Small-slope approximation in wave scattering by rough surfaces

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Where the wave scattering amplitude of a rough surface can be calculated by perturbation theory (Bragg scattering), and the slopes of the boundaries can be assumed small, it becomes possible to construct an approximation which holds for an arbitrary roughness height. To first order in the slope, the scattering amplitude can be factored into a geometric factor and a kernel which describes Bragg scattering. To a large extent, this approximation solves the problem of describing the scattering by rough surfaces by a common approach combining perturbation theory with the tangent-plane approximation.

1. INTRODUCTION

A scattering of waves by an irregular boundary arises in many branches of physics. For example, the diffraction of thermal molecular beams by the surfaces of single crystals (a process used to determine the characteristics of the corresponding interaction potentials) may be thought of as a scattering by an irregular potential barrier of infinite height.[1–4] The interaction of surface plasmons in a metal with boundary irregularities gives rise to optical emission, which is exploited to measure the properties of the plasmons.[5] A similar principle underlies the excitation of the plasmons which substantially enhance Raman scattering in a thin organic film.[6] The problem of wave scattering by irregular surfaces also arises in optics,[7,8] radio astronomy,[9] and radiophysics.[10–12] The practical importance of this problem has made it the subject of a special study in underwater acoustics.[13] Finally, it is one of the classical problems of diffraction theory. Although the theory of wave scattering by irregular surfaces has been the subject of many studies, nearly all of the analytic work has been confined to two approaches,[14,15] which employ essentially perturbation theory [16,17] and a semiclassical approximation[4,13,18,19] (either the Kirchhoff approximation or the tangent-plane method).

Not uncommonly, however, situations arise which cannot be dealt with by either of these approaches alone. An example is the scattering of sound or electromagnetic waves by the wavy surface of the sea. In these cases, the so-called two-scale surface model is used[10,13,20]; the scattering calculations in this model are carried out by combining the two classical approaches given above. The boundary irregularities are put in two classes: the scattering by large-scale smooth components is described in the Kirchhoff approximation, while the small-scale irregularies are dealt with by perturbation theory. This arbitrary classification of the irregularities in two groups unavoidably introduces in the theory at least one parameter whose choice is somewhat arbitrary. This is an inconvenience; it complicates the calculation of the corrections to the two-scale model; and it acts as a noise in an analysis of the inverse problem, i.e., in determining the spectrum of irregularities from the characteristics of the scattered field.

There is another way to approach the problem of wave scattering by irregular boundaries. This other approach is free of the disadvantage discussed above. It requires only that a parameter associated exclusively with the geometry of the boundary—its slope—be sufficiently small. This parameter is indeed small in many cases of practical interest (for the wave motion of the seas, for example, it is $\lesssim 0.1$). This alternative procedure allows one to write an asymptotic expansion of the wave scattering amplitude in this parameter. This expansion has a rather simple structure and is easily calculated. Since the assumptions underlying this approach to the problem are quite general, it is possible to outline a universal procedure for deriving an expression for scattering in the small-slope approximation, as in the approach which uses an expression describing the Bragg scattering of waves by boundary irregularities. In the statistical case, this approach reduces to a simple redefinition of the correlation function of the elevations. The corresponding expression for Bragg scattering can then be used (the redefined correlation function admittedly depends on the parameters of the incident and scattered waves).

After introducing this approach we examine in more detail the very simple problem of the scattering of scalar waves, for which the boundary condition is a Dirichlet condition, and we offer a direct derivation of the expansion in small slopes which does not lean on any general assumptions. In particular, we find in the latter case that the high-frequency limit for the scattering amplitude found by using only the first two terms of this expansion agrees exactly with the corresponding expression derived in the Kirchhoff approximation.

2. SMALL-SLOPE APPROXIMATION: THE GENERAL APPROACH

We introduce a Cartesian coordinate system with z axis directed vertically downward, and we denote by $\mathbf{r} = (x, y)$ the horizontal components of the radius vector $\mathbf{R} = (\mathbf{r}, z)$. We assume that the uniform half-space has an irregular boundary $z = \eta(\mathbf{r})$ at its bottom, and we assume that a monochromatic plane wave of frequency ω is incident in the upward direction on this boundary:

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$$\Psi_{in}=v_0^{-1/2}\exp(i\varkappa_0\mathbf{r}+iv_0z),\quad \varkappa_0^2+v_0^2=k^2=\omega^2/c^2,$$

where c = const is the propagation velocity (the normalization factor is chosen to normalize the energy flux density in the vertical direction). Here and below, $\varkappa$ and v are the horizontal and vertical components of the wave vector, $v=v(\varkappa)=(k^2-\varkappa)^{1/2}$, $\mathrm{Im}\,v\geqslant 0$, and the index 0 specifies the incident wave. Mathematically, the problem reduces to one of solving the Helmholtz equation $(\Delta+k^2)\Psi=0$ with the corresponding boundary condition and the radiation condition. We assume everywhere below that the boundary is noncompact and is, on the average, planar. In the region $z<\min\eta(\mathbf{r})$, the field can evidently be written as a superposition of plane waves, so that the total field Ψ is $\Psi=\Psi_{in}+\Psi_{sc}$ where

$$\Psi_{sc}=\int S(\varkappa,\varkappa_0)v^{-1/2}\exp(i\varkappa\mathbf{r}-ivz)\,d\varkappa. \tag{1}$$

The scattering amplitude S is completely analogous to the quantum-mechanical amplitude and has corresponding properties. In particular, a reciprocity theorem must hold $[S(\varkappa,\varkappa_0)=S(\varkappa_0,-\varkappa)]$, as must a corresponding unitarity condition.[21] In principle, knowledge of the scattering amplitude is equivalent to knowledge of the Green's function, into which it can easily be converted. Although finding the Green's function is frequently the ultimate goal of the solution of a given diffraction problem, the scattering amplitude is considerably more convenient to work with for a theoretical analysis.

Our approach is based on the simple properties of the scattering amplitude. Specifically, if the irregular boundary $z=\eta(\mathbf{r})$ is displaced as a whole horizontally by a distance a, $\eta(\mathbf{r})\to\eta(\mathbf{r}-\mathbf{a})$, the scattering amplitude evidently transforms in accordance with

$$S(\varkappa,\varkappa_0)\to S(\varkappa,\varkappa_0)\exp[-i(\varkappa-\varkappa_0)\mathbf{a}].$$

Analogously, lifting the boundary a distance h, $\eta(\mathbf{r})\to\eta(\mathbf{r})+h$, leads to the transformation

$$S(\varkappa,\varkappa_0)\to S(\varkappa,\varkappa_0)\exp[i(v+v_0)h].$$

Working from these elementary comments, we see, the scattering amplitude in the form of the expansion

$$S(\varkappa,\varkappa_0)=\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}+i(v+v_0)\eta(\mathbf{r})]\times\frac{d\mathbf{r}}{(2\pi)^2}e^{i\xi\mathbf{r}}\Phi(\varkappa,\varkappa_0;\xi;[\eta])\,d\xi, \tag{2}$$

where

$$\Phi=\delta(\xi)\Phi_0+\delta(\xi-\xi_1)\Phi_1\eta(\xi_1)+\delta(\xi-\xi_1-\xi_2)\Phi_2\eta(\xi_1)\eta(\xi_2)+\ldots.$$

Here the $\eta(\xi)$ are the Fourier components of the irregularities:

$$\eta(\mathbf{r})=\int\eta(\xi)\exp(i\xi\mathbf{r})\,d\xi.$$

An integration is to be carried out over the variables $\xi_1,\ldots,\xi_n$. The Φ_n are functions, symmetric with respect to ξ_i, which do not depend on the shape of the surface. Since Φ is essentially an arbitrary functional of the elevations η which is written as an integral-power series, it is legitimate to seek S in the form (2). We seen that a factor has been singled out in (2) which by itself provides the required transformation properties for S. The reason for the δ-functions in (2) is the law according to which S transforms under horizontal translations. Although, strictly speaking, this is not a consequence exclusively of the transformation property of S when the surface as a whole is raised, it is natural to assume (and this assumption is supported by the more specific calculations below) that the functional Φ must not transform at all under the substitution $\eta(\xi)\to\eta(\xi)+h\delta(\xi)$. It follows immediately that the functions Φ_n, $n\geqslant 1$, must vanish for $\xi_i=0$, $i=1,2,\ldots,n$. We can thus set

$$\Phi_n=\xi_1\ldots\xi_n\tilde{\Phi}_n,$$
$$\Phi=\delta(\xi)\tilde{\Phi}_0+\delta(\xi-\xi_1)\tilde{\Phi}_1\xi_1\eta(\xi_1)+\delta(\xi-\xi_1-\xi_2)\tilde{\Phi}_2\xi_1\eta(\xi_1)\xi_2\eta(\xi_2)+\ldots, \tag{3}$$

