Quantum decay of metastable current states at superconducting junctions

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(Submitted 15 January 1985)
Zh. Eksp. Teor. Fiz. **89**, 242–257 (July 1985)

The effective action for superconducting junctions with a direction conductivity is expressed in terms of the Green's functions of these junctions and calculated in the adiabatic approximation. In several cases (e.g., for pure SCS and SNS junctions at $T = 0$) the effective potential is not quasiclassical, and the dissipation is nonlinear. The rate of the quantum decay of metastable current states at superconducting weak links with a direct conductivity is calculated with exponential accuracy. This rate may differ significantly from that for tunnel junctions.

1. INTRODUCTION

A current state of a superconducting weak link is known to have a finite lifetime. The reason is that the phase difference between the order parameters of the superconductors in contact fluctuates about its equilibrium value, which is determined by the current which flows through the contact. At high temperatures the characteristic time for the decay of the current state is determined by thermal fluctuations of the phase difference. At low temperatures, thermal fluctuations are relatively unimportant, so that a quantum mechanism for the decay of the metastable state is dominant.

As it turns out, the electronic degrees of freedom can be ignored in a calculation of the rate of the quantum decay of current states of Josephson junctions only if these junctions have a large capacitance, which plays the role of a mass of the tunneling particle. The capacitance of the tunnel junctions used in the experiments of Refs. 1 and 2, on the other hand, was not very large. Under such conditions, transitions between different electronic states should generally be taken into account in a calculation of the lifetime of metastable current states.

Caldeira and Leggett[3,4] took into account the effect of transitions between harmonic-oscillator states of a heat reservoir on the probability for the decay of a metastable state of a quasiclassical degree of freedom at zero temperature. They showed that such transitions lead to a dissipation which may substantially reduce the probability for quantum tunneling.

A microscopic expression for the effective action was first derived by Ambegaokar *et al.*[5] by a tunnel-Hamiltonian method. The general expression for this action is nonlocal in time. If the current flowing through the contact is close to the critical value (this is the case in which it is easiest to experimentally detect and study the effects with which we are concerned here), the expression for the action simplifies substantially. In this case we can use the adiabatic approximation for the potential $V(\varphi)$ to calculate the decay rate, and the effect of the electronic degrees of freedom at $T = 0$ reduces to a renormalization of the capacitance, as Larkin and Ovchinnikov[6] have shown. Dissipation in such systems can be taken into account by introducing an auxiliary parameter, the shunt resistance, which is generally completely independent of the junction resistance. In this case the expression for the action of the tunnel junction in the adiabatic approximation is of the same form[7] as the corresponding expression of the phenomenological theory.[3,4] We wish to emphasize that if there is no shunt at $T = 0$, and if the frequency with which the phase difference changes is small, there is no dissipation at tunnel junctions.

At a nonzero temperature, it becomes slightly more complicated to calculate the lifetime of current states, since excited states must be taken into account. The fact that a metastable state has a finite lifetime means that the expression for the free energy of the system acquires an imaginary part. The temperature dependence of this quantity has been studied by Larkin and Ovchinnikov.[7]

In a theoretical description of quantum tunneling with dissipation, an effective action has accordingly been used. This action has been found either from the model of a particle plus a heat reservoir[3,4,8,9] or by a microscopic tunneling-Hamiltonian method,[5–7] which is known to be applicable only for tunneling junctions. Several experiments[10,11] have dealt with a macroscopic quantum tunneling in SQUIDs in which the weak link consists of point contacts with a direct (nontunneling) conductivity. Dissipation has a particularly strong effect on the probability of a quantum tunneling in weak links with a direct conductivity. Our basic purpose in the present paper is to carry out a microscopic calculation of the effective action and of the probability for the quantum decay of metastable current states in superconducting weak links of various types (point contacts and SNS bridges). The results show that the effective action and the probability for macroscopic quantum tunneling in such systems may be quite different from the corresponding quantities for tunnel junctions.

2. EFFECTIVE ACTION FOR SUPERCONDUCTING JUNCTIONS

As we have already mentioned, the microscopic expression for the effective action has been derived only for tunnel junctions[5–7] by a tunneling-Hamiltonian method. It is known quite well that this method is not suitable for describing the properties of superconducting weak links with a direct conductivity. For this reason, the corresponding results of Refs. 5–7 cannot be directly generalized to the case of

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junctions with a nontunneling conductivity. In this section of the paper we derive a general expression for the effective action of superconducting weak links which obviously also holds in the particular case of tunnel junctions.

A. Effective action on a Keldysh contour

For definiteness we consider the standard model of an SNS bridge: two bulk superconductors connected by a thin bridge of a normal metal of length d and cross-sectional area S. In the limit $d\to 0$ we find the model of superconducting contractions (SCS junctions). We describe the system by the ordinary Hamiltonian of superconductivity theory[12]:

$$\hat{H}=\int d\mathbf{r}\left\{\psi_\sigma^+(\mathbf{r})\left[\frac{1}{2m}(i\nabla-e\mathbf{A})^2-\mu+e\Phi\right]\psi_\sigma(\mathbf{r})-\frac{g}{2}\psi_\sigma^+(\mathbf{r})\psi_\sigma^+(\mathbf{r})\psi_{-\sigma}(\mathbf{r})\psi_{-\sigma}(\mathbf{r})\right\}+\hat{H}_{sc}. \quad (1)$$

Here ψ_0^+, ψ_σ are the ordinary operators which create and annihilate an electron with spin σ, $\mathbf{A}$ is the vector potential, Φ is the scalar potential, μ is the chemical potential, g is the effective constant of the BCS interaction, assumed equal to zero for a normal bridge, and $\hat{H}_{sc}$ is the part of the Hamiltonian which describes all possible electron-scattering processes. For simplicity we will omit this part of the Hamiltonian.

We write a general expression for the probability for a transition of a quantum-mechanical system from a state described by the density matrix $\Sigma_i w_i|i\rangle\langle i|$ to the state $|f\rangle$ over the time t_f-t_i:

$$W_f=\sum_i w_i\langle i|\hat{U}|f\rangle\langle f|\hat{U}^+|i\rangle, \quad (2)$$

where

$$\hat{U}=\hat{T}\exp\left(-i\int_{t_i}^{t_f}\hat{H}dt\right)$$

is the evolution operator. We are interested in the case in which the states $|f\rangle$ and $|i\rangle$ are separated by a high potential barrier, so that in the zeroth approximation in the transmission of this barrier these states may be assumed orthogonal. The total probability for a transition across the barrier over the time t_f-t_i is obviously the sum of W_f over all f:

$$W=\sum_i w_i\langle i|\hat{J}|i\rangle, \quad \hat{J}=\sum_f \hat{U}|f\rangle\langle f|\hat{U}^+. \quad (3)$$

The kernel of the operator $\hat{J}$ can be rewritten as

$$J=\int D^2\Delta D\Phi D\mathbf{A}\exp(iS_{C_0}),$$
$$iS_{C_0}=\ln \mathrm{Sp}\,\hat{T}_{C_0}\exp\left(-i\int_{C_0}\hat{H}_{\text{eff}}dt\right), \quad (4)$$

where

$$\hat{H}_{\text{eff}}=\int d\mathbf{r}\left\{\psi_\sigma^+(\mathbf{r})\left[\frac{1}{2m}(i\nabla-e\mathbf{A})^2-\mu+e\Phi\right]\psi_\sigma(\mathbf{r})+\left[\Delta^*(\mathbf{r},t)\psi_\sigma(\mathbf{r})\psi_{-\sigma}(\mathbf{r})+\text{H.a.}\right]+\frac{1}{g}|\Delta(\mathbf{r},t)|^2\right\}. \quad (5)$$

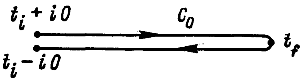

FIG. 1.

