Quark propagator and chiral condensate in an instanton vacuum

D. I. D'yakonov and V. Yu. Petrov

B. P. Konstantinov Leningrad Institute of Nuclear Physics
(Submitted 26 February 1985)
Zh. Eksp. Teor. Fiz. **89**, 361–379 (August 1985)

A new mechanism is proposed for the spontaneous breaking of chiral symmetry of strong interactions in the instanton vacuum of quantum chromodynamics. The mechanism is based on the collectivization of zero-fermion modes of individual instantons in a pseudoparticle medium. The quark propagator in an instanton medium is found, and it is shown that the massless pole of the free propagator cancels out, with the quark assuming a momentum-dependent effective mass. The parameters of the instanton medium found previously are used to obtain the value of the chiral condensate $\langle\bar{\psi}\psi\rangle$ and the effective mass of the quark, which are in good agreement with phenomenology.

1. INTRODUCTION

One of the most outstanding phenomena in quantum chromodynamics (QCD) is the spontaneous breaking of the chiral or γ_5 invariance of the theory. It is precisely this that leads to massless baryons, whilst pseudoscalar mesons become light Goldstone pseudoparticles (see, for example, Ref. 1). It would be exceedingly important for the physics of strong interactions to have an understanding of the mechanism responsible for the breaking of chiral invariance and to be able to calculate from first principles the value of the quark-antiquark condensate $\langle\bar{\psi}\psi\rangle$, the pion coupling constant f_π, and so on. This problem has recently become even more interesting in connection with the suggestion that baryons are solitons in the γ_5 phases of the chiral condensate.[2] This means that the physics of not only pseudoscalar mesons but also baryons is determined to a considerable extent by the dynamics of chiral symmetry breaking (in this connection, see the review literature[3]).

There have been published attempts to deduce the spontaneous breaking of chiral invariance from a quark-antiquark gluon exchange model[4] and from gluon exchange, improved by the summation of principal logarithms.[5] This work has shown that the condensate $\langle\bar{\psi}\psi\rangle$ is produced only if the strong-interaction coupling constant α_s is large enough, i.e., of the order of unity. However, perturbation theory, used to deduce the required result, then becomes invalid. The conclusion that appears to ensue from all this work is negative: either the Coulomb interaction between the quark and antiquark is insufficient to ensure a bound state with a negative mass, which precipitates into the condensate, or this occurs outside the framework of perturbation theory.

On the other hand, in chromodynamics, there are essentially nonperturbative fluctuations of the gluon field, i.e., instantons,[6] and, when these are taken into account, the result is a radical change in the situation as compared with the usual perturbation theory. In particular, we have to face the problem of spontaneous breaking of chiral invariance. The discovery of instantons was immediately followed by the recognition of the fact that, if instanton-type fluctuations in the gluon field are significant in the QCD vacuum, many of the features of strong interactions can be naturally explained. This includes the solution of the U_1 problem,[7–9] the appearance of the nonperturbative gluon condensate[10] $\langle F^2_{\mu\nu}\rangle$, a degree of justification for the bag model,[8,11] and the possibility of spontaneous breaking of chiral invariance.[8,12–14] Apart from the fact that the last four of these papers made use of a symmetry-breaking mechanism that, in our view, was incorrect, they were not based on any kind of rigorous theory of the instanton vacuum (because there is no such theory), and any attempt to obtain spontaneous chiral symmetry breaking without such a theory seems premature.

The difficulty in constructing the instanton vacuum is rooted in the infrared divergence of the integrals with respect to the instanton dimensions ρ (the instantons tend to inflate).[8] Allowance for the long-range dipole-dipole interaction between instantons,[16] which has also been investigated,[8] merely enhances the infrared catastrophe because the dipole-dipole interaction effectively leads to attraction, i.e., still greater inflation of the instantons.

At the same time, the recent analysis reported by Shuryak[16,17] has extended still further the range of known properties of the strong-interaction theory that can be naturally explained on the assumption of the dominance of instanton fluctuations in QCD vacuum. Moreover, Shuryak used phenomenological considerations to establish the basic characteristics of the instanton medium to which he referred as the "instanton liquid": the mean separation between pseudoparticles is $\bar{R}\sim(200\ \text{MeV})^{-1}$ and their average size is $\bar{\rho}\sim(600\ \text{MeV})^{-1}$. He showed that an instanton vacuum with these properties could explain the gluon and quark condensates[16] and the mass of the pseudoscalar nonet,[17] to obtain a reasonable value for the mass of the excited π' (Ref. 16), and so on.

Thus, for some time now, we have had sufficient indications that the instanton QCD vacuum is consistent with the required properties of strong interactions, but a theory of the instanton vacuum has not been available.

In view of this situation, we have suggested that the instanton vacuum might be constructed by using a modification of the Feynman variational principle.[18] The idea was to

 0038-5646/85/080204-11$04.00

calculate the QCD partition function on a trial Ansatz in the form of a superposition of gluon configurations of the instanton-anti-instanton type. Quantum fluctuations around the trial Ansatz were also taken into account. The resulting partition function was maximized with respect to the parameters (functions) of the trial Ansatz.

We found[19] an effective repulsion between instantons and anti-instantons, which became stronger than the dipole-dipole attraction even for pseudoparticle separations that were a few times greater than their dimensions. This leads to the stabilization of the pseudoparticle liquid (and to the removal of the infrared catastrophe), and the ratio of the mean separation $\bar{R}$ to the mean size $\bar{\rho}$ is found to be equal to the value required by the phenomenlogical analysis,[16] namely, $\bar{R}/\bar{\rho}\simeq 3$. The dimensional variables of the theory, the nonperturbative gluon condensate $\langle F^2_{\mu\nu}\rangle$, the mean instanton size $\bar{\rho}$, and so on were found as renormalization-invariant (and invariant under the choice of the regularization scheme) combinations of the cutoff and the coupling constant at the two-loop level. Moreover, variation of the instanton "profile function," used to find the best subbarrier transition with allowance for the pseudoparticle medium, led to a glueball mass gap even at the classical level.

Thus, the use of the variational principle leads to a reasonable instanton vacuum, and we have the possibility of studying the more complicated question of spontaneous breaking of chiral invariance when light quarks are introduced into this vacuum. The present paper is devoted to this question.

The logic of our paper is as follows. The Yang-Mills sector of QCD generates singular topological configurations of the instanton and anti-instanton type (the variational principle has led us to the recognition that such configurations must appear if only because they reduce the energy of the vacuum as compared with perturbation theory). The properties of the medium consisting of topological singularities such as the mean separation $\bar{R}$ and the mean dimension $\bar{\rho}$ over which the field falls with distance from the singularities, are largely determined by the gluon sector of the theory. When light quarks are introduced into the instanton medium, their high-frequency component produces some modification in the properties of the medium and participates together with the gluon part in determining the statistical ensemble of instantons, whereas the low-frequency component with frequencies $\lesssim 1/\bar{\rho}$ (which is dominated by zero fermion modes) is responsible for the appearance of the chiral condensate $\langle\bar{\psi}\psi\rangle$ and must be considered in a given statistical ensemble of pseudo-particles. The reaction of the low-frequency component to the properties of the medium reduces to only a reduction in the chemical potential for the pseudoparticles.

Although, for simplicity of calculation, we shall confine ourselves to a simple Ansatz in the form of the sum of the instantons and anti-instantons used in Ref. 19 [see (1)], our results will actually be more general: the essential point in our analysis is that the QCD vacuum be populated by topologically singular fields with mean separation $\bar{R}$ and mean dimension $\bar{\rho}$. We shall show that a medium consisting of topological singularities necessarily gives rise to the quark-antiquark condensate $\langle\bar{\psi}\psi\rangle$ which produces the breaking of the chiral invariance of the theory. We shall then calculate the value of this condensate in terms of $\bar{R}$ and $\bar{\rho}$. Parametrically, $\langle\bar{\psi}\psi\rangle\sim 1/\bar{R}^2\bar{\rho}$.

The new mechanism for the formation of $\langle\bar{\psi}\psi\rangle$ that we have proposed is based on the phenomenon of delocation of zero fermion modes of individual instantons in a pseudoparticle medium. This is fundamentally a collective effect. It arises only in the thermodynamic limit: $N\to\infty$, $V\to\infty$, $N/V\equiv\bar{R}^{-4}=\text{const}$, where $N=N_++N_-$ is the total number of pseudoparticles in the four-dimensional volume V. It is important to emphasize that, in contrast to other mechanisms, chiral breaking in topologically nontrivial fields does not require a coupling constant of the order of unity, although the scale of the breaking, for which the condensate $\langle\bar{\psi}\psi\rangle$ can be used as a measure, is large.

The first sections of this paper are devoted to the formulation of the problem and to an approximate (but physically clear) estimate of the condensate. The Green function for a quark in the instanton medium is found in the second part. We then show that the exact quark propagator remains finite in the chiral limit (the quark mass tends to zero), and the massless pole of the free propagator cancels out, with the quark assuming a momentum-dependent effective mass, which can also be expressed in terms of $\bar{R}$ and $\bar{\rho}$. The correlators for the meson currents in the instanton medium, the mass of the pion, and the pion coupling constant f will be determined in a subsequent paper.