where $\tilde{\Phi}_n=\tilde{\Phi}_n(\varkappa,\varkappa_0;\xi_1,\ldots,\xi_n)$. Since

$$\int i\xi\eta(\xi)\exp(i\xi\mathbf{r})\,d\xi=\nabla\eta,$$

series (3) turns out to be essentially an integral-power series in $\nabla\eta$; the kernels $\tilde{\Phi}_n$ are dimensionless. The question of the validity of our expansion (2), (3) reduces to the requirement that the functions $\tilde{\Phi}_n$ be bounded for all $\xi_1,\ldots,\xi_n$. In particular, the absence of singularities from $\tilde{\Phi}_n$ at $\xi_i=0$ is equivalent to our assumption that Φ is invariant under the transformation $\eta(\xi)\to\eta(\xi)+h\delta(\xi)$. The boundedness of Φ_n for all ξ_i ($|\tilde{\Phi}_n|<C_n$) means that we can estimate the successive terms of the series as quantities of order $C_n\varepsilon^n$, where $\varepsilon\sim|\nabla\eta|\ll 1$. This circumstance allows us to speak of expansion (2), (3) as an expansion in small slopes. It is easy to see that we are essentially nowhere making use of the scalar nature of the problem, so that the scattering amplitude S can in general be regarded as a matrix-valued amplitude describing the conversions of various types of waves into each other. Accordingly, the quantities $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}_n$ [and also $B(\varkappa,\varkappa_0)$; see below] are generally matrices.

It turns out that the kernels $\tilde{\Phi}_n$ in expansion (3) can be found easily by taking the limit of ordinary perturbation theory, with $\eta\to 0$, if we assume that we know the expansion of the scattering amplitude in an ordinary integral-power series in the elevations η (the latter expansion is found from ordinary perturbation theory; usually, no difficulties will arise). However, we are forced to appeal to an additional assumption. Specifically, expression (2) for Φ has some gauge arbitrariness, since we could add to $\tilde{\Phi}_n$ any arbitrary function which vanishes for $\varkappa-\varkappa_0=\xi_1+\ldots+\xi_n$. The substitution

$$\tilde{\Phi}_n\to\tilde{\Phi}_n+(\varkappa-\varkappa_0-\xi_1-\ldots-\xi_n)g,\quad \tilde{\Phi}_{n+1}\to\tilde{\Phi}_{n+1}-i(v+v_0)\tilde{\Phi}_n g,$$

where g is arbitrary, does not change the magnitude of the scattering amplitude in (2). In other words, if we single out from $\tilde{\Phi}_n$ a part which vanishes on this hypersurface, then an integration by parts will "correct" this part to a term of order $n+1$. Using this circumstance, we can eliminate from $\tilde{\Phi}_n$ the dependence on one of the integration variables (e.g., ξ_n), by setting $\xi_n=\varkappa-\varkappa_0-\xi_1-\ldots-\xi_{n-1}$ in $\tilde{\Phi}_n$. Substituting this expression for $\tilde{\Phi}_n$ into (2), we can perform in the resulting expression an operation which is the inverse of that which we have just described; specifically, we can integrate over the variable ξ_n. As a result, we find a factor $-i\nabla\eta$, which we can then eliminate by an integration by

parts. As a result, we find the following equation, which holds within quantities of order $(\nabla\eta)^n$:

$$\int \exp[-i(\varkappa-\varkappa_0)\mathbf{r}+i(\nu+\nu_0)\eta(\mathbf{r})]$$
$$\times\frac{d\mathbf{r}}{(2\pi)^2}e^{i\boldsymbol{\xi}\mathbf{r}}\delta(\xi-\xi_1-\ldots-\xi_n)$$
$$\times\widetilde{\Phi}_n(\varkappa,\varkappa_0;\xi_1,\ldots,\xi_n)\prod_{i=1}^{n}\xi_i\eta(\xi_i)\,d\xi_i\,d\xi$$
$$\approx-\frac{i}{\nu+\nu_0}\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}$$
$$+i(\nu+\nu_0)\eta(\mathbf{r})]\frac{d\mathbf{r}}{(2\pi)^2}e^{i\boldsymbol{\xi}\mathbf{r}}\delta(\xi-\xi_1-\ldots-\xi_{n-1})$$
$$\times\Phi_n(\varkappa,\varkappa_0;\xi_1,\ldots,\varkappa-\varkappa_0-\xi_1-\ldots-\xi_{n-1})$$
$$\times\prod_{i=1}^{n-1}\eta(\xi_i)\,d\xi_i\,d\xi. \qquad (4)$$

It is not difficult to see that this expression has the structure of the term of $(n-1)$-st order in expansion (2), (3).

