Resistance of a one-dimensional disordered system with a two-band spectrum

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The resistance of a one-dimensional system with a two-band charge carrier spectrum is analyzed in the one-electron approximation. The motion of the carriers is described by an equation of the Dirac type with a random potential and a fluctuating gap width. Since all the electron states in a one-dimensional disordered system are localized, the resistance at $T = 0$ K increases exponentially with the length x of the system. The logarithm of the mean resistance and the mean logarithm of the resistance are calculated. They are proportional to x, but the proportionality coefficients behave differently as functions of the electron energy and the parameters of the random perturbations. The reciprocal of the localization length of the electron states is characterized by the proportionality coefficient of the mean logarithm of the resistance. Exact expressions for the localization length are derived in several special cases.

1. INTRODUCTION

Broad electron bands in one-dimensional systems may occasionally be separated by a narrow forbidden band, in which case one must allow for the interaction between states in the two adjacent bands. The energy gap in the electron spectrum may be caused, e.g., by spin-orbit interaction of the electrons with the crystal lattice in narrow-gap semiconductors. In this case the spectrum and the electron states are described by a system of first-order equations of the Dirac type.[1] In principle, ion lithography[2] should make it possible to fabricate fine wires from narrow-gap semiconductors in which the carriers move one-dimensionally because of the small effective mass. Umklapp processes in quasi-one-dimensional compounds of the type TCNQ (Ref. 3) can also give rise to a narrow gap in an allowed band.

The random perturbations in a system with a two-band spectrum can be described by a Dirac Hamiltonian

$$H=\sigma_z p+\Delta(x)\sigma_x+U(x), \quad p=-i\partial/\partial x, \tag{1}$$

where σ_x and σ_y are the Pauli matrices, and the gap width Δ and potential energy U both fluctuate.

The Schrödinger equation with a weak random potential can be reduced to a two-band model similar to the one above by using "truncated equations."[4] In this case an additional term $\Delta'(x)\sigma_y$ appears in the Hamiltonian (1). It was shown in Ref. 5 that this term may give a small contribution to the carrier scattering in certain organic quasi-one-dimensional conductors with a half-filled band. The same equations also describe the propagation of electromagnetic waves in one-dimensional random media. The gap in the photon spectrum in this case may be caused, e.g., by a perturbation of a periodic structure composed of alternating layers with different dielectric constants (such structures occur in x-ray mirrors[6]). Random variations in the thickness of dielectric constant of the layers can be described in terms of fluctuations in Δ and U.

The Dirac equation reduces to the Schrödinger equation in the nonrelativistic limit. Moreover, it is obvious that all the results derived for the model Hamiltonian (1) should agree in the limit $\Delta\to\infty$ with the corresponding results for the one-band model. Anderson *et al.*[7] have argued that for conductors much longer than the electron mean free path, the distribution for the logarithm of the resistance $\ln\rho$ should be Gaussian regardless of the choice of model. The arguments used there have since been confirmed by an analysis of specific models. Fluctuations in the resistance of a one-dimensional conductor were studied in Refs. 8 and 9, where several specific models were considered (the Anderson model, binary alloy model, etc.). Mel'nikov[10] and Abrikosov[11] calculated the distribution function for the resistance and conductivity of a one-dimensional conductor by using the Born approximation to analyze carrier scattering by an isolated impurity. The results in Refs. 7–11 imply that in the weak-scattering limit, the mean free path L determines the growth exponent of the moments of the resistance for the one-dimensional model. The growth exponent for $\ln\rho$ is also determined by L.

The density of states $N(E)$ was studied in Refs. 12–14 for a one-dimensional system with the Hamiltonian (1). This model has the interesting feature that $N(E)$ has a Dyson singularity which causes the low-temperature magnetic susceptibility of quasi-one-dimensional conductors to behave anomalously. The spectrum and kinetic properties of one-dimensional conductors were studied in Refs. 15–17 for the case when the electron wave length is comparable to the lattice period.