2. PARTITION FUNCTION FOR QCD WITH LIGHT QUARKS

We shall consider N_f flavors of quarks ψ in the gluon field $A_\mu(x)$ which we shall divide into the classical and quantal parts $\bar{A}_\mu(x,\gamma)$ and $B_\mu(x)$, respectively. The field $\bar{A}_\mu(x,\gamma)$ will be taken in the form of the superposition of N_+ instantons (I) and N_- anti-instantons ($\bar{I}$) (the symbol γ_I will represent the set of collective coordinates characterizing the I-th pseudoparticle, the position of the center, the dimension, and the unitary orientation matrix will be denoted by $z_{I\mu}$, ρ_I, and U_I, respectively):

$$\bar{A}_\mu(x,\gamma)=\sum_{I=1}^{N_+}A_\mu^I(x-z_I,\rho_I,U_I)+\sum_{\bar{I}=1}^{N_-}A_\mu^{\bar{I}}(x-z_{\bar{I}},\rho_{\bar{I}},U_{\bar{I}}),\tag{1}$$

$$A_\mu^I=\frac{1}{2i}U_I(\tau_\mu^-\tau_\nu^+-\tau_\nu^-\tau_\mu^+)U_I^+\frac{(x-z_I)_\nu}{(x-z_I)^2}\frac{\rho_I^2}{(x-z_I)^2+\rho_I^2},\tag{2}$$

$$A_\mu^{\bar{I}}=\frac{1}{2i}U_{\bar{I}}(\tau_\mu^+\tau_\nu^--\tau_\nu^+\tau_\mu^-)U_{\bar{I}}^+\frac{(x-z_{\bar{I}})_\nu}{(x-z_{\bar{I}})^2}\frac{\rho_{\bar{I}}^2}{(x-z_{\bar{I}})^2+\rho_{\bar{I}}^2},\tag{3}$$

where $\tau_\mu^\pm$ are $N_c\times N_c$ matrices (N_c is the number of colors), whose top left-hand corner contains the matrices $(\boldsymbol{\tau},\mp i)$ ($\boldsymbol{\tau}$ is the Pauli matrix), and all the other elements are equal to zero. Expressions (2) and (3) constitute, respectively, an instanton and an anti-instanton in the singular gauge for an arbitrary group SU(N_c).

The QCD partition function is (we are using the Euclidean formulation)

$$Z=\int DB_\mu \exp\left\{-\frac{1}{2g^2(M)}\int d^4x\,\mathrm{Tr}\,F_{\mu\nu}^2(\bar{A}+B)\right\}\int D\psi D\psi^+ \times\exp\left\{\int d^4x\psi^+(i\hat{\nabla}(\bar{A}+B)+im)\psi\right\} \quad (4)$$

(a more accurate definition, including a fixed gauge and the extraction of collective coordinates, is given in Ref. 19).

This expression must be regularized and normalized. Following 't Hooft,[7] we shall use the Pauli-Willars regularization, dividing (4) by the expression in which the quantum fields B_μ and ψ have mass $M\to\infty$, and will normalize to the perturbative partition function in which the classical field is not present: $\bar{A}_\mu=0$. In this paper, we shall confine our attention to quantal fluctuations of B_μ and ψ in the single-loop (Gaussian) approximation. The regularized (index r) and normalized (index n) partition functions (4) can then be written in the form (see Refs. 8 and 19):

$$Z_{r,n}^{1-loop}=\frac{1}{N_+!N_-!}\int\prod_{I=1}^{N_++N_-} d\gamma_I J(\gamma_I)\exp\{-\beta(\bar{\rho})u_{int}(\gamma)\} \times\frac{\det(i\hat{\nabla}(\gamma)+im)\det(i\hat{\partial}+iM)}{\det(i\hat{\partial}+im)\det(i\hat{\nabla}(\gamma)+iM)}, \quad (5)$$

where $J(\gamma_I)$ is the (factorized) Jacobian of the transformation to the collective coordinates of the I-th pseudoparticle, evaluated in Refs. 7 and 20:

$$d\gamma_I J(\gamma_I)=d^4z_I dU_I(d\rho_I/\rho_I^5)(\rho_I\Lambda_{P.-V.})^{11/3N_c} \times C_{N_c}\beta(\rho_I)^{2N_c}(M/\Lambda_{P.-V.})^{2/3N_f},$$

$$C_{N_c}=\frac{4{,}66\exp(-1{,}68N_c)}{\pi^2(N_c-1)!(N_c-2)!},\qquad \beta(\rho_I)=\frac{8\pi^2}{g^2(\rho_I)} =\left(\frac{11}{3}N_c-\frac{2}{3}N_f\right)\ln\frac{1}{\rho_I\Lambda_{P.-V.}}. \quad (6)$$

The exponential factor in (5) represents the classical action defect and the nonfactorization of the determinants, and leads to the stabilization of the pseudoparticle medium. It was found and investigated in Ref. 19.

We now turn to the regularized and normalized fermion deteminant in the partition function (5). We shall introduce an intermediate mass M_1 and factorize the determinant ratio in (5) into "high-frequency" and "low-frequency" parts:[21]

$$\frac{\det(i\hat{\nabla}+im)\det(i\hat{\partial}+iM_1)}{\det(i\hat{\partial}+im)\det(i\hat{\nabla}+iM_1)}\,\frac{\det(i\hat{\nabla}+iM_1)\det(i\hat{\partial}+iM)}{\det(i\hat{\partial}+iM_1)\det(i\hat{\nabla}+iM)} =\mathrm{Det}_{low}\mathrm{Det}_{high}. \quad (7)$$

If the partition parameters M_1 is large enough, $\mathrm{Det}_{\mathrm{high}}$ includes large eigenvalues of the Dirac operator, so that it can be factorized with good precision into the product of the corresponding determinants evaluated in the field of the individual instantons. The nonfactorization correction can be monitored semiclassially[22] and is small if the packing parameter of the instanton medium is $\bar{\rho}/\bar{R}\ll1$. We have

$$\mathrm{Det}_{high}=\prod_{I=1}^{N_++N_-}(M\rho_I)^{-2/3}F(M_1\rho_I), \quad (8)$$

where the function $F(M_1\rho)$ is known for $M_1\rho\gg1$ (Ref. 23) and $M_1\rho\ll1$ (Ref. 13):

$$F(M_1\rho)=(M_1\rho)^{2/3}\left[1-\frac{2}{75}\frac{1}{(M_1\rho)^2}+\ldots\right],\qquad M_1\rho\gg1,$$
$$F(M_1\rho)=1{,}34(M_1\rho)[1+(M_1\rho)^2\ln(M_1\rho)^2+\ldots],\qquad M_1\rho\ll1. \quad (9)$$

If there are N_f quark flavors, the right-hand side of (8) must be raised to the power of N_f. Substituting (6), (7), and (8) in (9), we obtain

$$Z_{r,n}^{1-loop}=\frac{1}{N_+!N_-!}\int\prod_{I=1}^{N_++N_-}d^4z_I dU_I(d\rho_I/\rho_I^5)(\rho_I\Lambda)^{11/3N_c-2/3N_f} \times C_{N_c}\beta(\rho_I)^{2N_c}\exp\{-\beta(\bar{\rho})u_{int}(\gamma)\}F^{N_f}(M_1\rho_I)\,\mathrm{Det}_{low}(\gamma,m,M_1). \quad (10)$$

We thus see that the problem reduces to the evaluation of the fermion determinant $\mathrm{Det}_{\mathrm{low}}$ over the low frequencies [it is defined in (7)] and to the averaging of this determinant over the statistical ensemble of pseudoparticles, given by (10).

We note that, according to (7), the partition function (10) should not depend on the partition parameters M_1. However, we shall evaluate $\mathrm{Det}_{\mathrm{low}}$ approximately by including in it only the off-diagonal zero modes, which will lead to a degree of dependence on M_1. This dependence will actually be exceedingly weak in a wide range of variation of M_1, and this will be used as a check on the precision of our calculations.

3. DIAGONALIZATION OF ZERO MODES

It is well known that, in the instanton field, the massless Dirac operator has a zero mode for each type of quark, which is specified by a right-handed (for the instantons) or left-handed (for the anti-instanton) Weyl spinor.[7] When the quark masses are introduced, the zero eigenvalues shift by an amount equal to the mass, and the fermion determinant turns out to be proportional to m^{NN_f} (in the limit of the infinitely rarefied instanton medium) where $N=N_++N_-$ is the total number of pseudoparticles (for simplicity, we have confined our attention to the case of equal mass quarks). This result leads to the well-known paradox that the quark condensate, which is a derivative of the logarithm of the partition function with respect to mass, behaves as N/Vm in the chiral limit ($m\to0$).