We thus see that these transformations make it possible to eliminate from expansion (3) any preselected term of index greater than one, by writing $\widetilde{\Phi}_n$ as

$$\widetilde{\Phi}_n=\widetilde{\Phi}_n|_{\xi_n=\varkappa-\varkappa_0-\xi_1-\ldots-\xi_{n-1}}+(\widetilde{\Phi}_n-\widetilde{\Phi}_n|_{\xi_n=\varkappa-\varkappa_0-\xi_1-\ldots-\xi_{n-1}}),$$

by transforming the first term into a term of order $n-1$ in accordance with (4), and by transforming the second term into a term of order $n+1$ through an integration by parts. There is the further meaning that the functions Φ_n can be chosen such that n terms of expansion (2), (3) make it possible to calculate S with an accuracy up to $(n+1)$st order inclusively in the parameter $\nabla\eta$. We now seek the structure functions. To within quantities of first order in η, the scattering amplitude can be written in general as follows:

$$S_1(\varkappa,\varkappa_0)=S_0[\delta(\varkappa-\varkappa_0)+2i(\nu\nu_0)^{1/2}B(\varkappa,\varkappa_0)\eta(\varkappa-\varkappa_0)] \qquad (5)$$

(Bragg scattering). The dimensionless function $B(\varkappa,\varkappa_0)$ is determined by the particular scattering problem; the term $S_0\delta(\varkappa-\varkappa_0)$ describes reflection from a horizontal plane, and the factor of $(\nu\nu_0)^{1/2}$ is separated for convenience. In particular, for problems involving the scattering of scalar waves by a free boundary (the boundary condition is that the total field vanishes), we have

$$S_0=-1,\quad B(\varkappa,\varkappa_0)\equiv 1,$$

and in the case of the scattering of electromagnetic waves by an ideally conducting surface we have, in a basis of circularly polarized waves,

$$S_0=\sigma_1,\quad B(\varkappa,\varkappa_0)=\sum_{n=0}^{3}B_n\sigma_n,$$

where

$$B_{0,1}={}^1/_2[(k^2\pm\nu\nu_0)\varkappa\varkappa_0-\varkappa^2\varkappa_0^2]/|\varkappa||\varkappa_0|\nu\nu_0,$$
$$B_2={}^1/_2k(\nu-\nu_0)[\varkappa\varkappa_0]^{(z)}/|\varkappa||\varkappa_0|\varkappa\varkappa_0,$$
$$B_3={}^1/_2ik(\nu+\nu_0)[\varkappa\varkappa_0]^{(z)}/|\varkappa||\varkappa_0|\varkappa\varkappa_0,$$

σ_n arethe Pauli matrices, and the upper sign in $B_{0,1}$ refers to B_0. For the case $\eta(\mathbf{r})=h=\text{const}$, $\eta(\xi)=h\delta(\xi)$ in the limit $h\to 0$ we have

$$S_0e^{2i\nu h}\delta(\varkappa-\varkappa_0)\approx S_0(1+2i\nu h)\delta(\varkappa-\varkappa_0)$$
$$=S_0[\delta(\varkappa-\varkappa_0)+2i\nu B(\varkappa,\varkappa_0)h\delta(\varkappa-\varkappa_0)].$$

It follows that the condition

$$B(\varkappa,\varkappa)=1 \qquad (6)$$

always holds.

The further calculations can be outlined as follows. We identify in (2) terms of up to first order in η, assuming that there is no term with $n=1$ in (3). We have

$$S_1(\varkappa,\varkappa_0)=\Phi_0(\varkappa,\varkappa_0)\delta(\varkappa-\varkappa_0)+i(\nu+\nu_0)\Phi_0(\varkappa,\varkappa_0)\eta(\varkappa-\varkappa_0).$$

