpha c-2b)+\frac{3}{4}b\Big)J_{ik}J_{jk}+\alpha\delta_{ik}\Big(c(c\alpha-2b)\Big((J^2)_{ij}-\frac{\alpha}{2}J_{ij}\Big)-\frac{3}{8}bJ_{ij}\Big)+\frac{\alpha^2b}{16}\delta_{ij}\delta_{ik}-\frac{1}{\alpha}\sum_{l}\Big[acJ_{lk}J_{lj}\Big((J^2)_{il}-\frac{\alpha J_{il}}{2}\Big)+\frac{b}{2}J_{ij}J_{lk}J_{li}\Big]\Big\}-\frac{8}{\alpha^6}\sum_{i,j}\mathrm{STr}\,A_i^2\cdot\mathrm{STr}\,A_j^2\Big\{\Big[(3c\alpha-8b)^2-\frac{a^2}{2}+6b\Big]\times(J_{ij})^2+8c^2(J^2)_{ij}(J^2)_{ij}$$

$$-8c(3c\alpha-8b)(J^2)_{ij}J_{ij}-4c\left[3\alpha c+\frac{\alpha^4}{2}\frac{d^2}{d\alpha^2}\left(\frac{c}{\alpha}\right)\right]$$
$$\times(J^2)_{ii}J_{ij}+24c^2(J^2)_{ij}(J^2)_{ii}+32\delta_{ij}c\alpha^6$$
$$\times\frac{d^2}{d\alpha^2}\left(\frac{c}{\alpha^2}\right)(J^2)_{ii}+\sum_k\Big[2c\alpha^2\frac{d^2c}{d\alpha^2}(J^2)_{kk}(J^2)_{ij}$$
$$-\frac{4a}{\alpha^2}(4c\alpha J_{ik}(J^2)_{ik}J_{kj}$$
$$+2c\alpha(J^2)_{kj}(J_{ik})^2-(3c\alpha-4b)(J_{ik})^2J_{kj}\alpha)-\frac{12b}{\alpha}(J_{ik})^2J_{kj}\Big]$$
$$+\frac{2a^2}{\alpha^2}\sum_{k,l}J_{il}J_{jk}(J_{kl})^2\Big\}-\frac{32}{\alpha^6}\sum_i \mathrm{STr}\,A_i^2$$
$$\times\Big\{3c^2\alpha(J^2)_{ii}-3caJ_{ij}(J^2)_{ij}+(J_{ij})^3\Big[-\frac{3}{2}b+a(c\alpha-2b)\Big]$$
$$+\frac{c\alpha^3}{2}\frac{d^2c}{d\alpha^2}(J^2)_{ii}(J_{ij})^2+\frac{1}{2}\frac{a^2}{\alpha}J_{ik}J_{ij}(J_{kj})^2\Big\}.$$

In the formulas (27) the coefficients a, b, and c are defined by the expressions (A.1). The term $L_{23}[A]$ contains terms of the type $\mathrm{STr}A^2A^4$ and $\mathrm{STr}A^4$. In the present case of a system with broken time-reversal symmetry these terms do not give a contribution to the two-loop approximation. Therefore, the explicit form of $L_{23}[A]$ is not given here.

We draw attention to the different structures of $L_{21}[A]$ and $L_{22}[A]$. The term $L_{21}[A]$, like $L_{11}[A]$, in the long-wavelength limit can be obtained directly from the continuum σ-model if we make use of the parametrization (16) and make the replacement $W=2A/\alpha$. Therefore, we may expect in advance that the contribution arising from $L_{11}[A]$ and $L_{21}[A]$ should coincide with the results obtained for the continuum model by means of dimensional regularization.[3,4,6,9] The term $L_{22}[A]$ has an anomalous structure. If the contribution arising from the anomalous terms $L_{12}[A]$ and $L_{22}[A]$ contains logarithms, there are no grounds at all to suppose that the coefficients of these new logarithms will depend only on D_0.