The kinetic properties of disordered systems with a two-band spectrum are of interest. It is known[14,18] that all states in a one-dimensional disordered system are localized and have wave functions $\psi\sim\exp(-x/l_\psi)$ which decay exponentially with distance x, where l_ψ is the localization length. As a results, the static resistance at temperature $T=0$ increases exponentially with the length of the system. It is important to note that the resistance is not a self-averaging quantity—as x increases, the rms deviations increase faster than the mean value $\langle\rho\rangle$. The growth exponent of the resistance may therefore fail to characterize the localization length of the electron states in certain cases which we discuss below. The self-averaging quantity is actually the logarithm of the resistance, whose mean value is proportional to the length of the system: $\langle\ln\rho\rangle = l^{-1}x$, and l determines the

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localization length for states on the Fermi surface.

In this paper we use the one-electron model (1) to study the static resistance of a one-dimensional conductor at $T=0$.

2. DERIVATION OF THE EQUATIONS FOR ρ AND ln ρ

At $T=0$ the ohmic resistance is caused by electrons in states whose energies are close to the Fermi energy E. We consider a one-dimensional disordered system of length x in the one-electron approximation. Although the contacts also contribute to ρ, for sufficiently large $x\rho$ will be determined primarily by carrier scattering inside the conductor. We assume for simplicity that the contact material has an electron spectrum of the form (1) with $\Delta=0$, $U=U_0$, and that the Fermi level lies at the center of the band: $E_F{}^0=U_0$. In this case the Landauer formula[19]

$$\rho=R/T \tag{2}$$

gives the resistance (in units of $e^2/\pi\hbar$), where T and R are the transmission and reflection coefficients for states of energy $E=U_0$. When calculating ρ it is helpful to consider bilinear combinations

$$R_{\alpha\beta}=\psi_\alpha\psi_\beta^*$$

of the components of the wave function. The Dirac equation yields the equation

$$\dot{\hat{R}}=\Delta(x)(\sigma_y\hat{R}+\hat{R}\sigma_y)+i[E-U(x)](\sigma_z\hat{R}-\hat{R}\sigma_z) \tag{3}$$

for $\hat{R}$. If we expand $\hat{R}$ in terms of the Pauli matrices: $\hat{R}=R_0+R_a\sigma_a$ $(a=1,2,3)$ and make the change of scale $2x\to x$, we get

$$\begin{aligned}\dot{R}_0&=-\Delta(x)R_2,\\ \dot{R}_1&=-[E-U(x)]R_2,\\ \dot{R}_2&=-\Delta(x)R_0+[E-U(x)]R_1,\\ \dot{R}_3&=0.\end{aligned} \tag{4}$$

In the nonrelativistic limit we can apply a similarity transformation and reduce Eqs. (4) to the system given in Ref. 20 for bilinear combinations of the wave function and its derivative. The resistance of a conductor of length x is $\rho=[R_0(x)-1]/2$ if the initial conditions are taken to be $R(0)=(1,0,0,1)$. System (4) has the constant of motion

$$\begin{aligned}R_3&=\text{const}=1,\\ R_0^2-R_1^2-R_2^2&=\text{const}=1.\end{aligned} \tag{5}$$

We will use Eqs. (4) below to calculate $\langle\rho\rangle$ and its variance. We can calculate $\langle\ln\rho\rangle$ and the average localization length by using (5) defining the new variables

$$R_0=\operatorname{ch}\chi,\quad R_2+iR_1=\operatorname{sh}\chi e^{i\varphi}, \tag{6}$$

in terms of which the equations become

$$\begin{aligned}\dot{\chi}&=-\Delta(x)\cos\varphi,\\ \dot{\varphi}&=U(x)-E+\Delta(x)\operatorname{cth}\chi\sin\varphi,\\ \Delta(x)&=\Delta_0+\xi(x),\quad \Delta_0=\langle\Delta(x)\rangle.\end{aligned} \tag{7}$$

In order for localization effects to be appreciable, x must be large enough so that $\rho\gg1$. In this case $\chi\approx\ln\rho\gg1$ and we can set $\coth\chi=1$ in the second equation in (7):

$$\dot{\varphi}=U(x)-E+(\Delta_0+\xi(x))\sin\varphi. \tag{8}$$

We can interpret (8) as the equation of motion for a nonlinear pendulum in a viscous medium for which $\dot{\varphi}$ is small. Here $U(x)-E$ may be regarded as a torque, while $\xi(x)$ is a random parametric perturbation.