For several years, it was believed[8,12] that the resolution from this paradox lay in the so-called 't Hooft determinant interaction.[7] However, the 't Hooft many-quark "interaction" is not a new vertex arising in the theory, but merely a Green function in the instanton field; it does not vanish in the chiral limit and can in no way be used to obtain a Gorkov-type equation for the anomalous mean, in the present case $\langle\bar{\psi}\psi\rangle$. Another approach[13,14] (again erroneous, in our view) is based on the introduction of an effective momentum-dependent quark mass into the Lagrangian and the use of the "self-consistent" equation for this mass in the field of a single (!) instanton. However, the Lagrangian mass of the quark is zero, and the self-consistent equation of Refs. 13 and 14 cannot be deduced from the functional QCD integral.

In our view, the essence of the situation is that, when the instanton medium is present, the zero modes become collectivized in the field of the individual pseudoparticles and cease to be strictly zero modes. It is precisely this delocalization of zero modes that is the effect that leads to the appearance of the condensate $\langle\bar{\psi}\psi\rangle$. This mechanism is qualitatively different from that considered in Refs. 8 and 12–14. We emphasize that $\mathrm{Det}_{\mathrm{low}}$ never reduces to the product of determinants and must be evaluated in the field of all the pseudoparticles simultaneously. In particular, it must not be understood as the effective two-particle interaction between the pseudoparticles.

For simplicity, we begin by considering a fermion in the field of a single instanton $A_{1\mu}$ and a single antiinstanton $A_{2\mu}$, separated by a large but finite distance R. As $R\to\infty$, the Dirac operator $i\hat{\partial} + \hat{A}_1 + \hat{A}_2$ has two solutions ψ_1^R and ψ_2^L which are the zero eigenfunctions of the operators, i.e.,

$$(i\hat{\partial}+\hat{A}_1)\psi_1{}^R=0, \qquad (i\hat{\partial}+\hat{A}_2)\psi_2{}^L=0. \tag{11}$$

When the separation R between the pseudoparticles is finite, we can seek the eigenfunctions of the Dirac operator in the form of the superposition $\psi = c_1\psi_1^R + c_2\psi_2^L$. Combining the Dirac equation

$$(i\hat{\partial}+A_1+A_2)\,(c_1\psi_1+c_2\psi_2)=\lambda\,(c_1\psi_1+c_2\psi_2)$$

with ψ_1^+ and ψ_2^+, and using (11), we obtain

$$c_2(-i\delta)=\lambda c_1, \qquad c_1 i\delta=\lambda c_2,$$
$$\delta=\int d^4x\psi_1{}^+\hat{\partial}\psi_2=\int d^4x\psi_2{}^+\hat{\partial}\psi_1, \tag{12}$$

and hence $\lambda = \pm\delta$, where δ is the overlap integral of the zero modes ψ_1^R and ψ_2^L. The corresponding eigenfunctions have the form $\psi_\pm = (\psi_2^L \pm i\psi_1^R)/\sqrt{2}$.

As expected, the doubly-degenerate level with zero eigenvalue splits into two with eigenvalues $\lambda = \pm\delta$, where the eigenfunction corresponding to $\lambda = -\delta$ is obtained by operating with the matrix γ_5 on the function corresponding to $\lambda = +\Delta$.

The overlap integral δ can readily be evaluated, using the explicit form of the zero modes (i is the spin index and α the color index):

$$[\psi_1{}^R(x-z_1)]_i{}^\alpha=\varphi(x-z_1)\,(\hat{x}-\hat{z}_1)_{ii'}U_1{}^{\alpha\beta}\begin{cases}0, & i'=1.2\\ \varepsilon_{i'\beta}, & i'=3.4\end{cases},$$
$$[\psi_2{}^L(x-z_2)]_i{}^\alpha=\varphi(x-z_2,\rho_2)\,(\hat{x}-\hat{z}_2)_{ii'}U_2{}^{\alpha\beta}\begin{cases}\varepsilon_{i'\beta}, & i'=1.2\\ 0, & i'=3.4\end{cases} \tag{13}$$
$$\varphi(x,\rho)=\rho[\pi(2x^2)^{1/2}(x^2+\rho^2)^{3/2}]^{-1},$$

where $z_{1,2}$ are the centers, $\rho_{1,2}$ the dimensions, $U_{1,2}$ the unitary ($N_c\times N_c$) instanton (1) and anti-instanton (2) orientation matrices, and $\varepsilon_{i\beta}$ the 2×2 antisymmetric tensor. When $\rho_{1,2} < R = |z_1 - z_2|$, we find that (a more general expression is given in the Appendix)

$$\delta\approx-(2\rho_1\rho_2/R^4)R_\mu\,\mathrm{Tr}\,(U_2\tau_\mu{}^+U_1{}^+), \qquad \tau_\mu{}^+=(\tau,-i). \tag{14}$$

The zero modes can also be readily diagonalized in the case of an arbitrary number of instantons and anti-instantons. In fact, let ψ_I be the zero mode in the field of the I-th pseudoparticle, i.e., suppose that it satisfies the equation $(i\hat{\partial} + \hat{A}_I)\psi_I = 0$. We shall seek the eigenfunctions of the total Dirac operator $(i\hat{\partial} + \Sigma_I\hat{A}_I)\psi = \lambda\psi$ in the form of the superposition $\psi = \Sigma c_I\psi_I$. Substituting this expansion into the Dirac equation, canceling with ψ_J^+, and neglecting the overlap integrals of the form $\psi_J^+ A_K\psi_I$ with $I\neq J\neq K$, which contain small terms in the instanton density, we obtain

$$-T_{JI}c_I=\lambda S_{JK}c_K,$$
$$T_{JI}=\int d^4x\psi_J{}^+i\hat{\partial}\psi_I, \qquad S_{JK}=\int d^4x\psi_J{}^+\psi_K\approx\delta_{JK}. \tag{15}$$

We note that, because of the helical properties of the zero modes, the matrix T_{JI} contains zero modes belonging to "like" pseudoparticles, whereas S_{JI} contains modes belonging to "unlike" pseudoparticles. We shall retain in S_{JI} only the diagonal terms, since all the other terms are small with respect to the density of the instanton medium. The eigenvalues of the Dirac operator are thus the eigenvalues of the zero-diagonal $N\times N$ matrix T ($N = N_+ + N_-$ is the total number of pseudoparticles) consisting of the overlap integrals (15). Since $N\sim V^{(4)}$, it is meaningless to speak of the individual eigenvalues λ_n; it is more appropriate in this problem to use the density of states $\nu(\lambda)$.

4. EIGENVALUE DENSITY OF THE DIRAC OPERATOR

We now retain in the low-frequency part of the fermion determinant $\mathrm{Det}_{\mathrm{low}}$ [see (7)] only those eigenvalues that arise during the diagonalization of the zero modes, i.e., we retain only the eigenvalues of the matrix T (15). Nonzero eigenvalues appear in $\mathrm{Det}_{\mathrm{high}}$ when the partition parameters M_1 is small enough. We have

$$\mathrm{Det}_{low}(\gamma,m,M_1)=\prod_n^N\frac{\lambda_n+im}{\lambda_n+iM_1}=\exp\left\{\frac{1}{2}\sum_n\ln\frac{\lambda_n{}^2+m^2}{\lambda_n{}^2+M_1{}^2}\right\}$$
$$=\exp\left\{\frac{1}{2}\int d\lambda\ln\frac{\lambda^2+m^2}{\lambda^2+M_1{}^2}\sum_n\delta(\lambda-\lambda_n)\right\}$$
$$=\exp\left\{\frac{1}{2}\int d\lambda\nu(\lambda)\ln\frac{\lambda^2+m^2}{\lambda^2+M_1{}^2}\right\},$$
$$\nu(\lambda)=\sum_n\delta(\lambda-\lambda_n), \tag{16}$$

where $\nu(\lambda)$ is the spectral density or the density of states. In the limit of infinitely rarefied clouds of pseudoparticles, $\nu(\lambda) = N\delta(\lambda)$, since all the eigenvalues are equal to zero. In a real instanton medium, this δ-function smears out and acquires a certain width which is related to the mean overlap integral of the zero mode (15). To find the expression for $\nu(\lambda)$, we extend the chain of equations (16):

$$\mathrm{Det}_{low}(\gamma,m,M)=\exp\left\{\frac{1}{2}\int d\lambda\ln\frac{\lambda^2+m^2}{\lambda^2+M_1{}^2}\int\frac{ds}{2\pi}e^{i\lambda s}\right.$$
$$\left.\times\sum_n\exp(-is\lambda_n)\right\}=\exp\left\{\frac{1}{2}\int d\lambda\ln\frac{\lambda^2+m^2}{\lambda^2+M_1{}^2}\int_{-\infty}^{+\infty}\frac{ds}{2\pi}e^{is\lambda}\,\mathrm{Sp}\,e^{isT}\right\}, \tag{17}$$

where we have used the fact that λ_n are the eigenvalues of the matrix T, which consists of the overlap integrals (15). The density of states is therefore given by

$$\nu(\lambda)=\int_{-\infty}^{+\infty}\frac{ds}{2\pi}e^{is\lambda}\overline{\operatorname{Sp}e^{isT}}, \tag{18}$$

where the bar represents averaging over the statistical ensemble of instantons. The following general formula is valid:

$$\overline{\operatorname{Sp}e^{isT}}=N\exp\left\{is\frac{1}{N}\overline{\operatorname{Sp}T}+\frac{(is)^2}{2!}\left[\frac{1}{N}\overline{\operatorname{Sp}T^2}-\frac{1}{N^2}(\overline{\operatorname{Sp}T})^2\right]+\frac{(is)^3}{3!}\left[\frac{1}{N}\overline{\operatorname{Sp}T^3}-\frac{3}{N^2}\overline{\operatorname{Sp}T^2}\,\overline{\operatorname{Sp}T}+\frac{2}{N}(\overline{\operatorname{Sp}T})^3\right]+\ldots\right\}. \tag{19}$$