We now perform a direct calculation of the correction to the diffusion coefficient. As in the preceding Section, we calculate the self-energy part $\Sigma(\mathbf{k})$. The terms of the two-loop approximation arise from the terms of first order in $L_2[A]$ and second order in $L_1[A]$. The integrals obtained can be depicted by the graphs in Fig. 2. In this Figure, as in Fig. 1, the solid lines correspond to the propagator of the Lagrangian $L_0'[A]$ in (13), and the dashed lines correspond to the propagator of the Lagrangian $L_0''[B]$. Such graphs also arise in the analysis of spin models.[17] We note that the normal terms $L_{11}[A]$ and $L_{21}[A]$ and anomalous terms $L_{12}[A]$ and $L_{22}[A]$ give all possible graphs. It is not possible to assign each of these two types of terms to particular distinct types of graphs. The first ten graphs correspond to terms of second order in $L_1[A]$, and the others correspond to terms of first order in $L_2[A]$.

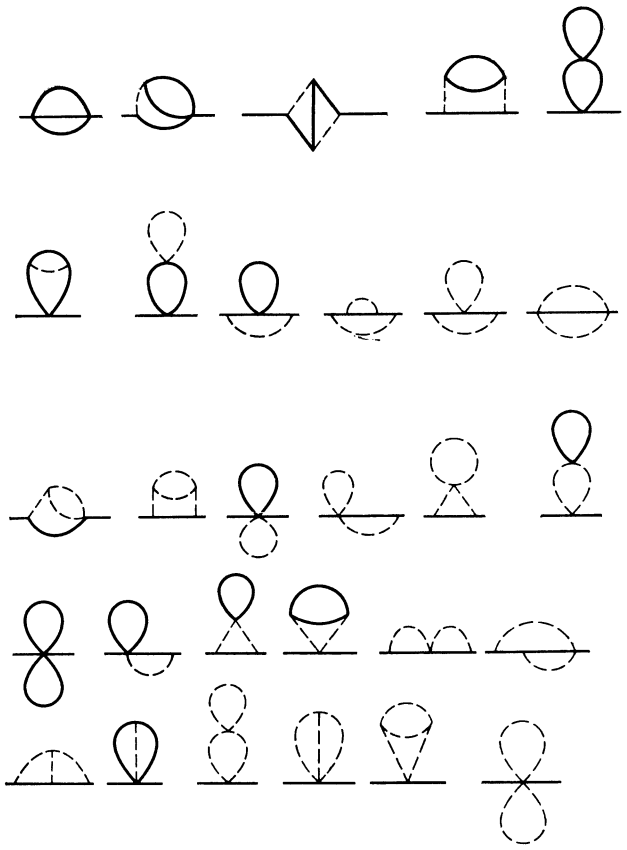

FIG. 2.

We consider first the contribution of the normal terms to $\Sigma_{21}(\mathbf{k})$. Calculating the Gaussian integrals by means of the formulas (22), we obtain after fairly simple computations

$$\Sigma_{21}(\mathbf{k})=\frac{2}{\alpha^8}\int D(\mathbf{k}_2)\Big\{D(\mathbf{k}_1)D(\mathbf{k}_2-\mathbf{k}_1+\mathbf{k})\Big[J^2(\mathbf{k}_1-\mathbf{k})+J(\mathbf{k}_1-\mathbf{k})J(\mathbf{k}_2+\mathbf{k})-2\alpha J(\mathbf{k}_1-\mathbf{k})+\frac{\alpha^2}{2}\Big]$$
$$+\frac{\alpha}{2}D(\mathbf{k}_1)\Big[2J^2(\mathbf{k}_1-\mathbf{k})$$
$$+J^2(\mathbf{k}_1-\mathbf{k}_2)+2J(\mathbf{k}_1-\mathbf{k})J(\mathbf{k}_1-\mathbf{k}_2)+J(\mathbf{k}_1-\mathbf{k})J(\mathbf{k}_2-\mathbf{k})-3\alpha J(\mathbf{k}-\mathbf{k}_2)-\frac{3}{2}\alpha J(\mathbf{k}_1-\mathbf{k}_2)+\frac{3}{4}\alpha^2\Big]$$
$$+\frac{\alpha^2}{4}J(\mathbf{k}_1)[2J(\mathbf{k}_1)+J(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k})]\Big\}\frac{d^2\mathbf{k}_1d^2\mathbf{k}_2}{(2\pi)^4}. \qquad (28)$$

In (28) the propagator $D(\mathbf{k})$ is determined by formula (23).