We consider a system with independent Gaussian totally uncorrelated random variables U and ξ:

$$\langle U(x)U(x')\rangle=2\Gamma\delta(x-x'),\quad \langle\xi(x)\xi(x')\rangle=2D\delta(x-x').$$

In general, there are many cases when one cannot neglect the finiteness of the correlation length for U and ξ. However, the characteristic features of localization in the two-band model (in particular, the behavior of the localization length l for states of energy $E\sim\Delta$) can be analyzed in the limit of vanishing correlation length.

3. MEAN VALUE AND DISPERSION OF THE RESISTANCE

The resistance ρ is expressible in terms of the zeroth component of the vector R satisfying system (4):

$$\langle\rho(x)\rangle=(\langle R_0(x)\rangle-1)/2.$$

Since Eqs. (4) are linear and the autocorrelation functions of U and ξ are assumed to be proportional to δ-functions, we can write the equations for $\langle R\rangle$ as

$$\langle\dot{\mathbf{R}}\rangle=\omega\langle\mathbf{R}\rangle,\quad \omega=\begin{pmatrix}D & 0 & -\Delta_0\\ 0 & -\Gamma & -E\\ -\Delta_0 & E & D-\Gamma\end{pmatrix}. \tag{9}$$

Equations (9) form a system of linear homogeneous equations with constant coefficients for $\langle R_0\rangle$, $\langle R_1\rangle$, and $\langle R_2\rangle$. The mean resistance thus grows exponentially for large x: $\langle\rho\rangle\sim\exp(\lambda x)$, where the growth exponent λ is the root of

$$(D-\lambda)(\Gamma+\lambda)(D-\Gamma-\lambda)-E^2(D-\lambda)-\Delta_0{}^2(\Gamma+\lambda)=0 \tag{10}$$

for which $\operatorname{Re}(\lambda)>0$ is largest. If $\Delta_0=0$, this root is independent of E and Γ: $\lambda=D$. For $E=0$ we have

$$\lambda=D+(\Gamma^2/4+\Delta_0{}^2)^{1/2}-\Gamma/2, \tag{11}$$

while for $E\to\infty$

$$\lambda\approx D+(\Gamma+D)\Delta_0{}^2/E^2. \tag{12}$$

We next calculate the variance of ρ by writing out the equations for bilinear combinations of the R_i. The symmetric tensor R_iR_k satisfies a closed system of linear equations. The constants of motion (5) imply that there are five independent components; we make the convenient choice

$$\begin{aligned}Q_1&=(3R_0{}^2-1)/2,\quad Q_2=(R_1{}^2-R_2{}^2)/2,\\ Q_3&=R_0R_1,\quad Q_4=R_0R_2,\quad Q_5=R_1R_2.\end{aligned}$$

After averaging the linear equations for Q_i over U and ξ, we find that

$$\dot{\langle Q\rangle}=\Omega\langle Q\rangle, \quad Q=(Q_1, Q_2, Q_3, Q_4, Q_5), \tag{13}$$

$$\Omega=\begin{pmatrix} 3D & -3D & -3\Delta_0 & 0 & 0 \\ -D & D-4\Gamma & \Delta_0 & 0 & -2E \\ -\Delta_0 & \Delta_0 & 4D-\Gamma & E & 0 \\ 0 & -E & 0 & D-\Gamma & -\Delta_0 \\ 0 & 2E & 0 & -\Delta_0 & D-4\Gamma \end{pmatrix}.$$

The mean square $\langle\rho^2\rangle = \langle Q_1/6\rangle \sim \exp(\Lambda x)$ increases exponentially, just as we found for $\langle\rho\rangle^2$. Here Λ is the eigenvalue of the matrix Ω with the largest real part. We note that we always have $\Lambda > 2\lambda$, i.e., the growth exponent of the squared variance $\langle\rho^2\rangle - \langle\rho\rangle^2$ is greater than the growth exponent for $\langle\rho\rangle^2$. The characteristic equation for the eigenvalues of Ω splits into a product of cubic and quadratic equations if $E = 0$ or $\Delta_0 = 0$. The solution for $\Delta_0 = E = 0$ is

$$\Lambda=2[D+(D^2+\Gamma^2+D\Gamma)^{1/2}-\Gamma], \tag{14}$$

while for $E = \Gamma = 0$ we have

$$\Lambda=4D+2\Delta_0. \tag{15}$$

The higher moments of ρ can be calculated by the same method. The fact that the growth exponents increase with moment order indicates that ρ is not a self-averaging quantity.