Here C_0 is the Keldysh contour[13] (Fig. 1), $\hat{T}_{C_0}$ is the ordering operator on this contour, and the trace is taken over the electron variables. In the derivation of $\hat{H}_{\text{eff}}$ in (5), we decouple the ψ^4 term in (1) by the usual procedure of introducing a scalar field

$$\Delta=|\Delta(\mathbf{r},t)|\exp[i\varphi(\mathbf{r},t)]$$

(the Hubbard-Stratonovich transformation).

We assume that a current I below the critical current I_c of the superconducting junction flows through the system. We assume that the phase distribution $\phi_0(\mathbf{r})$ corresponds to this situation. We write φ in the form

$$\varphi(\mathbf{r},t)=\varphi_0(\mathbf{r})+\varphi_1(\mathbf{r},t),$$

where φ_1 describes the phase fluctuations. Our problem is to determine how S_{C_0} in (4) depends on φ_1. The operation of taking the trace over the electron variables in (4) reduces to an evaluation of a Gaussian integral over the ψ fields, which is known to be equal to the determinant of the corresponding matrix. To calculate this determinant in its general form is a difficult problem. Furthermore, the result would contain an excess of information about the superconducting banks. On the other hand, the contribution of the fluctuations of the phase φ_1 to the effective action, in which we are interested, can be calculated by a simple approach, which can be outlined as follows. We make the gauge transformation

$$\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\frac{1}{2e}\nabla\varphi, \quad \Phi\to\Phi'=\Phi-\frac{1}{2e}\frac{\partial\varphi}{\partial t}, \quad (6)$$

and then formally replace $\varphi_1(\mathbf{r},t)$ by $\lambda\varphi_1(\mathbf{r},t)$, where the parameter λ describes the "turning on" of the fluctuations $(0\leqslant\lambda\leqslant 1)$. We differentiate the resulting expression for $S_{C_0}(\lambda)$ with respect to λ inside the trace:

$$\frac{\partial S_{C_0}(\lambda)}{\partial\lambda}=\frac{1}{2e}\int_{C_0}dt\int d\mathbf{r}\nabla\varphi_1\frac{i}{m}(\nabla_{\mathbf{r}}-\nabla_{\mathbf{r}'})_{\mathbf{r}\to\mathbf{r}'}\langle\hat{T}_{C_0}\psi\psi^+\rangle_\lambda. \quad (7)$$

The quantity $\partial S_{C_0}/\partial\lambda$ is thus expressed in terms of Green-Keldysh functions with identical arguments. Switching to quasiclassical Eilenberger functions,[14] integrated over the energy, on the contour C_0, we find

$$\frac{\partial S_{C_0}(\lambda)}{\partial\lambda}=\frac{1}{2e}\int_{C_0}dt\int d\mathbf{r}\nabla\varphi_1(\mathbf{r},t)\frac{ep_F^2}{4\pi}\mathrm{Sp}\hat{\tau}_3\int_{-1}^{1}d\alpha\hat{G}_\lambda(v_F\alpha;t,t), \quad (8)$$

where τ_3 is a Pauli matrix, and the integration over α implies an average over the directions of $\mathbf{v}_F$. Now integrating (8) over λ from 0 to 1, we find the expression for the effective action which we need. The integration over the variables $|\Delta|$, A, and Φ in (4) is carried out by the method of steepest descent. The condition that the contribution of the superconducting slopes is large allows us to replace $|\Delta(\mathbf{r},t)|$ by the equilibrium value of the order parameter in the system,

$\Delta(\mathbf{r})$, and in the integration over $\mathbf{A}$ and Φ on the saddle-point trajectory the following conditions hold, in complete analogy with the case of tunnel junctions (see Ref. 7, for example):

$$2e\Phi=\partial\varphi/\partial t, \quad 2e\mathbf{A}=-\nabla\varphi. \tag{9}$$

Adding to the action terms which reflect the energy of the external field and its interaction with the current I, we finally find

$$J=\int D\varphi \exp(iS_{C_0}[\varphi]), \tag{10}$$
$$S_{C_0}[\varphi]=S_{C_0}[\varphi_0]+\int_{C_0} dt\left\{\frac{C}{2e^2}\left(\frac{\partial\varphi}{\partial t}\right)^2+I\frac{\varphi}{e}\right.$$
$$\left.+\frac{\varphi(t)}{4e^2R}\int_0^1 d\lambda\int_{-1}^{1} d\alpha\,\mathrm{Sp}\,\hat{\tau}_3\hat{G}_\lambda(v_F\alpha;t,t)\right\}.$$

Here C represents the sum of the junction capacitance and a (possible) external capacitance, and $2[\varphi_0+\varphi(t)]$ is the phase difference between the order parameters of the superconducting banks. In deriving (9) we used current conservation, and we assumed that the mean free path is long (the "clean limit"). The bridge resistance in this case is determined by the familiar expression $R=\pi^2/p_F^2e^2S$, and the functions $\hat{G}$ for pure SCS junctions and SNS bridges are calculated in Refs. 15 and 16, respectively. Everywhere below we will assume that the term $S_{C_0}[\varphi_0]$, which is unimportant for our purposes, is zero. Using (8), we can obviously also find the effective action for other types of superconducting weak links whose functions $\hat{G}$ are known. For example, substituting the Green's functions for tunnel junctions into (8), we can easily reproduce the effective action of Refs. 5–7.