Comparing this expansion with (5), we find Φ_0:

$$\Phi_0(\varkappa,\varkappa_0)=2S_0(\nu\nu_0)^{1/2}B(\varkappa,\varkappa_0)/(\nu+\nu_0); \qquad (7)$$

here we have $\Phi_0(\varkappa,\varkappa)=S_0$ by virtue of (6). Calculating the Bragg scattering described by the function $B(\varkappa,\varkappa_0)$ in the lowest order, we can thus transform to the approximation of simple slopes by simply replacing the form factor $\eta(\varkappa-\varkappa_0)$ by an integral of the type in (2):

$$S^{(1)}(\varkappa,\varkappa_0)=S_0\frac{2(\nu\nu_0)^{1/2}B(\varkappa,\varkappa_0)}{\nu+\nu_0}$$
$$\times\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}+i(\nu+\nu_0)\eta(\mathbf{r})]\frac{d\mathbf{r}}{(2\pi)^2}. \qquad (8)$$

In contrast with expression (5), which describes the scattering correctly only if the Rayleigh parameter is small ($k\eta\cos\theta\ll 1$, where θ is the angle of incidence), expression (8) holds under the assumption $|\nabla\eta|\ll 1$ for an arbitrary value of the Rayleigh parameter (the criterion for the applicability of the approach formulated here requires some refinement; see §3),

We now find Φ_2. For this purpose we first write an expression for the scattering amplitude which holds to within quantities of second order in η:

$$S_2(\varkappa,\varkappa_0)=S_0\Big[\delta(\varkappa-\varkappa_0)+2i(\nu\nu_0)^{1/2}B(\varkappa,\varkappa_0)\eta(\varkappa-\varkappa_0)$$
$$+(\nu\nu_0)^{1/2}\int B_2(\varkappa,\varkappa_0;\varkappa')\eta(\varkappa-\varkappa')\eta(\varkappa'-\varkappa_0)\,d\varkappa'\Big], \qquad (9)$$

where the kernel B_2, like $B=B_1$, can be found easily by standard perturbation theory. Singling out terms through second order in η in (2) and (3), we find

$$\int\Big[-\frac{(\nu+\nu_0)^2}{2}\Phi_0+\Phi_2\Big]\eta(\xi_1)\eta(\xi_2)\delta(\varkappa-\varkappa_0-\xi_1-\xi_2)\,d\xi_{1,2}$$
$$=\frac{S_0(\nu\nu_0)^{1/2}}{2}\int(B_2(\varkappa,\varkappa_0;\varkappa-\xi_1)+B_2(\varkappa,\varkappa_0;\varkappa-\xi_2))\eta(\xi_1)\eta(\xi_2)$$
$$\times\delta(\varkappa-\varkappa_0-\xi_1-\xi_2)\,d\xi_{1,2}.$$

Since η is arbitrary, this relation determines the symmetric function Φ_2 to within an arbitrary term which vanishes in the case $\xi_1+\xi_2=\varkappa-\varkappa_0$. As we have already seen, however, this term can be transformed into a term of third order in $\nabla\eta$ and incorporated in Φ_3. Using (7), we can therefore set

$$\Phi_2(\varkappa,\varkappa_0;\xi_1,\xi_2)=S_0(\nu\nu_0)^{1/2}[B_2(\varkappa,\varkappa_0;\varkappa-\xi_1)$$
$$+B_2(\varkappa,\varkappa_0;\varkappa-\xi_2)+2(\nu+\nu_0)B(\varkappa,\varkappa_0)]/2.$$

Once the kernel Φ_2 has been determined in this way, we can

simplify the resulting expansion, by eliminating from (3) the $n = 2$, term which we have just calculated. According to (4), we find

$$\Phi_1(\varkappa, \varkappa_0; \xi) = -\frac{i}{\nu+\nu_0}\Phi_2(\varkappa, \varkappa_0; \xi, \varkappa-\varkappa_0-\xi)$$

$$=-\frac{iS_0(\nu\nu_0)^{1/2}}{2(\nu+\nu_0)}[B_2(\varkappa, \varkappa_0; \varkappa-\xi)+B_2(\varkappa, \varkappa_0; \varkappa_0+\xi)$$

$$+2(\nu+\nu_0)B(\varkappa, \varkappa_0)].$$

We have thus derived a small-slope expansion with an accuracy of order $(\nabla\eta)^2$. This procedure can be continued to calculate structure functions of any order. Specifically collecting from expansion (3), from which the $n = 2$ term is now absent, quantities of order η^3, and comparing the result with the known power-law expansion, we find Φ_3. We transform it, in accordance with (4), into a term with $n = 2$: etc.