We close this section by observing that the mean and variance of the resistance have a nonzero growth exponent even when the density of states $N(E)$ has a Dyson singularity.

4. MEAN LOCALIZATION LENGTH AND $\langle\ln\rho\rangle$

Because $\chi \approx \ln\rho$ satisfies the nonlinear equations (7), there is no closed equation for $\langle\chi\rangle$. The distribution density and hence also the mean value of χ can be found from the Fokker-Planck equation for the joint probability density for χ and φ:

$$\dot{W}(\chi,\varphi,x)=\left[E\frac{\partial}{\partial\varphi}+\Gamma\frac{\partial^2}{\partial\varphi^2}+\Delta_0\left(\cos\varphi\frac{\partial}{\partial\chi}-\frac{\partial}{\partial\varphi}\sin\varphi\right)\right.$$
$$\left.+D\left(\cos\varphi\frac{\partial}{\partial\chi}-\frac{\partial}{\partial\varphi}\sin\varphi\right)^2\right] W(\chi,\varphi,x). \tag{16}$$

To calculate $\langle\ln\rho\rangle$ and the localization length we multiply (16) by χ and integrate over φ and χ. The result is

$$\dot{\langle\chi\rangle}=-\Delta_0\langle\cos\varphi\rangle+D\langle\sin^2\varphi\rangle. \tag{17}$$

For large x the right-hand side of (17) tends to the constant value l^{-1}, so that

$$\langle\ln\rho\rangle\approx\langle\chi\rangle\approx l^{-1}x,$$

where l is the localization length for states of energy E. We can thus find $\langle\chi\rangle$ and l by using the stationary distribution function $P(\varphi)$ to carry out the averaging in the right-hand side of (17). Equation (16) implies that P satisfies the equation

$$\left(E+\Gamma\frac{d}{d\varphi}-\Delta_0\sin\varphi+D\sin\varphi\frac{d}{d\varphi}\sin\varphi\right)P(\varphi)=J, \tag{18}$$

where the integration constant J is determined by the normalization condition. In addition, $P(\varphi)$ is required to be nonnegative and periodic in φ.

The variable φ describes the phase of the oscillations of the bilinear combinations of the wave function components. The number of nodes of these components is related to the number of states in a system of length x. Following Ref. 12, we conclude that $J(E)$ is proportional to the number of states.

In the rest of this section we will derive solutions for the steady-state equation (18) and obtain expressions for the localization length in some special cases. The results will be discussed in Sec. 5.

a) Ultrarelativistic limit. In this case the energy E is large compared with Δ_0, Γ, D, and Eqs. (8) show that φ oscillates rapidly with frequency E, regardless of the relative magnitudes of Δ_0, Γ, D. The distribution $W(\chi,\varphi,x)$ is therefore almost independent of φ, and χ is normally distributed with mean $\langle\chi\rangle = x/l$. If we average the stationary Fokker-Planck equation (18) over the fast oscillations, we get

$$l^{-1}\approx\frac{1}{2}\left[D+\frac{\Gamma}{E^2}\left(\Delta_0^2-\frac{D^2}{2}\right)+\frac{D}{E^2}\left(\Delta_0^2-\frac{D^2}{4}\right)\right]$$

to the lowest nonvanishing order in $1/E$.