We assume that the current I is close to the critical value I_c. We consider the case of pure SCS and SNS bridges. In this case, the approximate agreement of I and I_c means $\chi=\pi/2-\varphi_0\ll 1$, since the critical current is reached at $\varphi_c=\pi/2$ (Refs. 17 and 18). The general expression for the function $\hat{G}(t,t)$ at such junctions is

$$\hat{G}(t,t)=\int_{C_0}\hat{X}(t,t')\hat{Y}(t',t)dt'. \tag{11}$$

The matrices $\hat{X}$ and $\hat{Y}$ are given by extremely lengthy expressions which are reproduced in Appendix 1. We must emphasize that the function $\hat{G}(t,t)$ is determined by expression (11), which is nonlocal in time, because of a retardation of the interaction. Expressions of this type ordinarily arise when an average is taken over some set of quantum variables which are interacting with a particular degree of freedom (influence functionals[19,20]). In our case, this degree of freedom is the quantity $\varphi(t)$, and the influence functional arises when an average is taken over the electron variables.

Here we are interested in the situation in which the characteristic frequencies of the change $\varphi(t)$ are small in comparison with $\min\{\Delta,\Delta_d\}$, where Δ is the equilibrium value of the modulus of the order parameter of the superconductor, and $\Delta_d=v_F/d$. In other words, we calculate the function $\hat{G}$ in the adiabatic approximation. In lowest order, the contribution to the action is determined by those terms in the general expressions (11), (A1)–(A3) which do not contain the functions $\hat{Q}_-^{R,A}$, $\hat{n}_-$. In this case we have $\hat{n}_+(\varepsilon)=\hat{1}\,\mathrm{th}(\varepsilon/2T)$, and the matrices $\hat{Q}_+^{R,A}$ are determined by the equilibrium expressions, in which we should replace φ_0 by $\varphi_0+\varphi(t)$. Substituting these terms into (10), and adding the term $I\varphi/e$, we find that the corresponding contribution to the action can be represented as an integral of the function $V(\varphi(t))$ over the contour C_0. In the limit $T\to 0$ we easily find the following expression for this function under the conditions $\Delta_d\gg\Delta$, $\chi\gg 1$:

$$V(\varphi)=\frac{\pi\Delta}{2e^2R}\left(\chi\varphi^2-\frac{\varphi^3}{2}\right)\theta(\chi-\varphi)$$
$$+\left[\frac{\pi\Delta\chi^3}{4e^2R}-\frac{2I(\chi-\varphi)}{e}\right]\theta(\varphi-\chi). \tag{12}$$

In the case $\Delta_d\ll\Delta$ we find

$$V(\varphi)=\frac{2\Delta_d}{3e^2R}[\varphi^2+2\pi(\chi-\varphi)\theta(\varphi-\chi)]. \tag{13}$$

In deriving (12) and (13) we made use of the fact that under the condition $I_c-I\ll I_c$ the quantity $\varphi_0+\varphi$ varies in a narrow interval of values near $\pi/2$; i.e., we have $\varphi\ll 1$. We will not go into detail here on the calculation of the contribution to the action from the next approximation in the adiabatic parameter in those terms in expressions (11) and (A1) which do not contain functions $\hat{Q}_-^{R,A}$, $\hat{n}_-$. We simply note that these calculations give rise to renormalization of the capacitance in expression (10):

$$C^*=C+\begin{cases}\pi\chi/2\Delta R, & \Delta_d\gg\Delta,\quad \chi\ll 1,\\ 1/2\Delta_d R, & \Delta_d\ll\Delta.\end{cases} \tag{14}$$

The difference between C^* and C evidently should be taken into account only when C is on the order of or less than C^*-C. Ordinarily (except in the case in which the dissipation is quite weak) the decay rate of the metastable state is independent of C^* to within exponential accuracy for such values of the capacitance; in other words, we are dealing with the strong dissipation limit. Nevertheless, we will take into account the difference between C and C^* here and below, since renormalization of the capacitance may prove important for determining the coefficient of the exponential function in the expression for the decay rate of the current states.

Let us examine in more detail the contribution to the action from those terms in (11) and (A1) which we have not yet considered. As we will see below, these are the terms which determine (in the leading approximation) the dissipation and the quantum noise in superconducting junctions. In this approximation the corresponding part of the function $\hat{G}(t,t)$ is

$$\hat{G}^a(t,t)=\frac{\mathrm{sign}\,\alpha}{8}\int dt'\{\hat{X}_1(t,t')\hat{Y}_1(t',t)+\hat{X}_2(t,t')\hat{Y}_2(t',t)\},$$
$$\hat{X}_1=\hat{Q}_-^R[(-1)^l+n]+\hat{Q}_-^A[(-1)^m-n],$$
$$\hat{X}_2=\hat{Q}_+^R\hat{n}_--\hat{n}_-\hat{Q}_+^A,\quad \hat{Y}_1=(\hat{Q}^R)^{-1}\hat{Q}_-^R\hat{n}_-(\hat{Q}_+^A)^{-1}$$
$$-(\hat{Q}_+^R)^{-1}\hat{n}_-\hat{Q}_-^A(\hat{Q}_+^A)^{-1},\quad \hat{Y}_2=(\hat{Q}_+^R)^{-1}$$
$$\times[(-1)^m+n]+(\hat{Q}_+^A)^{-1}[(-1)^l-n],$$
$$l=0(1),\ \mathrm{Im}\,t'>0(<0);\quad m=0(1),\ \mathrm{Im}\,t>0(<0). \tag{15}$$

The function $\hat{G}^a$ in (15) can be calculated easily in the approximation in which the response is linear in φ. In substituting the result into (10), we should replace φ by $\lambda\varphi$. Carrying out the integration over λ in (10), using the potential and kinetic terms in the limit $T\to 0$, we find

$$S_{c_0}[\varphi]=\int_{t_i}^{t_f} dt\varphi_-(t)\left\{-\frac{C^*}{e^2}\ddot{\varphi}_+(t)+V\left(\varphi_+-\frac{\varphi_-}{2}\right)\right.$$
$$-V\left(\varphi_++\frac{\varphi_-}{2}\right)\quad-\frac{1}{R_{\text{eff}}e^2}\dot{\varphi}_+(t)$$
$$\left.+\frac{i}{4\pi R_{\text{eff}}e^2}\int_{t_i}^{t_f} dt'\varphi_-(t')\int d\varepsilon\,|\varepsilon|\exp[-i(t-t')]\right\}, \tag{16}$$

$$R_{\text{eff}}=R\cos^2\varphi_0,$$

where

$$\varphi_+(t)={}^1/_2[\varphi(t+i0)+\varphi(t-i0)],\quad \varphi_-(t)=\varphi(t+i0)-\varphi(t-i0).$$

Expression (16) is of the same form as the effective action in Ref. 21 (in the limit $T=0$), which describes the behavior of a particle which is interacting with a large number of harmonic oscillators of the heat reservoir.