By direct calculation we can convince ourselves that $\Sigma_{21}(0)=0$. Expanding $\Sigma_{21}(\mathbf{k})$ in $\mathbf{k}$ and calculating the coefficient of $\mathbf{k}^2$, we can also see that only first powers of logarithms appear in this coefficient. These logarithms are gathered from the regions $k_2 \ll 1$, $k_1 \sim 1$ and $k_2 \sim 1$, $k_1 \ll 1$. Relabeling the integration variables and separating out the logarithmic terms, we reduce $\Sigma_{21}(\mathbf{k})$ (28) at small k to the form

$$\Sigma_{21}(\mathbf{k})=\frac{2}{\alpha^8}\sum_{\alpha,\beta}k_\alpha k_\beta \int D(\mathbf{k}_2)\frac{d^2\mathbf{k}_2}{(2\pi)^2}\int\limits_S\Big[D^2(\mathbf{k}_1)J_\alpha'(\mathbf{k}_1)J_\beta'(\mathbf{k}_1) \tag{29}$$
$$+\Big(\frac{\alpha^2}{4}J_{\alpha\beta}''(\mathbf{k}_1)+\alpha J_\alpha'(\mathbf{k}_1)J_\beta'(\mathbf{k}_1)\Big)$$
$$\times D(\mathbf{k}_1)+\frac{\alpha^2}{4}J_\alpha'(\mathbf{k}_1)J_\beta'(\mathbf{k}_1)\Big]\frac{d^2\mathbf{k}_1}{(2\pi)^2}.$$

In (29) k_α and k_β are components of the vector $\mathbf{k}$. In the integral over $\mathbf{k}_1$ the integration extends over the region S, which is the exterior of a circle with center at the coordinate origin and with radius tending to zero. It is not possible to integrate from the origin itself over the entire plane, since D has a pole at $\mathbf{k}=0$. To calculate the integral over $\mathbf{k}_1$ we use the identity

$$J_\alpha'(\mathbf{k}_1)D^2(\mathbf{k}_1)=\frac{dD(\mathbf{k}_1)}{dk_{1\alpha}}J^2(\mathbf{k}_1). \tag{30}$$

Using the identity (30), we integrate the first term in (29) by parts. Then the volume integral that arises cancels the contribution of the remaining terms in the square brackets. As a result the integral over $\mathbf{k}_1$ in (29) is reduced to an integral over a small circle with center at the coodinate origin. Evaluating this remaining integral (which effectively reduces to taking the limit $\mathbf{k}_1 \to 0$) and carrying out the integral over $\mathbf{k}_2$, we obtain

$$\Sigma_{21}(\mathbf{k})=\frac{\mathbf{k}^2}{(16\pi)^2\alpha^2}\frac{1}{D_0}\ln\frac{\tilde\omega_0}{\tilde\omega}. \tag{31}$$

In (31) D_0 is the diffusion coefficient, defined by the expressions (15), and $\tilde\omega_0$ is of the order of J and serves as the cutoff (the corresponding characteristic momentum is of the order of r_0^{-1}). We draw attention to the fact that in the calculation of the coefficient of the logarithm in (31) the contribution of the short distances turned out to be unimportant. Therefore, the integral considered could be calculated by the procedure of analytic continuation in the dimensionality of space from a dimensionality less than two, while rejecting the contribution of short distances at the outset.

The situation with the contribution of the anomalous terms $L_{12}[A]$ and $L_{22}[A]$ is different. Here short distances have a substantial influence on the magnitude of the coefficients of the logarithm that arises. Taking into account terms of first order in $L_{22}[A]$ and second order in $L_{12}[A]$ and calculating the Gaussian integrals by means of formulas (22), we can obtain the corresponding self-energy part:

$$\Sigma_{22}(\mathbf{k})=\frac{256}{\alpha^8}\int\Big\{ c^2D(\mathbf{k}_1)D(\mathbf{k}_2)\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)^2\Big[D(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\Big(J(\mathbf{k}_2+\mathbf{k})-\frac{\alpha}{2}\Big)^2+D(\mathbf{k}_1)\Big(J(\mathbf{k}_1-\mathbf{k}_2)-\frac{\alpha}{2}\Big)^2\Big]$$
$$+D(\mathbf{k}_1)D(\mathbf{k}_2)\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)$$
$$\times\Big[c^2\alpha\Big(J(\mathbf{k}_1-\mathbf{k})\Big(J(\mathbf{k}_2-\mathbf{k})+J(\mathbf{k}_1-\mathbf{k}_2)-\alpha\Big)^2+J(\mathbf{k}_1-\mathbf{k}_2)\Big(J(\mathbf{k}_1-\mathbf{k}_2)-\frac{\alpha}{2}\Big)\Big(J(\mathbf{k}_2-\mathbf{k})+J(\mathbf{k}_1-\mathbf{k})-\alpha\Big)$$
$$-b\Big(J(\mathbf{k}_2-\mathbf{k})-\frac{\alpha}{2}\Big)\Big(J(\mathbf{k}_1-\mathbf{k}_2)-\frac{\alpha}{2}\Big)\Big(\frac{3}{4}+4c\,(2J(\mathbf{k}_1-\mathbf{k})+J(\mathbf{k}_1-\mathbf{k}_2))\Big)\Big]+D(\mathbf{k}_1)J(\mathbf{k}_2)$$
$$\times\Big[\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)\Big(-\frac{3}{4}b\alpha J(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})+\frac{a^2}{8}J(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)+\frac{\alpha^2c}{8}\frac{d^2c}{d\alpha^2}J(\mathbf{k}_2)\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)$$
$$+\frac{\alpha^2c^2}{2}J(\mathbf{k}_2)\Big(3J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)-\frac{\alpha ac}{2}\Big(J(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\Big(J(\mathbf{k}_1-\mathbf{k})-\frac{\alpha}{2}\Big)+J(\mathbf{k}_2)\Big(2J(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})-\frac{\alpha}{2}\Big)\Big)\Big)$$
$$+\frac{\alpha^2c^2}{2}\Big(J^2(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})J(\mathbf{k}_2)-\alpha J(\mathbf{k}_2)J(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})+(J(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k})+J(\mathbf{k}_2))\frac{\alpha^2}{8}\Big)\Big]$$
$$+\frac{\alpha}{16}\Big[\alpha J(\mathbf{k}_1)J(\mathbf{k}_2)J(\mathbf{k}_1-\mathbf{k}_2)\Big(-3b+\alpha ac-\frac{a^2}{2}\Big)$$
$$+\alpha^2cJ^2(\mathbf{k}_1)J^2(\mathbf{k}_2)\Big(\frac{d^2c}{d\alpha^2}+6c\Big)+aJ^2(\mathbf{k}_1)J(\mathbf{k}_2)J(\mathbf{k}_1-\mathbf{k}_2)(a-6\alpha c)\Big]\Big\}\frac{d^2\mathbf{k}_1\,d^2\mathbf{k}_2}{(2\pi)^4}. \tag{32}$$

Using the relation (23), we can convince ourselves that $\Sigma_{22}(0)=0$. The fact that all the self-energy parts at $\mathbf{k}=0$ are equal to zero is a consequence of the law of conservation of particles. Expanding $\Sigma_{22}(\mathbf{k})$ (32) and calculating the coefficients of $\mathbf{k}^2$ we obtain, with logarithmic accuracy,

$$\Sigma_{22}(\mathbf{k})\approx\mathbf{k}^2\frac{8c^2}{\pi\alpha^4}\frac{1}{D_0}\ln\frac{\tilde\omega_0}{\tilde\omega}\int(J_x'(\mathbf{p}))^2J(\mathbf{p})\Big(J(\mathbf{p})-\frac{\alpha}{2}\Big)\frac{d^2\mathbf{p}}{(2\pi)^2}. \tag{33}$$

Formula (33) shows that, in contrast to the case of $\Sigma_{21}''(0)$, the coefficient of the logarithm in $\Sigma_{22}''(0)$ is formed by the region of large (of the order of r_0^{-1}) momenta (the prime denotes differentiation with respect to one of the momentum components).