b) States with energy at the center of the gap. If $E = 0$, $\Gamma \neq 0$, and $D \neq 0$ then we can derive an exact expression for l when $\Delta_0 = 0$. The stationary distribution density is given by

$$P(\varphi)=\frac{1}{N}\left(\frac{\Gamma+D}{\Gamma+D\sin^2\varphi}\right)^{1/2},$$
$$N=2K(k), \quad k=D/(\Gamma+D), \tag{20}$$

where $K(k)$ is the complete elliptic integral of the first kind. If we use the density (20) to carry out the averaging in (17) we find that

$$l^{-1}=2\Gamma\partial\ln K(k)/\partial\ln k^2, \tag{21}$$

which implies that

$$l\approx\frac{1}{2D}\ln\frac{D}{\Gamma} \tag{22}$$

in the limit $\Gamma \ll D$. In the opposite limit $\Gamma \gg 1$, (21) yields

$$l\approx\frac{2}{D}\left(1+\frac{25}{32}\frac{D}{\Gamma}\right). \tag{23}$$

c) Constant gap. If $\Delta = \Delta_0 = \text{const}$ then we can derive an exact expression for l for arbitrary E, Δ_0, and Γ. The stationary distribution density is given by

$$P(\varphi)=\frac{1}{N_\Gamma}\int_\varphi^{\varphi+2\pi} d\theta\exp[\mu(\theta-\varphi)+\delta(\cos\varphi-\cos\theta)], \tag{24}$$

where $\mu = E\Gamma$, $\delta = \Delta_0/\Gamma$, and

$$N_\Gamma=4\pi^2e^{-\pi\mu}|I_{i\mu}(\delta)|^2. \tag{25}$$

Here $I_\nu(z)$ is a modified Bessel function. Substituting (24) into (17), we readily find the expression

$$l^{-1}=\frac{\Gamma}{2}\frac{\partial\ln N_\Gamma}{\partial\ln\delta} \tag{26}$$

for l^{-1} in terms of the normalization factor.

The equation of the two-band model reduce to the Schrödinger equation with a random potential $U(x)$ if E and $\Delta = \Delta_0 \to \infty$ while $E - \Delta_0$ and Γ remain constant

$[(E-\Delta_0)/\Gamma$ can be arbitrary]. The result (26) can be transformed to yield the familiar expression[14] for the electron localization length in the quadratic spectrum model.

In the ultrarelativistic limit (26) reduces to

$$l^{-1}\approx\Gamma\Delta_0^2/2E^2, \tag{27}$$

while for $E=0$ we have

$$l^{-1}=\Gamma\delta I_1(\delta)/I_0(\delta). \tag{28}$$

For $E\ll\Gamma$ and $\Delta_0\ll\Gamma$ we have

$$l\approx\frac{2}{\Gamma}\frac{\Gamma^2-E^2}{\Delta_0^2}. \tag{29}$$

The localization length for states with energies close to the center of the gap is thus finite if $\Delta_0\neq 0$. The Dirac equation with $\Delta=\Delta_0+\xi=0$ implies that carrier scattering by the potential field merely changes the phase of the wave function without causing any of the states to become localized.

d) Fluctuating gap model. If the potential energy $U(x)=0$ then we can derive an exact expressions for the localization length for arbitrary E, Δ_0 and D. We replace φ in Eq. (8) by the new variable

$$\theta=\sinh^{-1}(\cot(\varphi)),\quad -\infty<\theta<\infty,$$

which is such that the random process appears additively in the stochastic equation,

$$\begin{aligned}\theta&=\operatorname{arcsh}(\operatorname{ctg}\varphi),\quad -\infty<\theta<\infty,\\ \dot\theta&=\varepsilon\operatorname{ch}\theta\pm\nu-\xi(x),\end{aligned} \tag{30}$$

where $\varepsilon=E/D$ and $\nu=\Delta_0/D$. The upper and lower signs in (30) correspond to $0<\varphi<\pi$ and $-\pi<\varphi<0$, respectively. The stationary distribution $P(\theta)$ has the form

$$P(\theta)=\frac{1}{N_D}\int_0^\infty d\tau\exp[\mp\nu(\tau-\theta)+\varepsilon(\operatorname{sh}\tau-\operatorname{sh}\theta)], \tag{31}$$

$$N_D=\pi^2[J_\nu^2(\varepsilon)+N_\nu^2(\varepsilon)], \tag{32}$$

where $J_\nu(z)$ and $N_\nu(z)$ are the Bessel and Neumann functions. Averaging (17) over the density (31), we find after some straightforward algebra that