In the two examples mentioned above the kernel $B_2(\varkappa, \varkappa_0; \varkappa')$ can be put in the form[21]

$$B_2(\varkappa, \varkappa_0; \varkappa') = -2\nu(\varkappa')B(\varkappa, \varkappa')B(\varkappa', \varkappa_0). \quad (10)$$

Expression (10) apparently holds for a wide range of scattering problems. In these cases, the small-slope expansion incorporating terms of order $(\nabla\eta)^2$ inclusively is

$$S^{(2)}(\varkappa, \varkappa_0) = S_0\frac{(\nu\nu_0)^{1/2}}{\nu+\nu_0}\int\frac{d\mathbf{r}\,d\xi}{(2\pi)^2}$$

$$\times\exp[-i(\varkappa-\varkappa_0-\xi)\mathbf{r}+i(\nu+\nu_0)\eta(\mathbf{r})]$$

$$\times\{2B(\varkappa, \varkappa_0)\delta(\xi)+i[\nu(\varkappa-\xi)B(\varkappa, \varkappa-\xi)B(\varkappa-\xi, \varkappa_0)$$

$$+\nu(\varkappa_0+\xi)B(\varkappa, \varkappa_0+\xi)B(\varkappa_0+\xi, \varkappa_0)+(\nu+\nu_0)B(\varkappa, \varkappa_0)]\eta(\xi)\}. \quad (11)$$

The expression in brackets vanishes if $\xi = 0$ and $\xi = \varkappa - \varkappa_0$. we can therefore divide it by $\xi(\varkappa - \varkappa_0 - \xi)$; we then immediately see that the second term in (11) transforms into a term of second order in (2) and (3). We do not find a singularity in $\tilde{\Phi}_2$ at $\xi_{1,2} = 0$.

We now consider a spatially uniform Gaussian statistical ensemble of elevations, $\eta(\mathbf{r})$. The mean scattering amplitude can be written in general as

$$\bar{S}(\varkappa, \varkappa_0) = \bar{V}(\varkappa)\delta(\varkappa-\varkappa_0),$$

where $\bar{V}$ is the so-called mean reflection coefficient. The average in (11) can be carried out without difficulty; as a result we find

$$\bar{V}(\varkappa) = S_0e^{-2\sigma^2\nu^2}\Big\{1-2\nu\int[\nu(\varkappa+\xi)B(\varkappa, \varkappa+\xi)$$

$$\times B(\varkappa+\xi, \varkappa)-\nu]G(\xi)\,d\xi\Big\}, \quad (12)$$

$$G(\xi) = \int W(\mathbf{r})e^{i\xi\mathbf{r}}\frac{d\mathbf{r}}{(2\pi)^2},$$

$$W(\mathbf{r}) = \overline{\eta(\mathbf{r}+\mathbf{a})\eta(\mathbf{a})}, \quad \sigma^2 = W(0) = \int G(\xi)\,d\xi,$$

where $G(\xi)$ is the spectrum of irregularities, and $W(\mathbf{r})$ is the corresponding correlation function. The first term in this expression is the same as the expression for $\bar{V}$ derived in the Kirchhoff approximation,[10–13] while the correction term reflects scattering by small-scale components of the irregularities.

Writing the scattering amplitude as a sum of the mean and fluctuating parts,

$$S(\varkappa, \varkappa_0) = \bar{V}(\varkappa)\delta(\varkappa-\varkappa_0)+\Delta S(\varkappa, \varkappa_0),$$

we can write

$$\overline{\Delta S(\varkappa_1, \varkappa_2)\otimes\Delta S^+(\varkappa_3, \varkappa_4)}$$

$$=Q(\varkappa_1, \varkappa_2; \varkappa_3, \varkappa_4)\delta(\varkappa_1-\varkappa_2-\varkappa_3+\varkappa_4).$$

Any second moment of the random fields can be axpressed in terms of Q by means of quadratures. In practice, we are interested in the scattering coefficients, which are related to Q in the following way:

$$m_s = \nu\nu_0Q(\varkappa, \varkappa_0; \varkappa, \varkappa_0).$$

The expression for m_s derived in the approximation $S \approx S^{(1)}$ is

$$m_s = S_0B(\varkappa, \varkappa_0)\otimes(S_0B(\varkappa, \varkappa_0))^+\frac{4\nu^2\nu_0^2}{(\nu+\nu_0)^2}\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}]$$

$$\times[\exp\{-(\nu+\nu_0)^2[W(0)-W(\mathbf{r})]\}$$

$$-\exp\{-(\nu+\nu_0)^2W(0)\}]\frac{d\mathbf{r}}{(2\pi)^2}. \quad (13)$$

The integral containing the statistical characteristics of the elevations is of precisely the same form as that used in the Kirchhoff approximation, and it is essentially of purely geometric origin. The scattering properties of the surface, on the other hand, enter m_s in the approximation $S \approx S^{(1)}$ in the form of a factor which has the same structure as when we used expression (5) for Bragg scattering:

$$m_s^{(B)} = S_0B(\varkappa, \varkappa_0)\otimes(S_0B(\varkappa, \varkappa_0))^+\cdot 4\nu^2\nu_0^2$$

$$\times\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}]W(\mathbf{r})\frac{d\mathbf{r}}{(2\pi)^2}. \quad (14)$$