We recall that the effective resistance of tunnel junctions at $T=0$ and at sufficiently low frequencies ($<2\Delta$) is infinite; i.e., there is no dissipation. In superconducting junctions with a direct conductivity, as we see from (16), the effective resistance at $T=0$ is the same order of magnitude as the resistance of the junction in its normal state. Furthermore, R_{eff} decreases (the dissipation increases) as the current approaches its critical value. In other words, superconducting junctions may be in a resistive state even at $T=0$, i.e., even if there are no quasiparticle excitations in the superconductors. To see the physical reason for this result, we consider a superconducting junction with a normal-metal bridge. When a voltage is applied across the junction, the electrons in the bridge are accelerated, and field energy is obviously expended for this purpose. The current in such a system, at low temperatures and low voltages, results from the flow of a superconducting condensate: The current of quaisparticles converts into a current of Cooper pairs by the familiar Andreev-reflection mechanism. These arguments hold for essentially any length (d) of the normal-metal bridge. On the other hand, we know quite well that at sufficiently small values of d the expression for the current through the junction is completely independent of the superconducting properties of the bridge (i.e., the bridge may be in either a normal or superconducting state).[1)]

The flow of the condensate in these superconducting junctions in the presence of a voltage is therefore dissipative. The dissipative component of the current, I^a, is a complicated function of $\varphi(t)$ in this case,[16,22] and it is generally not described by Ohm's law. In tunnel junctions, this mechanism for the transport of Cooper pairs does not operate in the first order of the expansion in the transmission, since in this case the transition of a Cooper pair from one superconductor to the other requires that the pair be "ruptured," at the cost of an energy 2Δ. In the following orders of the expansion in the transmission, however, a contribution is made to the current from the direct transition of Cooper pairs from condensate to condensate, so that the dissipative component of the current due to this mechanism should be nonvanishing in this case also. For similar reasons, the excess current seen on the voltage-current characteristics of superconducting junctions at high voltages is absent (again, only in first order in the transmission) in the case of tunnel junctions.

We should point out that a necessary condition for the validity of the approximation of a linear response in the calculation of the function $\hat{G}^a$ is $\varphi\ll\cos\varphi_0$. This condition clearly does not hold during the decay of current states in pure SCS and SNS junctions at low temperatures, since the final state of the system in this case is described by a quantity $\varphi>\chi$. Here the dissipative current is generally a nonlinear function of φ, and the expression for $S_{C_0}[\varphi]$ differs from (16). For our purposes it is convenient to express the effective action in terms of the function which actually determines the dissipative current I^a. Using expression (15) for $\hat{G}^a$ along with the analytic properties of retarded and advanced Green's functions, we find

$$S_{c_0}[\varphi]=\int_{C_0} dt\left\{\frac{C^*}{2e^2}\left(\frac{\partial\varphi}{\partial t}\right)^2-V(\varphi(t))\right.$$
$$\left.+i\int_{C_0} dt'\frac{F(\varphi(t),\varphi(t'))}{2\pi e(t-t')^2}\right\} \tag{17}$$

$$F(\varphi(t),\varphi(t'))=\int_0^1 d\lambda\varphi(t)f(\lambda\varphi(t),\lambda\varphi(t')),$$

where the function $f(\varphi(t),\varphi(t'))$ is related in the adiabatic approximation to the dissipative current I^a by

$$I^a(t)=-\left.\frac{\partial f(\varphi(t),\varphi(t'))}{\partial t'}\right|_{t'=t}. \tag{18}$$

The function $F(\varphi(t),\varphi(t'))$ will be shown below for several cases. At this point we simply note that

$$F(\varphi(t),\ \varphi(t))=0.$$

B. Analytic continuation of the action to the imaginary time axis

In calculating the probability for a transition between states separated by a high potential barrier, it is convenient to introduce an imaginary-time parameter to describe the system. There are no extremal trajectories describing the tunneling in real time. When we make the transition to an imaginary time, such trajectories arise, and they determine, to within exponential accuracy, the probability for the decay of the metastable state.

In the case of tunnel junctions, as Larkin and Ovchinnikov have shown,[6] the general expression for the effective action can be continued analytically since the Green's function for such junctions is known in first order in the transmission for arbitrary rates of change of φ. The expression for the transition probability which is continued analytically in this manner is actually the same as the expression for the

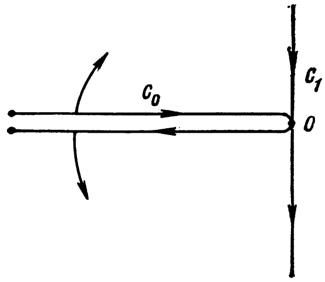

FIG. 2.

partition function of tunnel junctions.[5] For direct-conductivity junctions, the situation is more complicated. The Green's functions for such systems are rather difficult to calculate for arbitrary rates of change of φ (Refs. 23, 15, 16). Furthermore, the Green's functions which arise on the Keldysh contour when an average is taken over the electronic degrees of freedom are generally far from equilibrium near the junction. Accordingly, it is generally not possible to determine the tunneling probability directly from the expression for the effective action in terms of the temperature Green's functions of the system. If the phase fluctuation frequencies are low, however, the situation simplifies considerably. For example, the expression which we derived in the adiabatic approximation for the effective action, (17), can be continued in a comparatively simple way to the imaginary time axis.

We assume $t_f = 0$, and we assume that the time $-t_i$ is large in comparison with the characteristic tunneling frequencies; i.e., $t_1 \to -\infty$. We now "straighten out" the contour C_0, as shown in Fig. 2. The exponential function in the functional integral (10) then becomes $\exp(-S_E[\varphi(\tau)])$, where τ is the "time" on contour C_1. It is trivial to perform the analytic continuation of the potential and kinetic terms in action (17). The last term in (17) is also written in a form convenient for analytic continuation. As a result we find the expression

$$S_E = \int_{-\infty}^{\infty} d\tau \left[\frac{C^*}{2e^2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + V(\varphi(\tau)) + \int_{-\infty}^{\infty} d\tau' \frac{F(\varphi(\tau),\varphi(\tau'))}{2\pi e(\tau-\tau')^2}\right]. \tag{19}$$

Expression (19) can be derived by taking our approach of introducing a parameter λ in the expression for the partition function of the system. In the differentiation with respect to λ, we find equations analogous to (7) and (8), but with the effective action expressed in terms of the Matsubara Green's functions. The last term in brackets in (19) can be written

$$\frac{\varphi(\tau)}{e} \int_0^1 d\lambda I_E^a(\lambda\varphi(\tau)),$$

where $I_E^a(\varphi(\tau))$ is the analytically continued expression for the dissipative current at the junction. As we have already pointed out, this approach is valid only if the deviation from equilibrium is small, while the method which we have used is suitable for describing superconducting junctions with an arbitrary distribution function.