$$l^{-1}=-\frac{D}{2}\frac{\partial\ln N_D}{\partial\ln\varepsilon}. \tag{33}$$

We stress that for potential fluctuations (case *c* above), l^{-1} is expressed in terms of the logarithmic derivative of the normalization factor of the steady-state density with respect to $\ln(\Delta_0/\Gamma)$, whereas for gas fluctuations the derivative is with respect to $\ln(E/D)$.[1)] This analogy between expressions (26) and (33) can be traced to the fact that in the latter case, the equation

$$\dot{\tilde\varphi}=-i\Delta_0+iE\cos\tilde\varphi+i\xi$$

for φ follows by continuing Eq. (30) for θ analytically to the imaginary axis. The above result then reduces to the equation for φ for the case of potential fluctuations if we make the substitutions $i\Delta_0\to E$, $iE\to\Delta_0$, $i\xi\to U$, and $\tilde\varphi\to\varphi+\pi/2$. Equation (33) thus follows in principle from (26) by analytic continuation.

Recalling the asymptotic formulas for Bessel functions of large argument, we find that

$$l^{-1}\approx\frac{D}{2}\left[1+\frac{1}{E^2}\left(\Delta_0^2-\frac{D^2}{4}\right)\right] \tag{34}$$

in the ultrarelativistic limit. In the nonrelativistic approximation the Dirac equation reduces to the Schrödinger equation, where Δ_0 plays the role of the particle mass and $\xi(x)$ is the potential energy. Equation (33) then reduces to the familiar expression for the localization length given in Ref. 14.

For half-integral values $\nu=\Delta_0/D$ we can express the localization length as a ratio of polynomials of degree $\nu-1/2$ in the variable $\varepsilon^2=(E/D)^2$. For example, for $\nu=3/2$ we have

$$Dl/2=(1+\varepsilon^2)/(3+\varepsilon^2). \tag{35}$$

For $\Delta_0=D/2(\nu=1/2)$, $l=2/D$ is independent of E. In the limit $E\ll D$ and $\Delta_0\ll D$ we have the asymptotic approximation

$$l\approx\frac{2}{\Delta_0}\operatorname{th}\left(\frac{\Delta_0}{D}\ln\frac{2D}{E}\right) \tag{36}$$

for (33). For $\Delta_0=0$ the density of states $N(E)$ has a Dyson singularity and l becomes infinite for states with energies at the center of the gap, as was noted in Ref. 14. As $\Delta_0\to 0$ we find from (36) that

$$l\approx\frac{2}{D}\ln\frac{2D}{E}. \tag{37}$$

The Dyson singularity disappears if the gap Δ_0 is nonzero [cf. (36)] or if the potential $U(x)$ is random (22).

5. DISCUSSION

In this paper we study how the mean value and dispersion of the resistance depend on the Fermi energy E, the unperturbed gap width Δ_0, and the potential (Γ) and gap (D) parameters characterizing the random fluctuations. We recall that the growth exponent λ of $\langle\rho\rangle$ is always less than the growth exponent $\Lambda/2$ for the root-mean-square deviation (this reflects the fact that ρ is not a self-averaging variable). It was shown in Ref. 7 that $\ln\rho$ is self-averaging; it is proportional to the length x of the system with proportionality coefficient equal to $1/l$, where l is the localization length of the corresponding electron state.

The results in the previous sections imply that the de-

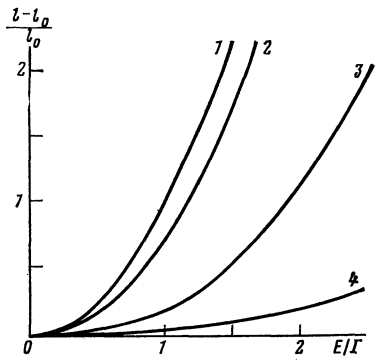

FIG. 1. Localization length as a function of energy for a nonfluctuating gap width; $\Delta_0/\Gamma=0.05$ (1), 0.5 (2), 2.5 (3), and 5.0 (4).

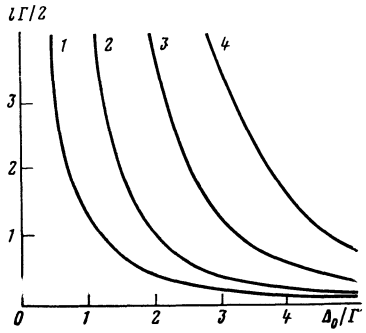

FIG. 2. Localization length as a function of gap width for a nonfluctuating gap; $E/\Gamma = 0$ (1), 2 (2), 4 (3), 6 (4).