For large $\alpha \gg 1$ the quantity $\Sigma_{22}''(0)$ contains higher powers of α^{-1}. However, allowance for this term for $r_0 \gg 1$ does not represent an excess of accuracy, since the contributions of the next approximations will contain higher powers of r_0^{-1}. But if we consider the case $\alpha \lesssim 1$, while preserving the inequality $D_0 \gg 1$, then $\Sigma_{21}''(0)$ and $\Sigma_{22}''(0)$ become equal in order of magnitude.

6. CONTRADICTION WITH THE RESULTS OF THE RENORMALIZATION-GROUP METHOD

In the preceding Sections we have developed a perturbation theory in the long range of the interaction for a system of disordered metallic granules. The chief merit of the calculation offered is that there is no necessity to cut off the integrals artificially at large momenta, since the region of integration is a cell of the reciprocal lattice. The formulas (25), (31), and (33) make it possible to write in explicit form the correction acquired by the diffusion coefficient D as a result of the interaction of the diffusion modes. Substituting (25), (31), and (33) into (21) and comparing with (15), we find

$$\tilde{D}=\tilde{D}_0(1-\delta)-\frac{1}{4\tilde{D}_0}(1-\gamma)\ln\frac{\tilde{\omega}_0}{\tilde{\omega}}, \tag{34}$$

where

$$\tilde{D}=\frac{D}{16\pi}, \quad \tilde{D}_0=\frac{D_0}{16\pi},$$

$$\delta=\frac{c}{D_0}\int\frac{[J_x'(p)]^2}{J-J(p)}\frac{d^2p}{(2\pi)^2},$$

$$\gamma=\frac{2^9\pi c^2}{J^2}\int[J_x'(p)]^2J(p)[J-J(p)]\frac{d^2p}{\pi^2}.$$

In (34) we have taken the limit $\tilde{\omega}\to 0$ in the expressions for δ and γ. If we neglect the quantities δ and γ, formula (34) gives the well known result obtained by means of dimensional regularization.[3,4,6] In this situation all the coefficients of the logarithms should depend only on $\tilde{D}_0$, as should be expected from the existence of the renormalization group that was investigated in Refs. 1, 4, 6, and 9. The quantities δ and γ are small for $J \gg 1$. However, to take them into account does not constitute an excess of accuracy, since the higher terms of the perturbation-theory series contain higher powers of r_0^{-1}. For $J \lesssim 1$ the parameter γ is of order unity.

The appearance of finite coefficients δ and γ that depend on the structure of the lattice contradicts the existence of the renormalization group. In the present limit of a long range r_0 of the interaction between the granules the expression (34) obtained is sufficient to display this contradiction. Indeed, replacing the model (2) by the continuum σ-model and performing the usual transformations in the scheme of the renormalization-group method we arrive at the well known formula.

$$g=g_0\left(1-g_0^2/2\ln\frac{\Omega_0}{\tilde{\omega}}\right)^{-1/2}, \tag{35}$$

where g_0 is the effective charge, proportional to the resistivity, and Ω_0 is the cutoff parameter. The dependence of the coefficient of the logarithm in (34) on the structure of the lattice would not contradict the existence of the renormalization group only if it were possible to force the quantities γ and δ into g_0 and Ω_0 in (35). We shall attempt to do this by expanding the denominator in (35) in powers of the logarithm and comparing the coefficients of the logarithms in (34) and (35). In practice, only the parameter γ is important, since $\delta \ll \gamma$ always holds. A simple calculation makes it possible to establish the following correspondence:

$$g_0=\frac{1}{\tilde{D}_0}(1-\gamma)^{1/3}, \tag{36}$$

$$\Omega_0=\tilde{\omega}_0\exp\varkappa, \quad \varkappa=2\tilde{D}_0^2\frac{1-(1-\gamma)^{1/3}}{1-\gamma}.$$

In order of magnitude the parameter $\varkappa$ is equal to

$$\varkappa\sim\begin{cases} r_0^4, & J\gg 1 \\ \tilde{D}_0^2, & J\ll 1 \end{cases}. \tag{37}$$

In the limit $r_0 \gg 1$, $D_0 \gg 1$ the parameter $\varkappa$ is always large. This means that the cutoff should occur at frequencies much higher than $\tilde{\omega}_0 \sim J$. But this cannot happen, since the continuum limit of the lattice model exists only for $\tilde{\omega} \ll \tilde{\omega}_0$. This indicates that the assumption of the existence of the renormalization group contradicts the formula (34) obtained. Since the expression (34) was obtained by a rigorous microscopic calculation, while the existence of the renormalization group was justified by application of the procedure of dimensional regularization, the correctness of which has not been proven, the conclusion that there is no renormalization group becomes unavoidable.