The calculation of the dissipative contribution to the effective action S_E is thus reduced to the problem of analytically continuing the expression for the current I^a, i.e., that of finding the function $F(\varphi(t),\varphi(t'))$. Using (17) and (18) in the adiabatic approximation, we find a relation which we can use to express F in terms of I^a:

$$\left.\frac{\partial^2 F(x_1,x_2)}{\partial x_1 \partial x_2}\right|_{x_1=x_2=\varphi(t)} = -I^a(t)\left(\frac{\partial\varphi}{\partial t}\right)^{-1}. \tag{20}$$

Without any loss of generality we can seek the function F in the class of functions which are symmetric with respect to the arguments $\varphi(t)$, $\varphi(t')$ (otherwise, we could always put the result in symmetric form). In the case of a linear dispersion-free dissipation we would have

$$I^a(t) = \frac{1}{eR_{\text{eff}}}\frac{\partial\varphi}{\partial t}. \tag{21}$$

We can easily determine the function F from (20) and (21). As a result we find the well-known form of the effective action[3,4]:

$$S_E = \int_{-\infty}^{\infty} d\tau \left\{ \frac{C^*}{2e^2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + V(\varphi(\tau)) + \frac{1}{4\pi e^2 R_{\text{eff}}} \int d\tau' \left[\frac{\varphi(\tau)-\varphi(\tau')}{\tau-\tau'}\right]^2 \right\}. \tag{22}$$

In the case of pure superconductors, in the approximation of a linear response, we find, according to (16),

$$R_{\text{eff}} = R\cos^2\varphi_0. \tag{23}$$

As we have already mentioned, it is legitimate to restrict the calculation of I^a to the linear response only if φ_0 is not too close to $\pi/2$. At a current close to the critical value, this situation may arise at tunnel junctions with short circuits. In this case, R is the resistance of the short circuit. The potential $V(\varphi)$ for such systems is

$$V(\varphi) = \frac{1}{e}\left\{[k(I_c-I)]^{1/2}\varphi^2 - \frac{k}{3}\varphi^3\right\}, \quad k = -\left.\frac{\partial^2 I}{2\partial\varphi^2}\right|_{I=I_c}, \tag{24}$$

where

$$I(\varphi) = \frac{\pi\Delta\sin\varphi}{eR_I}(\cos\varphi + 2r),$$

$$\cos\varphi_c = \left(r^2 + \frac{1}{2}\right)^{1/2} - r, \quad r = \frac{R_I}{2R}, \tag{25}$$

R_I is the resistance of the dielectric barrier, and $2\varphi_c$ is the phase difference which parametrizes the current state $I = I_c$. The quantity C^* in this case is given by

$$C^* = C + \frac{\pi\cos\varphi_c}{2\Delta R\sin^4\varphi_c} + \frac{\pi(3-\cos 2\varphi_c)}{32\Delta R_I}, \quad \varphi_c \geqslant \frac{\pi}{4}. \tag{26}$$

An effective action in the form (22) and (24) also describes SCS junctions with a large impurity concentration. The Green's functions for such junctions were calculated in Ref.

23. Substituting these functions into (8), and carrying out an analytic continuation, we find the effective action (22) with potential (24), where (see Appendix 2)

$$k=\pi\Delta/eR_d\sin 2\varphi_c\approx 3{,}40\Delta/eR_d,\quad C^*-C\sim(\Delta R_d)^{-1},\quad (27)$$

and R_d is the resistance of the junction in its normal state. In this case, in determining the dissipative contribution to the action for a current close to the critical value, we have restricted the analysis to a calculation of I^a in the linear approximation in φ, since the value of φ_c for dirty SCS junctions is not very close to $\pi/2$ ($\varphi_c\approx 0.983$). There would evidently be no difficulty in writing an effective action for tunnel junctions with short circuits in the case of very dirty superconductors. As in the pure limit, the values of C^* and k are determined by the sum of the contributions from the tunnel barrier and the short circuit, and $R_{\rm eff}$ is equal to the effective resistance of the short circuit.

An effective action of the type in (22), (24) thus describes different types of superconducting weak links for $T\to 0$ and $I_c-I\ll I_c$. The theory constructed here can be used to calculate the values of the parameters in (22) and (24) for junctions of this type. As we have already mentioned, however, in pure SCS and SNS junctions and also in wide, extremely dirty SNS junctions,[24] the current state $I=I_c$ corresponds at $T=0$ to $\varphi_c=\pi/2$. In this case the potential $V(\varphi)$ is not described by (24), even at a current close to I_c [see (12) and (13)]. Furthermore, the dissipation in such systems is nonlinear. To determine the nature of the dissipative term in the effective action we should find the function F. As before, we use (20) to determine this function.

At pure SCS and SNS junctions, at frequencies less than the reciprocal of the inelastic relaxation time of the electronic states of the superconductor (γ), the current I^a is given by[22]

$$I^a(t)=\frac{1}{(\varphi-\chi)^2+\varkappa^2}\frac{\partial\varphi}{\partial t},\quad \chi\ll 1,\quad \varkappa=\frac{\gamma}{\Delta}. \quad (28)$$

In this case the function F can be found easily. Using (20), we find

$$F=\frac{1}{2eR}[\sigma(\tau)-\sigma(\tau')]^2,\quad \sigma=\operatorname{Arsh}\frac{\chi-\varphi}{\varkappa}. \quad (29)$$

The effective action is

$$S_E[\varphi]=\int d\tau\left\{\frac{C^*}{2e^2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2+V(\varphi)\right.$$

$$\left.+\frac{1}{4\pi Re^2}\int d\tau'\left[\frac{\sigma(\tau)-\sigma(\tau')}{\tau-\tau'}\right]^2\right\}. \quad (30)$$

At higher frequencies, the expression for I^a becomes more complicated. In the adiabatic approximation, it is

$$I^a(t)=\frac{2i}{eR}\int\frac{d\omega}{2\pi}\frac{\varphi_\omega}{\omega}\int_0^1\alpha d\alpha\tilde\Delta^2\ln\left|\frac{(\chi-\varphi)^2+\varkappa^2}{(\chi-\varphi)^2-\omega^2/\tilde\Delta^2}\right|e^{-i\omega t},$$

$$\tilde\Delta=\begin{cases}\Delta, & \Delta_d\gg\Delta\\ \Delta_d\alpha, & \Delta_d\ll\Delta\end{cases}. \quad (31)$$

For this case, we restrict the analysis to an estimate of the dissipative term in the effective action. At frequencies $\omega\gg\gamma$ we can replace $(\chi-\varphi)^2$ in (31) by some constant $\tilde\chi^2$ which is of order χ^2. After this replacement, the expression for I^a becomes linear in φ. To determine the function F it is convenient to use the Fourier representation. As a result we find the dissipative contribution to the action to be

$$\frac{1}{2e}\int\frac{d\omega}{2\pi}\alpha_E(\omega)\varphi_\omega\varphi_{-\omega}, \quad (32)$$

where

$$\alpha_E=\frac{\Delta^2}{eR|\omega|}\ln\frac{\tilde\chi^2+\omega^2/\Delta^2}{\tilde\chi^2+\varkappa^2},\quad \Delta_d\gg\Delta, \quad (33)$$

$$\alpha_E=\frac{\Delta_d^2}{2eR|\omega|}\left[\left(1-\frac{\omega^4}{\Delta_d^4\tilde\chi^4}\right)\ln\left(1+\frac{\omega^2}{\Delta_d^2\tilde\chi^2}\right)+\frac{\omega^4}{\Delta_d^4\tilde\chi^4}\ln\left(\frac{\omega^2}{\Delta_d^2\tilde\chi^2}\right)+\frac{\omega^2}{\Delta_d^2\tilde\chi^2}\right],\quad \Delta_d\ll\Delta.$$

In the case of an expression for I^a which is nonlinear in φ, the dissipative term can be written in the form (32) by replacing φ_ω by σ_ω, where $\sigma=\sigma(\varphi)$ is some function of φ.