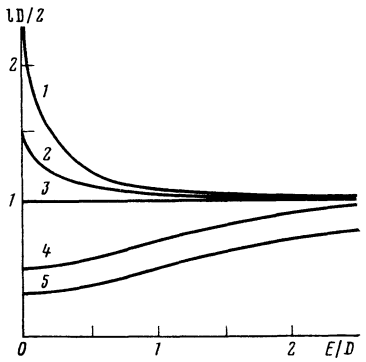

FIG. 3. Localization length vs energy for the fluctuating gap model; $\Delta_0/D = 0$ (1), 1/3 (2), 1/2 (3), 1 (4), 3/2 (5).

pendences $\lambda^{-1}(E)$ and $l(E)$ differ qualitatively as well as quantitatively. For example, λ^{-1} is always finite, while l becomes infinite when the electron energy lies at the center of the gap ($\Gamma = \Delta_0 = 0$). Under these same conditions, $N(E)$ has a Dyson singularity. The singular behavior of l for $E \to 0$ follows directly from Eq. (4). Indeed, for $E = 0$ there is an additional constant of the motion: $R_1 = \text{const} = 0$. The distribution density $P(\varphi)$ is nonzero only at the points 0 and π, whereas (17) implies that l^{-1} tends to zero. For small $E \ll D$, P has sharp peaks for $\varphi \approx 0$, π, which causes l to behave logarithmically [cf. (37)]. We can explain this qualitatively by noting that because $N(E)$ is higher at low energies (Dyson singularity), the wave function is a superposition of many overlapping states and the localization length is large. On the other hand, we have shown that the Dyson singularity disappears if the gap width is nonzero ($\Delta_0 \neq 0$) or if the potential energy fluctuates [cf. (22)].

We note that according to the above model, the behavior of λ^{-1} and l differs qualitatively even if the carriers are only weakly scattered ($E \to \infty$). According to (12), the probabilities for scattering by the statistically independent fluctuations in U and ξ contribute additively to λ, i.e., the Mattissen rule is valid, in contrast to the case for l^{-1} [cf. (19)]. We showed in Sec. 4a that $\langle \ln \rho \rangle$ has a Gaussian distribution function in the weak-scattering limit. We have already stressed that in the nonrelativistic limit $\Delta_0 \to \infty$ our results coincide with the results for the one-band model. For instance, if we assume weak scattering $\Gamma \ll E - \Delta_0$, then (23) yields $l^{-1} = \Delta_0 \Gamma / 2(E - \Delta_0)$ for the localization length; this result agrees with l found in the one-band model for scattering by a Gaussian random potential $U(x)$ with a correlation function of the form $\langle U(x) U(x') \rangle = 2\Gamma \delta(x - x')$.

As an illustration, Fig. 1 shows how $(l - l_0)/l_0$ depends on energy for several nonfluctuating gap widths (l_0 is the localization length of states with $E = 0$). Figure 2 shows the dependence $l\Gamma/2(\Delta_0/\Gamma)$ for several values of E, including $E = 0$.

Figure 3 shows how l depends on E for the fluctuating gap model. We see that the fluctuations in U and Δ lead to qualitatively different dependences $l(E)$. For potential fluctuations l increases with E for all mean gap widths Δ_0. However, if Δ fluctuates and $U \equiv 0$, $l(E)$ is decreasing for $\Delta_0 < D/2$ and increasing for $\Delta_0 > D/2$. The length l is independent for E for $\Delta_0 = D/2$. It was shown in Ref. 23 that the form of the state density function $N(E)$ changes at this value of Δ_0.

We note in closing that Eq. (18) can be solved without difficulty in the general case when both $U(x)$ and $\Delta(x)$ fluctuate and the electron energy is arbitrary. However, because the result for the localization length cannot be expressed in terms of standard special functions, numerical methods are needed to analyze l.

[1)]The normalization factor $N_D(E)$ is proportional to the number of electron states with energies between $-E$ and $+E$.

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