In the above investigation the assumption of a long range r_0 of the interaction was essential. Intuitively, it seems clear that the question of the existence of the renormalization group is not connected with the magnitude of the range of the interaction. Apparently, the renormalization group does not exist for $r_0 \sim 1$ as well. To verify this assertion, however, would require calculations of higher orders of the perturbation theory, and this does not seem possible.

7. CONCLUSION

It has been shown above that a two-dimensional supersymmetric σ-model with broken time-reversal symmetry, describing a system of metallic granules with impurities, is not renormalizable. Despite the formal similarity, the model under consideration differs strongly in its properties from spin models on a lattice. This difference is a consequence of the noncompactness of the group of supermatrices and the presence of anticommuting Grassman elements as the elements of the matrices. Although for this model in the continuum limit (just as for spin models) it is possible to carry out a renormalization procedure using dimensional regularization, the renormalization-group method is not applicable here. The whole point is that global rotational symmetry is broken in this model. However this symmetry-breaking is manifested only when short distances are considered. The use of dimensional regularization causes information about

short distances to be discarded, and as a result, until now, the symmetry breaking has not been noticed in the treatment of two-dimensional models of localization. There are no grounds to suppose that the situation will change if we consider models that are invariant under time reversal.

Of course, the investigation performed serves as a refutation only of the renormalization group that was proposed in Refs. 1 and 3. It may be conjectured that there exists some more complicated renormalization group, not yet known to us.

According to the scaling hypothesis that was used in Ref. 1, the change of the conductivity of a finite sample with change of the size depends only on the conductance and the magnitude of the change of the size. The established absence of a renormalization group implies that the conductance is not the only quantity determining the properties of a finite sample. Large deviations from the average value are also important.

The hypothesis of the existence of a renormalization group made it possible to conclude that the kinetic coefficients have a power-law behavior near the mobility edge in a space with dimensionality $d>2$. This result is contradicted by an exact analysis in a model on a Cayley tree,[14] in which the minimum metallic conductivity first predicted by Mott (see, e.g., Ref. 15) was obtained. In view of the fact that the renormalization-group hypothesis turns out to be incorrect, there are no grounds to believe in scaling near the mobility edge. The existence of a minimum metallic conductivity and a maximum dielectric permittivity in the region of localization seems to the author to be more natural.

Very strong statements followed from the existence of the renormalization group for $d=2$. According to Refs. 1 and 2, in two-dimensional space all states are localized for an arbitrarily weak random potential. An even more surprising statement was made in Ref. 18, in which it was predicted that the conductivity becomes infinite in two-dimensional systems that are invariant under time reversal but have broken central symmetry. The absence of the renormalization group means that we cannot regard these results as justified either. It seems more natural that there exist in all cases a mobility edge and a minimum metallic conductivity.

Localization in wires with arbitrary weak disorder is not in doubt. Renormalization-group ideas, however, are not necessary to establish this fact.

In conclusion, the author thanks P. B. Vigman, A. I. Larkin, and D. E. Khmel'nitskiĭ for discussion of the results of the work.

APPENDIX

We shall give formulas for the values of the integrals arising in the calculation of $M(Z)$ (7). Introducing for brevity the notation

$$\int \varphi(Z,Q)\, e^{\alpha\, \mathrm{STr}\, AQ}\, dQ \equiv \langle \varphi(Z,Q)\rangle_Q$$

for arbitrary functions $\varphi(Z,Q)$, and

$$c=\frac{1-e^{-8\alpha}}{32\alpha^2},\qquad b=\frac{1}{8}\frac{d}{d\alpha}(\alpha^2 c),\qquad a=\alpha^2\frac{dc}{d\alpha}, \tag{A.1}$$

we write the necessary integrals in the form

$$\langle \mathrm{STr}\, B(Q-\Lambda)\rangle_Q=0, \tag{A.2}$$

$$\langle (\mathrm{STr}\, AQ)^2\rangle_Q=\frac{1}{\alpha}\mathrm{STr}\, A^2, \tag{A.3}$$