The quantity $\alpha_E(\omega)$ serves as a generalized Matsubara susceptibility, which characterizes the response of the system to a generalized force $\sigma(\tau)$. Here we have the well-known relation (see Refs. 12 and 25, for example)

$$\alpha_E(\omega)=\alpha(i|\omega|), \quad (34)$$

where $\alpha(\omega)$ is a Fourier component of the generalized susceptibility, which describes the kinetic properties of the system in the linear approximation. In determining the dissipative contribution to the effective action S_E in this approach we should find the functions $\alpha(\omega)$ and $\sigma(\varphi)$ corresponding to the current I^a. In particular, for a current I^a of the form in (28) we have $\alpha_E(\omega)=|\omega|/eR$, and $\sigma(\varphi)$ is given in (29). In the case of a linear dissipation, Eqs. (32) and (34) reproduce the results derived by Leggett.[26]

3. DECAY RATE OF THE CURRENT STATES

We turn now to the determination of the decay rate Γ of the metastable current states at superconducting links. In the quasiclassical approximation this decay rate is

$$\Gamma=Be^{-A}.$$

We will restrict the present analysis to a calculation of Γ with exponential accuracy; i.e., we will determine A. We know that we have[27] $A=S_E[\tilde\varphi(\tau)]$, where $\tilde\varphi(\tau)$ is the extremal trajectory for the action $S_E[\varphi(\tau)]$, which contains points $\varphi>\varphi_c-\varphi_0$. We first consider the case of pure SCS and SNS junctions. The potential $V(\varphi)$ is then given by (12) and (13), while C^* is given by (14). The potential $V(\varphi)$ in (12) and (13) is definitely not quasiclassical near the point $\varphi=\chi$. The extremal trajectory $\tilde\varphi(\tau)$ in the absence of dissipation can be found easily. In this case, with $\chi\ll 1$, we have

$$A_0=\begin{cases}\dfrac{32-14\sqrt{2}}{15}\left(\dfrac{\pi C^*\Delta}{R}\right)^{1/2}\dfrac{\chi^{5/2}}{e^2}, & \Delta\ll\Delta_d\\ 2\left(\dfrac{C^*\Delta_d}{3R}\right)^{1/2}\dfrac{\chi^2}{e^2}, & \Delta\gg\Delta_d\end{cases}. \tag{35}$$

For small instanton frequencies (smaller than or comparable to γ), the effective action for pure SCS and SNS junctions is of the form (30). Let us assume that $\varkappa$ is so small that the condition $\ln(\chi/\varkappa)\gg 1$ holds. Under the condition $A_0\gg\ln^2(\chi/\varkappa)/e^2R$, the dissipative term can be calculated on the extremal trajectory $\tilde{\varphi}(\tau)$ found in the absence of dissipation. In a first approximation we find

$$A=A_0+\beta\ln^3(\chi/\varkappa)/\pi e^2R. \tag{36}$$

The numerical coefficient here is $\beta=4$ if $|\ln\chi|\ll\ln(\chi/\varkappa)$ and $\beta=1$ if $\ln(\chi/\varkappa)\gg\ln(\chi^2/\varkappa)$. We note that the second term in (36) may prove larger than the first. There thus exists a region of parameter values in which the dissipative term has only a weak effect on the extremal trajectory, but the decay rate Γ of the metastable state (more precisely, the quantity A) is determined by precisely this term. As we have already mentioned, expression (36) holds under the conditions

$$\ln\frac{\chi}{\varkappa}\gg 1,\quad RC^*\gg\begin{cases}\ln^4(\chi/\varkappa)/\Delta\chi^5, & \Delta\ll\Delta_d\\ \ln^4(\chi/\varkappa)/\Delta_d\varkappa^4, & \Delta\gg\Delta_d\end{cases}, \tag{37}$$

$$RC^*\gg\begin{cases}(\varkappa^2\Delta\chi)^{-1}, & \Delta\ll\Delta_d\\ \Delta_d/\varkappa^2\Delta^2, & \Delta\gg\Delta_d\end{cases}. \tag{38}$$

If inequality (38) does not hold, expression (36) remains valid if we replace $\varkappa$ by a quantity of order $A_0e^2/C^*\chi^2$. In this case, of course, inequalities (37) must hold when the same substitution is made in them.

For superconductors with large values of γ, the inequality $\varkappa\gg\chi$ may hold in a very narrow current interval near I_c. In this case the dissipative term in the action (30) becomes quadratic in φ; i.e., (30) becomes (22) with $R_{\rm eff}=R\varkappa^2$. There is of course also a linear, dispersion-free dissipation when a suitably selected shunt is connected to the junction.[7] In the case of wide SNS junctions ($\Delta_d\ll\Delta$) under these conditions we can calculate A for an arbitrary relation between the kinetic and dissipative terms in the action. The extremal trajectory $\tilde{\varphi}$ is found easily:

$$\tilde{\varphi}_\omega=\frac{8\pi\Delta_d\sin\omega\tau_0}{3R\omega\Omega(\omega)},\quad \Omega(\omega)=C^*\omega^2+e\alpha_E(\omega)+\frac{4\Delta_d}{3R}, \tag{39}$$

where $\alpha_E(\omega)=|\omega|/eR_{\rm eff}$, and the parameter τ_0 is found with the help of the self-consistency equation:

$$\tilde{\varphi}(\tau)|_{\tau=\pm\tau_0}=\chi. \tag{40}$$

Substituting (39) into the effective action, we find

$$A=\frac{8\pi\Delta_d\chi\tau_0}{3e^2R}-2\pi\left(\frac{4\Delta_d}{3eR}\right)^2\int_0^\infty d\omega\frac{\sin^2\omega\tau_0}{\omega^2\Omega(\omega)}. \tag{41}$$