Comparing the latter expression with (13), we can say that the transformation from the small-perturbation method to the small-slope approximation reduces to a redefinition of the correlation function,

$$W(\mathbf{r}) \to W^{(B)}(\mathbf{r}) = \frac{1}{(\nu+\nu_0)^2}$$

$$\times[\exp\{-(\nu+\nu_0)^2[W(0)-W(\mathbf{r})]\}-\exp\{-(\nu+\nu_0)^2W(0)\}],$$

after which we should use expression (14), which follows from ordinary perturbation theory. An analogous rule can be formulated easily for Q [and even for η in (8)]. This procedure would appear to be appropriate only when we are using the first approximation in the expansion in small slopes, in which case the geometric effects which stem from large-scale irregularities and the "actual" Bragg scattering, by irregularities with horizontal scale dimensions on the order of the wavelength, are factored out.

3. CASE OF SCALAR WAVES AND A FREE BOUNDARY

Since the arguments used in §2 in the derivation of the small-slope expansion and in the calculation of the structure functions Φ_n were quite general in nature, we would like to

derive this expansion by some other methods, which does not make explicit use of those general considerations. In the present section of the paper we show how this can be done in the very simple scalar problem with a Dirichlet boundary condition (this situation corresponds to the scattering of sound waves by a free surface or to the scattering of *TE* electromagnetic waves by an ideal surface in the two-dimensional case). We employ the Rayleigh equation, which can be found by calculating the scattered field at $z=\eta(\mathbf{r})$ directly from Eq. (1) and substituting the result into the boundary condition $\Psi|_{z=\eta(\mathbf{r})}=0$

$$\int S(\varkappa',\varkappa_0)\frac{1}{\nu'^{1/2}}\exp[i\varkappa'\mathbf{r}-i\nu'\eta(\mathbf{r})]d\varkappa'$$
$$=-\frac{1}{\nu_0^{1/2}}\exp[i\varkappa_0\mathbf{r}+i\nu_0\eta(\mathbf{r})].$$

This equation is valid if the Rayleigh hypothesis applies to the irregularities.[22] However, this is not an important point since we are involved in deriving representations (2), (3) and in calculating the functions Φ_n without discussing the convergence properties of the expansion.[23] Multiplying the Rayleigh equation by $\exp[-i\varkappa\mathbf{r}+i\nu\eta(\mathbf{r})]/\nu^{1/2}$, and integrating over $\mathbf{r}$, we can easily put in the form

$$S(\varkappa,\varkappa_0)=-\left(\frac{\nu}{\nu_0}\right)^{1/2}\int\exp[-i(\varkappa-\varkappa_0)\mathbf{r}+i(\nu+\nu_0)\eta(\mathbf{r})]\frac{d\mathbf{r}}{(2\pi)^2}$$
$$+\int S(\varkappa',\varkappa_0)\left(\frac{\nu}{\nu'}\right)^{1/2}$$
$$\times\frac{\varkappa+\varkappa'}{\nu+\nu'}\int\exp[-i(\varkappa-\varkappa')\mathbf{r}'+i(\nu-\nu')\eta(\mathbf{r}')]$$
$$\times\nabla\eta(\mathbf{r}')\frac{d\mathbf{r}'}{(2\pi)^2}d\varkappa'. \qquad (15)$$

Here we have used the following identity, which is easily found through integration by parts:

$$\int\exp[-i(\varkappa-\varkappa')\mathbf{r}]\{\exp[i(\nu-\nu')\eta(\mathbf{r})]-1\}\frac{d\mathbf{r}}{(2\pi)^2}$$
$$=-\frac{\varkappa+\varkappa'}{\nu+\nu'}\int\exp[-i(\varkappa-\varkappa')\mathbf{r}+i(\nu-\nu')\eta(\mathbf{r})]\nabla\eta\frac{d\mathbf{r}}{(2\pi)^2}.$$