All the leading terms in the density can readily be collected together in this series. To obtain a rought estimate for $\nu(\lambda)$, we retain only the first nonzero term in (19). This is given by

$$N\exp\frac{(is)^2}{2!}\frac{1}{N}\overline{\operatorname{Sp}T^2}=N\exp\left(-\frac{1}{2}s^2\varkappa^2\right). \tag{20}$$

Substituting this in (18), we obtain

$$\nu(\lambda)=N(2\pi\varkappa^2)^{-1/2}e^{-\lambda^2/2\varkappa^2}. \tag{21}$$

As can be seen, the erstwhile zero eigenvalues are now smeared out to a width $\varkappa$ (20). To obtain the required $\mathrm{Det}_{\mathrm{low}}$, we must substitute the above function $\nu(\lambda)$ into the general expression (16) and then integrate with respect to λ. We note that the result is finite as $m\to 0$.

Let us now calculate the width $\varkappa$ of the smeared-out eigenvalues. According to (20) and (15),

$$\varkappa^2=\frac{1}{N}\overline{\operatorname{Sp}T^2}=\frac{2}{N}\overline{\sum_{I,\bar I}\int d^4x\psi_I^+ i\hat\partial\psi_{\bar I}\int d^4y\psi_{\bar I}^+ i\hat\partial\psi_I}$$
$$=\frac{2N_+N_-}{N}\int\frac{d^4z_I}{V}\frac{d^4z_{\bar I}}{V}(d\rho_I)(d\rho_{\bar I})\,dU_{\bar I}\,dU_I\int d^4x\psi_I^+(x-z_I,\rho_I,U_{\bar I})\,i\hat\partial_x\psi_{\bar I}(x-z_{\bar I},\rho_{\bar I},U_{\bar I})\int d^4y\psi_{\bar I}^+(y-z_{\bar I},\rho_{\bar I},U_{\bar I})\,i\hat\partial_y\psi_I(y-z_I,\rho_I,U_I).$$

we now average in this formula over the positions, orientations, and dimensions of the instanton and anti-instanton. We shall use the symbol $(d\rho_I)$ to denote averaging over the dimensions of the pseudoparticles with a δ-shaped weighting function $\mu(\rho)$ obtained in Ref. 19. For the purposes of estimates, we replace all the dimensions ρ_I by the average dimension $\bar\rho$ of the pseudoparticles in the medium. Using the Fourier transforms of the zero modes and the expression for $T_{I\bar I}(p)$ given in the Appendix, we obtain

$$\varkappa^2=\frac{N}{2V}\int\frac{d^4p}{(2\pi)^4}dU_I\,dU_{\bar I}T_{I\bar I}(p)\,T_{\bar I I}(p)$$
$$=\frac{N}{2V}\int\frac{d^4p}{(2\pi)^4}dU_I\,dU_{\bar I}\times 4[\varphi'(p,\bar\rho)]^4\operatorname{Tr}(U_{\bar I}p^+U_I^+)\operatorname{Tr}(U_Ip^-U_{\bar I}^+)$$
$$=\frac{4N}{VN_c}\int\frac{d^4p}{(2\pi)^4}[\varphi'(p,\bar\rho)]^4p^2$$
$$\approx\frac{N}{VN_c}\bar\rho^2\cdot 6.6=\frac{6{,}6}{N_c}\frac{\bar\rho^2}{\bar R^4}, \tag{22}$$

where we have used the formulas for integration with respect to the Haar measure normalized to unity:

$$\int dU=1,\qquad \int dU\,U_{\alpha\beta}U_{\gamma\delta}^+=\frac{1}{N_c}\delta_{\alpha\delta}\delta_{\beta\gamma}, \tag{23}$$

$$\operatorname{Tr}(p^+p^-)=p_\mu p_\nu\operatorname{Tr}(\tau_\mu^+\tau_\nu^-)=p_\mu p_\nu 2\delta_{\mu\nu}=2p^2. \tag{24}$$

As expected, $\varkappa\to 0$ in the infinitely rarefied medium ($N/V\to 0$, $\bar R\to\infty$) and the density of states (21) tends to the δ-function: $\nu(\lambda)=N\delta(\lambda)$.

5. QUARK-ANTIQUARK CONDENSATE

By definition (4), in the chiral limit the quark-antiquark condensate $\langle\bar\psi\psi\rangle$ is the derivative of the logarithm of the partition function with respect to the quark mass m as $m\to 0$ (we are confining our attention to the case of one quark flavor, $N_f=1$). We recall that the connection between the condensate in Minkowski space and the Euclidean condensate is

$$\langle\bar\psi\psi\rangle_M=-i\langle\psi^+\psi\rangle_E=-\frac{1}{V^{(4)}}\frac{\partial\ln Z}{\partial m}\bigg|_{m\to 0}. \tag{25}$$

It follows from this definition that the necessary condition for the appearance of the spontaneous condensate in the chiral limit is that the logarithm of the partition function depends nonanalytically on the mass, i.e., on the perturbation that explicitly breaks the symmetry:

$$\ln Z=c_0+c_1|m|+O(m^2). \tag{26}$$

Since the dependence on the current mass m appears only in $\mathrm{Det}_{\mathrm{low}}$ (16), we have

$$\langle\bar\psi\psi\rangle=-\frac{1}{V}\int d\lambda\nu(\lambda)\frac{m}{\lambda^2+m^2}\bigg|_{m\to 0}=-\frac{1}{V}\pi\nu(0)\operatorname{sign}m, \tag{27}$$

which is in accordance with (26), where $c_1=\pi\nu(0)$.

It is striking that the current mass m of the quark appears in (16) only as m^2, which is a manifestation of γ_5 invariance. For any finite number N of pseudoparticles, this fact signifies the conservation of chiral invariance. However, in the thermodynamic limit ($N\to\infty$, $V\to\infty$, $N/V=\text{const}$), the density of states $\nu(\lambda)$ is finite for $\lambda=0$, the partition function begins to depend nonanalytically on m, and the symmetry is broken. Thus, the necessary and sufficient condition for a nonzero chiral condensate is that the spectral density of the Dirac operator $\nu(\lambda)$ be zero for $m=0$. It is clear that this is a general result.

If the QCD vacuum contains a finite density of topologically singular gluon configurations with $\int 2\operatorname{Tr}F_{\mu\nu}\tilde F_{\mu\nu}dx=\pm 32\pi^2$, the Atiyah-Singer index theorem shows that the Dirac operator has a zero mode for each topological singularity. When the medium consisting of singularities is taken account of, the result is a spreading of the zero modes over a certain range of eigenvalues, the width of which is of the order of the mean overlap integral $\varkappa$ of the zero modes corresponding to the individual singularities. The quantity $\nu(0)$, in terms of which the condensate is expressed, can then be immediately estimated as the total number N of the previous zero modes divided by $\varkappa$. It is interesting to note that, if, instead of the instanton-anti-instanton

Ansatz (1)–(3), we take a more general Ansatz (minimizing the vacuum energy by choosing the best shape of the subbarrier transition), in which the last factor in (2) and (3) is replaced by some function $f(x^2,\rho^2)$, where $f(0)=1$ (Ref. 19), we find that our results are numerically only slightly different. Actually, the zero fermion modes in the field of an individual "fremon"[19] will be somewhat different: the solution of the Dirac equation will now be the functions (13) with $\varphi(x,\rho)$ replaced with

$$\varphi(x,\rho)=\frac{\text{const}}{x^4}\exp\left\{-\int_{x^2}^{\infty}\frac{3f(x'^2,\rho^2)\,dx'^2}{2x'^2}\right\}. \qquad (28)$$

For the instanton, $f(x^2,\rho^2)=1/(1+x^2/\rho^2)$ and (28) is replaced with (13). The striking feature is that the behavior of $\varkappa(x,\rho)$ for $x\to 0,\ \infty$ is universal: $\varphi(x\to\infty)\sim x^{-4}$, since $f(x^2,\rho^2)$ should decrease as $x\to\infty$, and $\varphi(x\to 0)\sim x^{-1}$, which is guaranteed by the topological character of the singularities in the gluon Ansatz [$f(0)=1$]. Since the zero modes must also be normalized to unity, we arrive at the conclusion that the mean overlap integral of the zero modes (22), which determines the entire physics of spontaneous breaking of chiral invariance, will be only slightly different numerically when gluon configurations more general than instantons are examined.