The integrals in (39) and (41) can be expressed in terms of the integral sine and integral cosine. We will not reproduce here the corresponding equations, which are extremely lengthy. The condition $\chi\ll 1$ can be used to simplify expression (41):

$$A=\frac{\pi C^*\chi^2}{e^2p}\left|\omega_0^2-\frac{\omega_\eta^2}{4}\right|^{1/2},\quad \omega_\eta\lesssim\omega_0\chi^{-1/2},$$
$$A=\frac{\pi q(1-q/2)\chi^2}{2e^2R_{\rm eff}},\quad \omega_\eta\gtrsim\omega_0\chi^{-1/2}. \tag{42}$$

Here

$$p=\begin{cases}\pi-2\,\mathrm{arctg}\dfrac{\omega_\eta}{(4\omega_0^2-\omega_\eta^2)^{1/2}}, & \omega_\eta<2\omega_0\\ \ln\left[\dfrac{\omega_\eta+(\omega_\eta^2-4\omega_0^2)^{1/2}}{\omega_\eta-(\omega_\eta^2-4\omega_0^2)^{1/2}}\right], & \omega_\eta>2\omega_0\end{cases},$$

$$\omega_0^2=4\Delta_d/3RC^*,\quad \omega_\eta^{-1}=R_{\rm eff}C^*,$$

and q is determined from the equation $q[1-C-\ln(\chi q)]=1$, where $C\approx0.577$ is the Euler constant.

We see that action (30) can be used to describe pure superconducting junctions with sufficiently large values of RC^* or at currents close to the critical value. In a real experimental situation, these conditions may not hold. If not, we use an approximate expression for the dissipative term, (32), (33), to determine A. For SNS junctions under the condition $\Delta_d\ll\Delta$, the extremal trajectory $\tilde{\varphi}(\tau)$ and the value of A are determined from (39)–(41), where $\alpha_E(\omega)$ is of the form in (33) in the expression for $\Omega(\omega)$. For SCS junctions ($\Delta_d\gg\Delta$), it is more difficult to determine $\tilde{\varphi}(\tau)$ for all τ because of the cubic term in the potential (12). However, we do not need to go through this calculation in order to calculate A. The function $\tilde{\varphi}_\omega$ in the case $\Delta_d\gg\Delta$ can be written

$$\tilde{\varphi}_\omega=4eI\sin(\omega\tau_0)\{\omega[C^*\omega^2+\alpha_E(\omega)+u(\tilde{\varphi})]\}^{-1}, \tag{43}$$

where $u(\tilde{\varphi})\sim\Delta\chi/R$, and $\alpha_E(\omega)$ is found from (33).

We consider the most interesting case, that of strong dissipation. To determine the parameter τ_0, we substitute (43) into the self-consistency equation (40). It is easy to see that the function $u(\tilde{\varphi})$ is important in the integral over the frequency only at $\omega\lesssim\Delta\chi\tilde{\chi}^2$. This frequency region is of minor importance in the integral (40). Also ignoring the term containing C^*, we find, to within a factor of order unity,

$$\tau_0\approx(\Delta\chi)^{-1}. \tag{44}$$

Equation (44) justifies our assumption that the most important instanton frequencies in this situation are of order (or less than) $\Delta\chi$.

It is also straightforward to estimate A:

$$A=b/e^2R, \tag{45}$$

where b is a numerical factor on the order of unity. Expression (45) for A does not contain χ (in this estimate, for course). A more accurate determination of b for SCS junctions in the case of a weak dissipation of the type in (32), (33) would have no special meaning since, as we have already mentioned, Eqs. (32) and (33) themselves are approximate in this case. The condition for the applicability of the weak dissipation limit is

$$RC^* \ll (\Delta\chi^3)^{-1}. \qquad (46)$$

Our analysis also applies to SNS junctions in the case $\Delta_d \ll \Delta$. In this case, we can again use estimate (45) for A, while we need to replace Δ in (44) and (46) by Δ_d.

We conclude with a few comments regarding the case of junctions with effective action (22), (24). The probability for quantum-mechanical tunneling in systems with an action of this sort has been the subject of many papers.[3,4,6–9,28] Exact expressions have been derived for A for the case of weak and strong dissipation. The intermediate case has also been discussed in several papers.[4,9,28] The quantity A was found numerically in Ref. 9, while an interpolation expression was proposed in Ref. 28 for describing the case of an intermediate dissipation. Here we derive an (essentially exact) simple expression for A which holds for an arbitrary relation between the effective mass and the effective viscosity.

We replace $\tilde{\varphi}(\tau)$ by $\alpha\tilde{\varphi}(\beta\tau)$. Substituting this function into effective action (22), (24), we find $A(\alpha,\beta)$. The optimum values of the parameters α and β for the given $\tilde{\varphi}(\tau)$ are found from the condition for an extremum of $A(\alpha,\beta)$. We then immediately find

$$A=\frac{9M}{k^2}\left(\frac{2C^*}{e^3}\right)^{1/2}[k(I_c-I)]^{5/4}\left(\frac{z}{2}+y\right)(1-y^2)^2, \qquad (47)$$

where the function $y(z)$ has the simple form

$$y^{-1}(z)=(z^2+5)^{1/2}+z, \qquad z=\frac{N}{e^2R_{\text{eff}}}\left(\frac{2C^*}{e^3}\right)^{1/2}[k(I_c-I)]^{1/4}, \qquad (48)$$

and the quantities

$$M=K_1^{1/4}K_2^{5/4}K_3^{-2}, \qquad N=K_4(K_1K_2)^{-1/2}$$

are invariants of the two-parameter transformation group of the function $\alpha\tilde{\varphi}(\beta\tau)$:

$$K_1=\int_{-\infty}^{\infty}d\tau\left(\frac{\partial\tilde{\varphi}}{\partial\tau}\right)^2, \qquad K_2=\int_{-\infty}^{\infty}d\tau\tilde{\varphi}^2(\tau), \qquad K_3=\int_{-\infty}^{\infty}d\tau\tilde{\varphi}^3(\tau),$$

$$K_4=\frac{1}{2\pi}\iint_{-\infty}^{\infty}d\tau\,d\tau'\left[\frac{\tilde{\varphi}(\tau)-\tilde{\varphi}(\tau')}{\tau-\tau'}\right]^2.$$

Strictly speaking, the parameters M and N are functionals of the function $\tilde{\varphi}(\tau)$, but they change by only a few percent as z changes from 0 to ∞. In the absence of dissipation we would have $M=5^{3/2}/6\approx1.863$, $N=9.5^{1/2}$. $\zeta(3)\pi^{-3}\approx0.780$, while in the limit of strong dissipation we would have $M=2^{5/2}\pi/9\approx1.975$, $N=2^{-1/2}\approx0.707$.