$$\langle (\mathrm{STr}\, B(Q-\Lambda))^2\rangle_Q=c((\mathrm{STr}\, B\Lambda)^2-(\mathrm{STr}\, B)^2), \tag{A.4}$$

$$\langle (\mathrm{STr}\, AQ)^2\cdot \mathrm{STr}\, B(Q-\Lambda)\rangle_Q = \frac{1}{\alpha^2}(-\mathrm{STr}\,\Lambda A^2 B+c\alpha\, \mathrm{STr}\, A^2\cdot \mathrm{STr}\,\Lambda B), \tag{A.5}$$

$$\langle (\mathrm{STr}\, B(Q-\Lambda))^3\rangle_Q=\frac{3}{2\alpha^2}[a(\mathrm{STr}\,\Lambda B\cdot \mathrm{STr}\, B^2 - \mathrm{STr}\,\Lambda B^2\cdot \mathrm{STr}\, B)+2b((\mathrm{STr}\,\Lambda B)^2-(\mathrm{STr}\, B)^2)\mathrm{STr}\,\Lambda B], \tag{A.6}$$

$$\langle (\mathrm{STr}\, AQ)^4\rangle_Q-3\langle (\mathrm{STr}\, AQ)^2\rangle_Q^2 = \frac{3}{\alpha^3}(-\mathrm{STr}\, A^4+c\alpha(\mathrm{STr}\, A^2)^2), \tag{A.7}$$

$$\begin{aligned}&\langle (\mathrm{STr}\, AQ)^2(\mathrm{STr}\, B(Q-\Lambda))^2\rangle_Q \\ &-\langle (\mathrm{STr}\, AQ)^2\rangle_Q\langle \mathrm{STr}\, B(Q-\Lambda))^2\rangle_Q \\ &=\frac{1}{\alpha^3}\Big[\mathrm{STr}\, A^2B^2-\mathrm{STr}\,(AB)^2 \\ &\qquad +b\,\mathrm{STr}\, A^2(3(\mathrm{STr}\,\Lambda B)^2-(\mathrm{STr}\, B)^2) \\ &+a\Big(\frac{1}{2}\mathrm{STr}\, A^2\cdot \mathrm{STr}\, B^2-\mathrm{STr}\, A^2B\cdot \mathrm{STr}\, B\Big) \\ &+2(a-4b)\,\mathrm{STr}\, A^2B\Lambda\cdot \mathrm{STr}\, B\Lambda\Big],\end{aligned} \tag{A.8}$$

$$\begin{aligned}&\langle (\mathrm{STr}\, AQ)^4\,\mathrm{STr}\, B(Q-\Lambda)\rangle_Q-6\langle (\mathrm{STr}\, AQ)^2\rangle_Q\langle (\mathrm{STr}\, AQ)^2 \\ &\times \mathrm{STr}\, B(Q-\Lambda)\rangle_Q=\frac{3}{\alpha^4}[3\,\mathrm{STr}\,\Lambda BA^4+3b(\mathrm{STr}\, A^2)^2\,\mathrm{STr}\,\Lambda B \\ &\qquad +(a-4b)(\mathrm{STr}\, A^4\cdot \mathrm{STr}\,\Lambda B+2\,\mathrm{STr}\, A^2\cdot S\,\mathrm{Tr}\,\Lambda BA^2)],\end{aligned} \tag{A.9}$$

$$\begin{aligned}&\langle (\mathrm{STr}\, AQ)^6\rangle_Q-15\langle (\mathrm{STr}\, AQ)^2\rangle_Q\langle (\mathrm{STr}\, AQ)^4\rangle_Q \\ &\qquad +3\langle (\mathrm{STr}\, AQ)^2\rangle_Q^3 \\ &=\frac{45}{\alpha^5}[\mathrm{STr}\, A^6+b(\mathrm{STr}\, A^2)^3+(a-4b)\,\mathrm{STr}\, A^2\cdot \mathrm{STr}\, A^4].\end{aligned} \tag{A.10}$$

In formulas (A.2)–(A.10) the supermatrices A and B are determined by the expressions (8) and (9). Integrals containing an odd number of supermatrices A are equal to zero.

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