We must now examine the value of the condensate $\langle\bar\psi\psi\rangle$. Substituting the expressions for $\nu(\lambda)$ given by (21) into the general formula (27), with the width $\varkappa$ taken from (22), we obtain

$$\langle\bar\psi\psi\rangle=-\frac{N}{V}\left(\frac{\pi}{2\varkappa^2}\right)^{1/2}=-\left(\frac{\pi N_c}{13{,}2}\frac{N}{V}\right)^{1/2}\frac{1}{\bar\rho}$$
$$=-\left(\frac{\pi N_c}{13{,}2}\right)^{1/2}\frac{1}{\bar R^2\bar\rho}. \qquad (29)$$

The pseudoparticle density N/V is directly related to the nonperturbative gluon condensate. The numerical result is[10] $N/V\approx\langle F_{\mu\nu}{}^2/32\pi^2\rangle\approx(200\text{ MeV})^4$. Substituting this and $\bar R/\bar\rho\simeq 3$ (see the Introduction), we obtain for $N_c=3$

$$\langle\bar\psi\psi\rangle\approx-(273\,\text{MeV})^3. \qquad (30)$$

We note that, since the instanton density is theoretically proportional to the number of colors,[19] the quark condensate is also proportional to N_c, as assumed. We must now agree that the quantity $g^{8/9}\langle\bar\psi\psi\rangle$ will be a renormalization-invariant combination at the two-loop level. The estimates given in the present paper correspond to the normalization point $\sim 1/\bar\rho\approx 600$ MeV.

The results of the last two sections can readily be generalized to the case of several quark flavors. Consider N_f flavors with mass matrix of the general from (m_L and m_R are $N_f\times N_f$ matrices):

$$\psi^+\left(im_L\frac{1+\gamma_5}{2}+im_R\frac{1-\gamma_5}{2}\right)\psi. \qquad (31)$$

It is readily shown that Det_{low} in (16) then generalizes to the following expression:

$$\text{Det}_{low}=\exp\left\{\frac{1}{2}\int_{-\infty}^{+\infty}d\lambda\nu(\lambda)\,[\ln\det{}_{N_f}(\lambda^2+m_Lm_R)\right.$$
$$\left.-\ln\det{}_{N_f}(\lambda^2+M_1{}^2)]\right\}=\exp\left\{-\frac{1}{2}\int_{-\infty}^{+\infty}d\lambda\nu(\lambda)\int_0^{\infty}\frac{dt}{t}\right.$$
$$\left.\times\text{Tr}_{N_f}\{\exp[-t(\lambda^2+m_Lm_R)]-\exp[-t(\lambda^2+M_1{}^2)]\}\right\}, \qquad (32)$$

where λ^2 and M_1^2 are $N_f\times N_f$ matrices proportional to the unit matrix. We shall now expand $\nu(\lambda)$ for small λ, and retain only the term $\nu(0)$ since we are interested only in terms proportional to the condensate. Integrating (32) with respect to λ, we obtain the following contribution to $\ln\text{Det}_{\text{low}}$:

$$-\frac{1}{2}\nu(0)\sqrt{\pi}\int_0^{\infty}\frac{dt}{t^{3/2}}\text{Tr}\,(e^{-tm_Lm_R}-e^{-tM_1{}^2})$$
$$=\pi\nu(0)\,[\text{Tr}\,(m_Lm_R)^{1/2}-N_fM_1{}^2].$$

Thus, for an arbitrary mass matrix (31), the dependence of the QCD partition function on the masses is [cf. (26)]

$$\ln Z=\text{const}+\pi\nu(0)\,\text{Tr}\,(m_Lm_R)^{1/2}+O(m_{L,R}^2). \qquad (33)$$

As expected, this expression is nonanalytic in the quark masses. By differentiating it with respect to the masses, we establish the γ_5 phases of the condensate. For example, in the case of one flavor and mass matrix of the form

$$\psi^+i(m_1+im_2\gamma_5)\psi$$
$$=\psi^+i\left[(m_1+im_2)\frac{1+\gamma_5}{2}+(m_1-im_2)\frac{1-\gamma_5}{2}\right]\psi,$$
$$m_1=m\cos\alpha,\qquad m_2=m\sin\alpha$$

we have $\text{Tr}(m_Lm_R)^{1/2}=(m_1^2+m_2^2)^{1/2}$, and hence

$$\langle\bar\psi\psi\rangle=-\frac{1}{V}\frac{\partial\ln Z}{\partial m_1}=-\frac{1}{V}\pi\nu(0)\cos\alpha,$$
$$\langle\bar\psi i\gamma_5\psi\rangle=-\frac{1}{V}\frac{\partial\ln Z}{\partial m_2}=-\frac{1}{V}\pi\nu(0)\sin\alpha,$$

as should be the case since the γ_5 phase of the mass fixes the γ_5 phase of the condensate. Analogous relationships arise in the case of several quark flavors.

To conclude this section, we make a remark about the joining of the high and low frequencies into which we divided the fermion determinant by using the fictitious parameter M_1 [cf. (7)]. Det_{high} depends on M_1 through the function $F(M_1\rho)$ in (9), and Det_{low} is given by (16). It is clear that, in the parametrically wide region $\varkappa<M_1<1/\bar\rho$ [we recall that $\varkappa\sim(1/\bar\rho)\,(\bar\rho/\bar R)^2\ll 1/\bar\rho$, see (22)], the product $\text{Det}_{\text{high}}\,\text{Det}_{\text{low}}$ is a slowly-varying function of M_1. Actually, in this region, we must use the second formula in (9), and the main dependence on M_1 in the determinant product cancels out because $\int\nu(\lambda)d\lambda=N$. The M_1-dependent corrections are expanded into a series in $(M_1\rho)^2$ and $\varkappa^2/M_1^2$ for Det_{high} and Det_{low}, respectively. The corrections are small in the indicated range of M_1, and their sum has a very flat maximum which ensures that $\mathrm{Det}_{\mathrm{high}}$ and $\mathrm{Det}_{\mathrm{low}}$ are satisfactorily joined (see Ref. 21 and further details).

The diagonalization of the zero modes is thus seen to ensure that the fermion determinant does not vanish as $m^{N_f N}$, as in the infinitely verified medium, but has the finite limit $\sim \varkappa^{N_f N}$ as $m \to 0$.

6. QUARK GREEN FUNCTION IN THE "PSEUDOPARTICLE" REPRESENTATION

Having explained the basic function of the physics of spontaneous breaking of chiral invariance in a medium consisting of topological singularities of the gluon field, we now proceed to a more systematic theory based on the evaluation of Green's functions in the medium. This is necessary, above all, to enable us to derive a more accurate expression for $\langle \bar{\psi}\psi \rangle$ and, later, to obtain the observed hadronic current correlators.

We must first define the quark Green function:

$$S_{ij}(x-y) \equiv \langle \psi_i(x) \psi_j^+(y) \rangle, \tag{34}$$

where i and j represent the set of spinor and color indices. This function satisfies the equation

$$(i\hat{\nabla}_x + im)_{ki} S_{ij}(x-y) = -\delta_{kj}\delta^{(4)}(x-y). \tag{35}$$

The free Green function then has the form

$$S_{ij}^0(p) = \int d^4(x-y) \langle \psi_i(x) \psi_j^+(y) \rangle_0 e^{-i(p,(x-y))} = \left(\frac{im + \hat{p}}{m^2 + p^2} \right)_{ij}. \tag{36}$$

Let us construct the Green function in the external gluon field in the form of a superposition of instantons and antiinstantons: $A_\mu = \Sigma_I A_\mu^I$. We have

$$S = -(i\hat{\nabla} + im)^{-1} = S_0 + S_0 \hat{A} S_0 + S_0 \hat{A} S_0 \hat{A} S_0 + \dots .$$

This series can be regrouped by summing all the powers of the external field produced by one instanton, then two instantons, and so on. The exact Green function then takes the form of a series over the exact Green functions S_I in the field of the individual instantons:

$$S = S_0 + \sum_I (S_I - S_0) + \sum_{I \neq J} (S_I - S_0) S_0^{-1} (S_J - S_0)$$
$$+ \sum_{I \neq J,\, J \neq K} (S_I - S_0) S_0^{-1} (S_J - S_0) S_0^{-1} (S_K - S_0) + \dots . \tag{37}$$

In principle, the exact Green function S_I in the field of an instanton in the limit as $m \to 0$ is unknown:[24] it has a part that is singular in the mass and is related to the zero modes, whereas the nonsingular part transforms into the free Green function for momenta $\gtrsim 1/\rho$, where ρ is the instanton dimension. Since we are now interested in the physics of chiral symmetry breaking, and this, as we have seen, is related to zero modes, we shall now adopt a model for $S_I - S_0$, in which only the contribution of the zero mode is retained. Our results will be numerically uncertain in momentum space by the amount $p \sim 1/\rho$. Thus, we put

$$(S_I - S_0)_{ij}(x, y) \approx -\psi_{Ii}(x) \psi_{Ij}^+(y)/im, \tag{38}$$

where ψ_I is the zero mode of (13). We now substitute this into the exact formula (37):

$$S_{ij}(x, y) \approx S_{ij}^0(x, y) + \sum_I \frac{\psi_{Ii}(x) \psi_{Ij}^+(y)}{-im}$$
$$+ \sum_{I \neq J} \frac{\psi_{Ii}(x)}{-im} \int d^4z \psi_{Ik}^+(z) (-i\hat{\partial} - im)_{kl} \psi_{Jl}(z) \frac{\psi_{Jj}^+(y)}{-im} + \dots . \tag{39}$$