4. CONCLUSION

This microscopic analysis yields an expression for the effective action describing superconducting weak links in terms of Green's function on a Keldysh contour. A calculation of these functions in the adiabatic approximation shows that the effective action of superconducting junctions with a direct conductivity contains a dissipative contribution even in the case $T=0$, because electrons are accelerated near the junction. In several cases (extremely dirty SCS junctions or tunnel junctions with short circuits), at a current close to the critical level, the effective potential is a cubic parabola, as in the case of tunnel junctions, and the dissipation is linear. The theory constructed here can be used to determine the parameters in the effective action of such systems. In particular, the effective resistance may be either less than or greater than the resistance of the junction in its normal state. The second of these situations may have occured in the experiments of Ref. 11, where the tunneling probability was observed to be higher than in the case $R_{\text{eff}}=R_N$, where R_N is the junction resistance in the normal state. For pure SCS junctions in the presence of an oxide film we have

$$R_N^{-1}=R_I^{-1}+R^{-1}, \qquad R_{\text{eff}}=R\cos^2\varphi_c,$$

where φ_c is given by (25). In particular, for $R_I \ll R$ we have $R_{\text{eff}} \gg R_N$.

In pure SCS and SNS junctions at low temperatures, the effective potential is not quasiclassical; it is quite different from a cubic parabola [see (12) and (13)]; and the dissipation is not linear. We have calculated (with exponential accuracy) the decay rate of metastable current states at such junctions. When there is a definite relation among the parameters of the system, we can switch between the weak and strong dissipation regimes by varying the current (I) through the junction (or by varying the critical current).

We wish to thank A. I. Larkin, K. K. Likharev, and Yu. N. Ovchinnikov for several useful discussions.

APPENDIX 1

The matrices $\hat{X}$ and $\hat{Y}$ in (11) are written in the form

$$\hat{X}(t,t')={}^1/_2[\hat{P}^R(-1)^l+\hat{P}^A(-1)^m+\hat{P}],$$

$$\hat{Y}(t',t)={}^1/_2[(\hat{Q}_+^R)^{-1}(-1)^m+(\hat{Q}_+^A)^{-1}(-1)^l-(\hat{Q}_+^R)^{-1}\hat{Q}_+(\hat{Q}_+^A)^{-1}],$$

$$\hat{P}^{R,A}=\hat{1}+\hat{Q}_-^{R,A}\,\text{sign}\,\alpha, \qquad \hat{P}=\hat{Q}_-\,\text{sign}\,\alpha, \qquad (A1)$$

$$\check{Q}_\pm=\begin{pmatrix}\hat{Q}_\pm^R & \hat{Q}_\pm\\ \hat{0} & \hat{Q}_\pm^A\end{pmatrix}=\frac{1}{2}(\check{C}^+\check{A}_1\check{C}\pm\check{C}\check{A}_2\check{C}^+),$$

$$l=0(1),\ \text{Im}\,t'>0(<0); \qquad m=0(1),\ \text{Im}\,t>0(<0),$$

where

$$\check{A}_j=\begin{pmatrix}\hat{A}_j^R & \hat{A}_j\\ \hat{0} & \hat{A}_j^A\end{pmatrix}, \qquad \hat{A}_j=\hat{A}_j^R\hat{n}_j-\hat{n}_j\hat{A}_j^A,$$

$$\hat{A}_j^{R(A)}=\hat{S}_j(t)\int\hat{g}^{R(A)}(\varepsilon)\exp[-i\varepsilon(t-t')]\frac{d\varepsilon}{2\pi}\hat{S}_j^+(t'),$$

$$\hat{n}_j=\hat{S}_j(t)\int n(\varepsilon)\exp[-i\varepsilon(t-t')]\hat{S}_j^+, \qquad n(\varepsilon)=\text{th}\frac{\varepsilon}{2T}, \qquad (A2)$$

It is easy to derive the following relations:

$$\hat{g}^{R(A)}(\varepsilon)=\frac{(\varepsilon\pm i\gamma)\hat{\tau}_3+\Delta i\hat{\tau}_2}{[(\varepsilon\pm i\gamma)^2-\Delta^2]^{1/2}}, \qquad \hat{S}_j(t)=\hat{1}\cos\left[\frac{\varphi_0+\varphi(t)}{2}\right]+i(-1)^j\hat{\tau}_3\sin\left[\frac{\varphi_0+\varphi(t)}{2}\right], \qquad \hat{C}(\varepsilon,\varepsilon';\alpha)$$

$$=\left(\hat{1}\cos\frac{\varepsilon}{2\Delta_d\alpha}+i\hat{\tau}_3\sin\frac{\varepsilon}{2\Delta_d\alpha}\right)\delta(\varepsilon-\varepsilon'), \quad j=1,2.$$

$$\hat{Q}_{\pm}={}^{1}/_{2}(\hat{Q}_{+}{}^{R}\hat{n}_{\pm}-\hat{n}_{\pm}\hat{Q}_{+}{}^{A}+\hat{Q}_{-}{}^{R}\hat{n}_{\mp}-\hat{n}_{\mp}\hat{Q}_{-}{}^{A}), \quad \text{(A3)}$$

where

$$\hat{n}_{\pm}={}^{1}/_{2}(\hat{C}\hat{n}_{1}\hat{C}^{+}\pm\hat{C}^{+}\hat{n}_{2}\hat{C}).$$

APPENDIX 2

To determine the potential $V(\varphi)$ in the effective action for superconducting junctions in the adiabatic approximation it is sufficient to know how the steady-state Josephson current depends on the phase difference. For extremely dirty SCS junctions, this dependence can be written as a sum over Matsubara frequencies [see, e.g., Eq. (27) in Ref. 18]. We note that this expression can be written in the form

$$I(\varphi)=\frac{\pi\Delta}{eR_{d}}\int_{0}^{\varphi}d\theta\frac{\cos\varphi}{\cos\theta}\,\mathrm{th}\,\frac{\Delta\cos\varphi}{2T\cos\theta}. \quad \text{(A4)}$$

In the case $T\ll\Delta$, in which we are interested here, integral (A4) can be evaluated easily; we find

$$I(\varphi)=\frac{\pi\Delta}{2eR_{d}}\ln\left[\frac{1+\sin\varphi}{1-\sin\varphi}\right]\cos\varphi. \quad \text{(A5)}$$

The maximum value of $I_c = I(\varphi_c)$ is reached at the point $\varphi_c \approx 0.983$. Using (A4), we can easily find expression (27) for k. The renormalization of the capacitance $C^* - C$ is determined by the coefficient of the ω^2 term in the expansion of the dissipation-free part of the linear response, $I^r(\omega)$. Artemenko *et al.*[23] have calculated I^r for the case of dirty SCS junctions. The exact expression for $C^* - C$ in this case is extremely lengthy, and we will not reproduce it here.

[1)]The reason for this result is that all the terms other than the gradient terms can be ignored in the Eilenberger equations near the junction (provided only that the dimensions of the junction are sufficiently small). See, e.g., Refs. 15 and 18.

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Translated by Dave Parsons