Neglecting the less singular terms containing the mass in the numerator (we are interested in the chiral limit!), we note that only "unlike" pseudoparticles can participate in the remaining matrix elements. This means that, in (39), we can sum without the restriction $I \neq J$, and so on (it is satisfied automatically). As a result, the series (39) becomes a geometric progression and we obtain

$$S_{ij}(x, y) \approx S_{ij}^0(x, y) + \sum_{I,J} \psi_{Ii}(x) \left(\frac{1}{T - im} \right)_{IJ} \psi_{Jj}^+(y), \tag{40}$$

where the matrix T_{IJ} represents the overlap integral of the zero modes, which we already know [cf. (15)]:

$$T_{IJ}(z_I - z_J, \rho_I, \rho_J, U_I, U_J)$$
$$= \int d^4z \psi_{Ik}^+(z - z_I, \rho_I, U_I) (i\hat{\partial}_z)_{kl} \psi_{Jl}(z - z_J, \rho_J, U_J). \tag{41}$$

To obtain the quark propagators for the instanton medium, we must average (40) over the statistical ensemble of pseudoparticles. Since the "extreme" pseudoparticles I and J appear twice in (40) [in the zero modes and in $(T - im)^{-1}$], it is convenient to extract them and consider the Green function in the pseudoparticle representation by introducing the functions

$$\overline{\left(\frac{1}{T - im} \right)}_{IJ}$$
$$= \begin{cases} -\dfrac{1}{im} \delta_{IJ} - D_{IJ}(z_I - z_J, U_I, U_J); & I, J \text{ — "like"} \\ -F_{IJ}(z_I - z_J, U_I, U_J); & I, J \text{ — "unlike"} \end{cases} \tag{42}$$

The bar in this expression indicates averaging over the positions and orientations of all the pseudoparticles encountered in the geometric progression, with the exception of the extreme I and J. The possibility that the "internal" pseudoparticles will coincide with I and/or J will be taken into account separately (see Section 7). A more explicit expression for the above functions is

$$F_{IJ} = \frac{1}{im} T_{IJ} \frac{1}{im} + \sum_{K, L \neq I, J} \overline{\frac{1}{im} T_{IK} \frac{1}{im} T_{KL} \frac{1}{im} T_{LJ} \frac{1}{im}}$$
$$+ \dots (T \text{ — odd}),$$

$$D_{IJ} = \sum_K \overline{\frac{1}{im} T_{IK} \frac{1}{im} T_{KJ} \frac{1}{im}} + \dots \quad (T \text{ — even}). \tag{43}$$

Our immediate problem is to evaluate these fundamental functions of the theory. We shall do this under the following approximation. We shall assume that the pseudoparticles

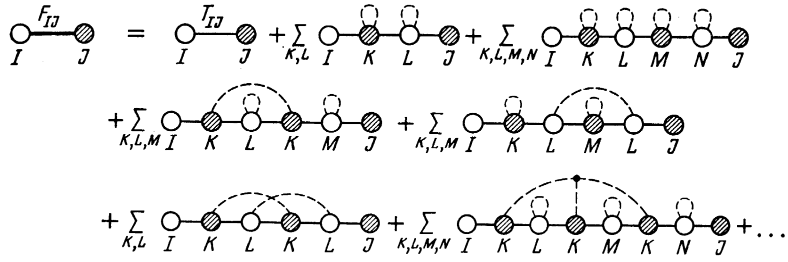

FIG. 1.

are uncorrelated. The inclusion of correlations gives the next correction in the packing parameter $\bar{\rho}/\bar{R}$ of the medium [actually $(\bar{\rho}/\bar{R})^4$], which will be assumed small. The dimensions of the pseudoparticles will be replaced by the average dimension $\bar{\rho}$. It is shown in Ref. 19 that the distribution over the dimensions ρ is narrow and tends to a δ-function when the number N_c of colors is large. Finally, we shall assume that $N_c \gg 1$. Corrections of the order of $1/N_c$ are amenable to systematic inclusion and appear to be very small in the real world ($N_c = 3$).

To facilitate the evaluation of F and D, we shall formulate a diagram method. We shall use a circle, a shaded circle, and a line joining such circles to represent an instanton, an anti-instanton, and the overlap integral T_{IJ} (41), respectively. We associate the factor $1/im$ with each circle. Dashes entering these circles will represent averaging over the positions and orientations of the pseudoparticles. Whenever a given pseudoparticle is encountered only once in the geometric progression (43), we shall indicate this by dashes leaving the corresponding circle and then reentering it. Whenever the same pseudoparticle is encountered several times, we shall join all the circles representing the pseudoparticles by dashes.

In this notation, the expression for, say F_{IJ} (43) assumes the form shown in Fig. 1. This series can be summed with the aid of a special Dyson equation (Fig. 2), in which thick lines joining the pseudoparticles represent the required functions F and D. In contrast to Fig. 1, each inner circle in Fig. 2 is assigned the factor im since the functions F and D in the graphs of Fig. 2 have the factors $1/im$ "at the ends" [cf. (43)]. For the same reason, the first inner circle is assigned the factor 1.

We note that graphs with intersecting dashed lines (such as the last graph in Fig. 2) are of the order of $1/N_c$ as compared with "planar" graphs because of the complicated averaging over the pseudoparticle orientations, but they also contain an additional small numerical quantity due to integration with respect to the angles. Nonplanar graphs will be neglected. Planar graphs, on the other hand, can readily be summed.

In fact, introducing the notation $D_{KK} \equiv \gamma$ for the quantity $D_{KL}(z_K - z_L, U_K, U_L)$ at $K = L$, we find that the Dyson equation assumes the form

$$F_{IJ}(z_I-z_J, U_I, U_J) = \frac{1}{im} T_{IJ}(z_I-z_J, U_I, U_J)\frac{1}{im}$$
$$+\frac{N}{2V}\int d^4z\, dU_K \frac{1}{im} T_{IK}(z_I-z_K, U_I, U_K)$$
$$\times[1+im\gamma+(im\gamma)^2+\ldots]D_{KJ}(z_K-z_J, U_K, U_J). \tag{44}$$

The factor $N/2V$ is due to averaging over the positions of the antiinstanton K and summation over all the antiinstantons, the number of which is $N_- = N/2$. An analogous Dyson equation (but without the free term) is obtained for the function D_{IJ}:

$$D_{IJ}(z_I-z_J, U_I, U_J) = \frac{N}{2V}\int dz_K\, dU_K \frac{1}{im} T_{IK}(z_I-z_K, U_I, U_K)$$
$$\times\frac{1}{1-im\gamma} F_{KJ}(z_K-z_J, U_K, U_J). \tag{45}$$

We note that (44) and (45) are nonlinear equations because γ is found from D_{IJ}. To solve these equations, we must first establish the structure of F and D as functions of the orientation U. This can readily be done after the first iterations in (44) and (45). Transforming to the momentum representation for S and D

$$F(D)_{IJ}(p) = \int d^4(z_J-z_I)\, e^{i(p,z_J-z_I)} F(D)_{IJ}(z_I-z_J),$$

we seek the solution in the form

$$F_{I\bar{I}}(p) = \mathrm{Tr}(U_{\bar{I}}p^+U_I^+)f(p^2), \quad F_{\bar{I}I}(p) = -\mathrm{Tr}(U_I p^- U_{\bar{I}}^+)f(p^2), \tag{46}$$
$$D_{II'}(p) = \mathrm{Tr}(U_{I'}1_2U_I^+)d(p^2), \quad D_{\bar{I}\bar{I}'}(p) = \mathrm{Tr}(U_{\bar{I}'}1_2U_{\bar{I}}^+)d(p^2),$$

where $p^\pm = p_\mu \tau_\mu^\pm$, and $\tau_\mu^\pm$ and 1_2 are 2×2 matrices in the

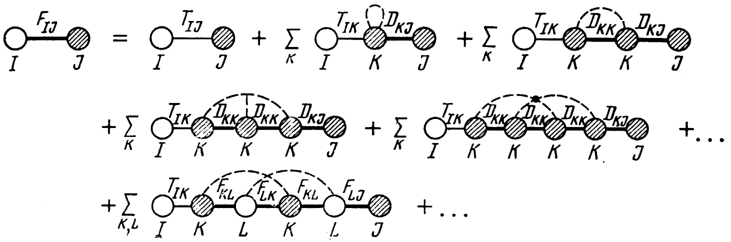

FIG. 2.

top left-hand corner of the $N_c \times N_c$ matrix in which all the other elements are zeros. Substituting (46) and the explicit form of $T_{IJ}(p, U_I, U_J)$ given in the Appendix into (44) and (45), and averaging over the orientations with the aid of (23) and (24), we obtain the following set of algebraic equations for the scalar functions $f(p)$ and $d(p)$:

$$f(p) = -2i\left(\frac{1}{im}\right)^2 \varphi'^2(p) - 2i\frac{N}{2VN_c}\frac{1}{im}\frac{1}{1-im\gamma}\varphi'^2(p)\,d(p),$$

$$d(p) = 2i\frac{N}{2VN_c}\frac{1}{im}\frac{1}{1-im\gamma}p^2\varphi'^2(p)f(p), \tag{47}$$

$$\gamma = 2\int\frac{d^4p}{(2\pi)^4}d(p). \tag{48}$$

We must first solve (47) by expressing $f(p)$ and $d(p)$ in terms of γ, and then find the number γ from the self-consistency condition (48). It is readily verified that (48) has a solution for which

$$1/(1-im\gamma) \equiv m\varepsilon, \quad \varepsilon = 0(m^0). \tag{49}$$

In view of this, it is convenient to express f and d in terms of ε (instead of γ) and also in terms of the function $M(p)$ which, as we shall see in the next section, can be interpreted as the effective mass of the quark:

$$M(p) \equiv \frac{\varepsilon N}{VN_c}\varphi'^2(|p|, \rho)\,|p| = \begin{cases} \dfrac{2\pi^2\varepsilon N\bar{\rho}^2}{VN_c}, & p \ll 1/\bar{\rho}, \\[2ex] \dfrac{72\pi^2\varepsilon N}{VN_c\bar{\rho}^4}\dfrac{1}{p^6}, & p \gg 1/\bar{\rho}. \end{cases} \tag{50}$$

In terms of these quantities, the solution of (47) and (48) is

$$d(p^2) = \frac{VN_c}{N\varepsilon}\frac{2i}{m^2}\frac{M^2(p)}{M^2(p)+p^2}, \quad f(p^2) = \frac{VN_c}{N\varepsilon}\frac{2i}{m^2}\frac{M(p)}{M^2(p)+p^2}, \tag{51}$$

and the self-consistency condition (48), which is the equation for ε, is

$$\int\frac{d^4p}{(2\pi)^4}\frac{M^2(p)}{M^2(p)+p^2} = \frac{N}{4VN_c}(1-m\varepsilon). \tag{52}$$

It follows from this equation that ε does, in fact, tend to a finite limit as $m \to 0$. Parametrically, $\varepsilon \sim N_c^{1/2}\bar{R}^{\,2}/\bar{\rho}$. The numerical solution of (52) for $m = 0$, $N/V = (200\text{ MeV})^4$, $\bar{\rho} = (600\text{ MeV})^{-1}$, and $N_c = 3$ gives $\varepsilon = (85\text{ MeV})^{-1}$. The higher-order terms in the expansion for ε in terms of m can be obtained by iterating (52). Substituting the above value of ε into the expression for the effective mass (50), we find that $M(p=0) = 345$ MeV, which is in good agreement with the generally accepted mass of the constituent quark. Parametrically, $M(p=0) \sim N_c^{-1/2}\bar{R}^{\,-2}\bar{\rho}$, i.e., we have stability with respect to N_c.

7. QUARK PROPAGATOR IN THE INSTANTON MEDIUM

We can now see the above Green function in the pseudoparticle representation (42) to obtain the quark propagator in the medium with the aid of formula (40). When we sum

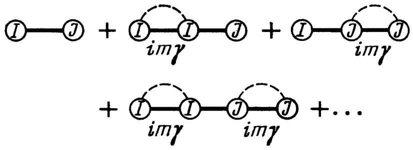

FIG. 3.

over the "extreme" pseudoparticle I, J in (40), we must take into account the following possibilities: (1a) I and J are the same instanton, (1b) I and J are the same anti-instanton, (2a) I and J are different instantons, (2b) I and J are different anti-instantons, (3a) I is an instanton and J and anti-instanton, and (3b) I is an anti-instanton and J and instanton. Case (1) involves the quantity $D_{II} \equiv \gamma$ (see the previous section). However, we must remember that the pseudoparticle I can be encountered several times in the expansion (43). Summation over these possibilities leads us, in the case of the derivation of the Dyson equations (44), (45), to a geometric progression: $\gamma + im\gamma^2 + (im\gamma)^2\gamma = \gamma/(1 - im\gamma)$. In cases (2) and (3), we must also remember that the extreme pseudoparticles I and J can be encountered as many times as convenient (cf. Fig. 3). This leads to the factor $(1 - im\gamma)^{-2}$. We note that, when the extreme pseudoparticles I, J are repeated, they must not be confused because, otherwise, the graph becomes nonplanar and this ensures that $1/N_c$ becomes small.

Let us now use the above remarks to rewrite (40) in the momentum representation:

$$S_{ij}(p) = \left(\frac{\hat{p}}{p^2}\right)_{ij} - \frac{N}{2V}\left(\frac{1}{im} + \frac{\gamma}{1-im\gamma}\right)\left[\int dU_I\psi_{Ii}(p)\psi_{Ij}^+(p) + \int dU_{\bar{I}}\psi_{\bar{I}i}(p)\psi_{\bar{I}j}^+(p)\right] - \left(\frac{N}{2V}\right)^2\frac{1}{(1-im\gamma)^2}\left[\iint dU_I\, dU_{I'}\psi_{Ii}(p)\psi_{I'j}^+(p)D_{II'}(p) + \iint dU_{\bar{I}}dU_{\bar{I}'}\psi_{\bar{I}i}(p)\psi_{\bar{I}'j}^+(p)D_{\bar{I}\bar{I}'}(p)\right]$$
$$-\left(\frac{N}{2V}\right)^2\left(\frac{1}{1-im\gamma}\right)^2 \times\int dU_I\, dU_{\bar{I}}[\psi_{Ii}(p)\psi_{\bar{I}j}^+(p)F_{I\bar{I}}(p) + \psi_{\bar{I}i}(p)\psi_{Ij}^+(p)F_{\bar{I}I}(p)]. \tag{53}$$

Be definition (49),

$$(1-im\gamma)^{-2} = m^2\varepsilon^2,\ 1/im + \gamma/(1-im\gamma) = -i\varepsilon,\ \varepsilon = 0(m^0).$$

Since F, $D \propto 1/m^2$ [see (51)], it follows immediately that the propagator (53) remains finite as $m \to 0$. To evaluate (53), we must use the zero-mode density matrices, given in the Appendix, and formulas (46) and (51) for the functions D and F. Averaging with the aid of (23) and (24), we obtain the propagator

$$S(p) = \frac{\hat{p}}{p^2} + \frac{iM(p)}{p^2} - \frac{iM(p)}{p^2}\frac{M^2(p)}{M^2(p)+p^2} - \frac{\hat{p}M^2(p)}{M^2(p)+p^2} = \frac{iM(p)+\hat{p}}{M^2(p)+p^2}. \tag{54}$$

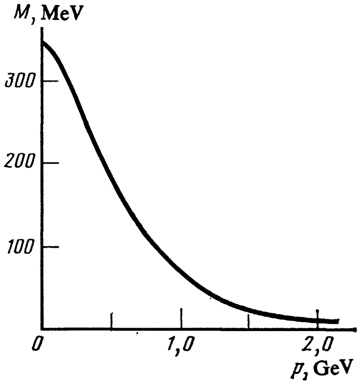

FIG. 4. Plot of the function $M(p)$; $1/\bar{R} = 200$ MeV and $1/\bar{\rho} = 600$ MeV.

We note the remarkable fact that the pole of the free propagator at $p^2 = 0$ has canceled out and the quark propagator has assumed the form of the propagator of a massive particle with effective mass $M(p)$ (50), whose graph is shown in Fig. 4. The last point is, of course, a manifestation of the spontaneous breaking of chiral invariance in the instanton medium. In fact, the chiral condensate is, by definition,

$$\langle\bar{\psi}\psi\rangle_M = -i\langle\psi^+\psi\rangle_E = i\,\mathrm{Sp}\,S(x,x) = i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Sp}\,\frac{iM+\hat{p}}{M^2+p^2}$$

$$= -4N_c\int\frac{d^4p}{(2\pi)^4}\frac{M(p)}{M^2(p)+p^2}. \qquad (55)$$

This formula yields the following numerical value: $\langle\bar{\psi}\psi\rangle = -(255\text{ MeV})^3$ [cf. the cruder estimate (30)]. Phenomenologically, $\langle\bar{\psi}\psi\rangle \approx -(240\text{–}250\text{ MeV})^3$.

8. DISCUSSION

In this paper, we have explained the physics of spontaneous breaking of chiral invariance in a vacuum populated by topologically singular gluon fields. The mechanism responsible for this phenomenon, which we have discovered, is the delocalization of the zero fermion mode belonging to individual pseudoparticles and the formation of a finite density of states with zero eigenvalues. This is different from mechanisms proposed previously in the literature. It is curious that the theory of light quarks in the instanton vacuum acquires, in our approach, many of the features of the theory of disordered systems.

In deriving the quark propagator, we introduce a number of approximations, formulated in Section 6. The most serious of these is the use of the approximate Green function in the field of one instanton, which means that, strictly speaking, we cannot pretend that we have an accurate calculation in the momentum region $p \sim 1/\bar{\rho}$. However, for small momenta and small eigenvalues of the Dirac operator, our calculations, based on the diagonalization of the zero modes, are parametrically justified. It is remarkable that the cancelation of the pole of the free propagator at $p^2 = 0$ and the generation of the effective quark mass $M(p)$ occur precisely in the region of low momenta. Although we do not pretend that we have achieved confinement, we do, nevertheless, note that singularities due to the massless quark have disappeared from the theory. It will be shown in our next paper that the instanton vacuum reproduces the qualitative properties of the hadronic spectrum and, apparently, provides a good qualitative description of the hadronic current correlators in the entire Euclidean momentum range.

We are indebted to L. N. Lipatov, N. G. Ural'tsev, É. V. Shuryak, and M. I. Éĭdes for useful discussions.

APPENDIX

We now list the formulas for the zero-mode density matrix and the zero-mode overlap integrals used in this paper. The zero fermion modes in the field of the instanton I and anti-instanton $\bar{I}$ are given by (13). The density matrices for two instantons ($i, j = 1, 2, 3, 4$—spinors, $\alpha, \beta = 1, 2, \ldots, N_c$—color indices) are as follows:

$$\psi_I(x-z_I)_{i\alpha}\psi_{I'}^+(x-z_{I'})_{j\beta} = \varphi(x-z_I)\varphi(x-z_{I'})$$
$$\times\left[(\hat{x}-\hat{z}_I)U_I\left(1_2+\frac{i}{4}\tau^a\eta_{\mu\nu}^a\sigma_{\mu\nu}\right)U_{I'}^+(\hat{y}-\hat{z}_{I'})\frac{1-\gamma_5}{2}\right]_{ij}^{\alpha\beta},$$
$$\varphi(x) = \rho[\pi(2x^2)^{1/2}(x^2+\rho^2)^{3/2}]^{-1}. \qquad (A.1)$$

where $\eta_{\mu\nu}^a$ are the 't Hooft symbols[7] and $\sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu]$. We now transform to the momentum representation, substituting

$$\psi(k) = \int d^4x e^{-ikx}\psi(x), \qquad \psi^+(k) = \int d^4x e^{ikx}\psi^+(x).$$

In terms of the Fourier components, the density matrix (A1) assumes the form

$$\psi_I(k_1)_{i\alpha}\psi_{I'}^+(k_2)_{j\beta} = \frac{\varphi'(|k_1|)}{|k_1|}\frac{\varphi'(|k_2|)}{|k_2|}$$
$$\times\left[\hat{k}_1U_I\left(1_2+\frac{i}{4}\tau^a\eta_{\mu\nu}^a\sigma_{\mu\nu}\right)U_{I'}^+\hat{k}_2\frac{1-\gamma_5}{2}\right]_{ij}^{\alpha\beta}, \qquad (A2)$$

$$\varphi'(|k|) = \frac{\pi}{\sqrt{2}}\rho^2\frac{d}{dz}[I_0(z)K_0(z)-I_1(z)K_1(z)]_{z=|k|\rho/2}$$
$$= \begin{cases} -\pi\sqrt{2}\dfrac{\rho}{|k|}, & k\rho\ll 1, \\ -6\pi\sqrt{2}\dfrac{1}{k^4\rho^2}, & k\rho\gg 1. \end{cases} \qquad (A3)$$

Similarly, the density matrix for two anti-instantons is

$$\varphi_{\bar{I}}(k_1)_{i\alpha}\psi_{\bar{I}'}^+(k_2)_{j\beta} = \frac{\varphi'(k_1)}{|k_1|}\frac{\varphi'(k_2)}{|k_2|}\left[\hat{k}_1U_{\bar{I}}\left(1_2\right.\right.$$
$$\left.\left.+\frac{i}{4}\tau^a\eta_{\mu\nu}^a\sigma_{\mu\nu}\right)U_{\bar{I}'}^+\hat{k}_2\frac{1+\gamma_5}{2}\right]_{ij}^{\alpha\beta}. \qquad (A4)$$

The density matrix for an instanton and an anti-instanton is

$$\psi_I(k_1)_{i\alpha}\psi_{\bar{I}}^+(k_2)_{j\beta} = \frac{\varphi'(k_1)}{|k_1|}\frac{\varphi'(k_2)}{|k_2|}(-i)\left[\hat{k}_1\gamma_\mu\hat{k}_2\frac{1+\gamma_5}{2}\right]_{ij}$$
$$\times[U_I\tau_\mu^-U_{\bar{I}}^+]^{\alpha\beta}. \qquad (A5)$$

The density matrix for an anti-instanton and an instanton is

$$\psi_{\bar{I}}(k_1)_{i\alpha}\psi_I^+(k_2)_{j\beta} = \frac{\varphi'(k_1)}{|k_1|}\frac{\varphi'(k_2)}{|k_2|}i\left[\hat{k}_1\gamma_\mu\hat{k}_2\frac{1-\gamma_5}{2}\right]_{ij}$$
$$\times[U_{\bar{I}}\tau_\mu^+U_I^+]^{\alpha\beta}. \qquad (A6)$$

For the definition of the overlap integral of zero modes, see (41). Transforming to the momentum representation and using (A5) and (A6), we obtain

$$T_{I\bar{I}}(p, U_I, U_{\bar{I}})$$

$$=\int d^4(z_{\bar{I}}-z_I)\exp[i(p,(z_{\bar{I}}-z_I))]T_{I\bar{I}}(z_{\bar{I}}-z_I, U_I, U_{\bar{I}})$$

$$=-2i[\varphi'(p)]^2\operatorname{Tr}(U_{\bar{I}}p^+U_I^+),\quad p^+=p_\mu\tau_\mu^+; \qquad (A7)$$

$$T_{\bar{I}I}(p, U_{\bar{I}}, U_I)=2i[\varphi'(p)]^2\operatorname{Tr}(U_Ip^-U_{\bar{I}}^+),\quad p^-=p_\mu\tau_\mu^-. \qquad (A8)$$

We note that the matrix T is Hermitian: $T_{I\bar{I}}(p)=T^*_{\bar{I}I}(p)$.

[1]D. I. D'yakonov and M. I. Eĭdes, in Fizika elementarnykh chastits. Materially XVI Zimneĭ shkoly LIYaF (Elementary Particle Physics, Proc. Sixteenth Winter School of the Leningrad Institute of Nuclear Physics), Nauka, Leningrad, 1981, p. 123.
[2]E. Witten, Nucl. Phys. B **223**, 422 (1983).
[3]D. I. D'yakonov and V. Yu. Petrov, Preprint LIYaF-967, 1984.
[4]P. I. Fomin and V. A. Miransky, Phys. Lett. B **79**, 166 (1976).
[5]V. A. Miransky, V. P. Gusynin, and Yu. A. Sitenko, Phys. Lett. B **100**, 157 (1981); H. D. Politzer, Nucl. Phys. B **117**, 397 (1976); Phys. Lett. B **116**, 171 (1982).
[6]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. M. Tupkin, Phys. Lett. **59**, 85 (1975); A. M. Polyakov, Nucl. Phys. B **121**, 429 (1977).
[7]G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).
[8]C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D **17**, 2717 (1978); **19**, 1826 (1979); **20**, 3279 (1979).
[9]R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37**, 172 (1976).
[10]M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **147**, 385, 448, 519 (1979).
[11]E. V. Shuryak, Phys. Lett. B **79**, 135 (1978).
[12]D. G. Caldi, Phys. Rev. Lett. **39**, 121 (1977).
[13]R. D. Carlitz, Phys. Rev. D **17**, 3225 (1978); R. D. Carlitz and D. B. Creamer, Ann. Phys. (N.Y.) **118**, 429 (1979).
[14]N. A. McDougall, Oxford University Preprint 33/82, 1982; C. E. I. Carneiro and N. A. McGougall, Nucl. Phys. B **245**, 293 (1984).
[15]D. Foerster, Phys. Lett. B **66**, 279 (1977).
[16]E. V. Shuryak, Nucl. Phys. B **203**, 93, 116, 140 (1982).
[17]E. V. Shuryak, Nucl. Phys. B **214**, 237 (1983).
[18]R. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* [Russian translation, Mir, Moscow, 1968, Chap. 2], McGraw-Hill, 1965.
[19]D. I. Dyakonov and V. Yu. Petrov, Nucl. Phys. B **245**, 259 (1984).
[20]C. Bernard, Phys. Rev. D **19**, 3013 (1979).
[21]D. I. Dyakonov and V. Yu. Petrov, Phys. Lett. B **147**, 351 (1984).
[22]D. I. Dyakonov, V. Yu. Petrov, and A. V. Yung, Phys. Lett. B **130**, 385 (1983); Yad. Fiz. **39**, 240 (1984) [Sov. J. Nucl. Phys. **30**, 150 (1984)].
[23]A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Usp. Fiz. Nauk **136**, 553 (1982) [Sov. Phys. Usp. **25**, 195 (1982)].
[24]L. S. Brown, R. D. Carlitz, D. B. Creamer, and C. Lee, Phys. Rev. D **17**, 1583 (1979).

Translated by S. Chomet