We see that the integral term in (15) contains the small parameter which we need—the slope—and that this equation can be solved by iterations. The result can easily be put in the form in (2); we find the following representation for the functional Φ:

$$\Phi=\sum_{n=0}^{\infty}\Phi^{(n)},\quad \Phi^{(n)}(\varkappa,\varkappa_0;\xi,[\eta])$$
$$=-\left(\frac{\nu}{\nu_0}\right)^{1/2}\int e^{-i\xi\mathbf{r}}\frac{d\mathbf{r}}{(2\pi)^2}\prod_{m=1}^{n}\frac{(\varkappa_{m+1}+\varkappa_m)\nabla\eta_m}{\nu_{m+1}+\nu_m}$$
$$\times\exp[i(\varkappa_{m+1}-\varkappa_m)(\mathbf{r}-\mathbf{r}_m)-i(\nu_{m+1}-\nu_m)(\eta_{\mathbf{r}}-\eta_m)]$$
$$\times\frac{d\mathbf{r}_m\,d\varkappa_m}{(2\pi)^2}, \qquad (16)$$

where $\eta_m=\eta(\mathbf{r}_m)$, $\varkappa_{n+1},\equiv\varkappa$, $\nu_{n+1}\equiv\nu$ and $\Pi\equiv1$ for $n=0$. Obviously, Φ does not change as a result of the vertical displacement $\eta\to\eta+h$; it thus follows directly, as we have already mentioned, that an expansion of Φ in an integral-power series is actually an expansion in gradients of the elevations. The situation is equivalent to the absence of singularities form the structure functions $\tilde{\Phi}_n$ for $\xi_i=0$. We see from (16) that in the expansion of Φ in the integral-power series the corresponding kernels are rational-fraction funcions of $\varkappa$, ν, $\varkappa_m$, ν_m; their denominators consist of products of binomials of the form $\nu_{m+1}+\nu_m$. It follows that $\tilde{\Phi}_n$ has no singularities for any finite real $\varkappa$. A direct calculation of the functions $\tilde{\Phi}_n$ with the help of (16) shows that the first two terms of the expansion of S in the small slopes can in fact be written in the form (11) with $S_0=-1$ and $B=1$.

Transforming to the equation for the high-frequency limit, $k\to\infty$, with

$$\nu(\varkappa-\xi)+\nu(\varkappa_0+\xi)-\nu-\nu_0\approx(\varkappa/\nu-\varkappa_0/\nu_0)\xi,$$

we can easily put (11) in the form

$$S(\varkappa,\varkappa_0)=-\frac{1}{(\nu\nu_0)^{1/2}}\left(\nu+\frac{\varkappa(\varkappa-\varkappa_0)}{\nu(\nu+\nu_0)}\right)$$
$$\times\exp[-i(\varkappa-\varkappa_0)\mathbf{r}+i(\nu+\nu_0)\eta(\mathbf{r})]\frac{d\mathbf{r}}{(2\pi)^2}.$$

Interestingly, this expression for the scattering amplitude is identical to the expression derived in this problem by the Kirchhoff approximation,[21] although, generally speaking, the agreement should extend only to quantities of order $(\nabla\eta)^2$. The requirement that the terms in (11) fall off systematically with increasing order leads to a further restriction in the case of grazing angles of incidence or scattering, as can be shown. As result, the condition for the applicability of the small-slope approximation in our problem becomes

$$|\nabla\eta|\ll1 \text{ for } \chi,\ \chi_0\geqslant1, \qquad (17a)$$

$$|\nabla\eta|\ll\min(\chi,\chi_0) \text{ for } \chi\ll1 \text{ or } \chi_0\ll1, \qquad (17b)$$

where χ_0 and χ are the grazing angles of the incident and scattered waves. The latter condition arises formally because the quantities $\nu+\nu_1$, $\nu_0+\nu_1$, which appear in (16) because of the corresponding denominators, are small at grazing angles. Condition (17b), which means that there is no geometric shadowing of the rays of the incident wave or of the scattered wave by irregularities, should remain in force for other scattering problems also, although there is the possibility in principle that further restrictions will arise, in this case because of the structure of the function $B(\varkappa,\varkappa_0)$.

The method used in this section to derive an expansion of the type in (2) for the simple scalar problem can obviously be generalized in a straightforward way to other scattering problems.

4. CONCLUSION

This study has shown that the amplitude (S) for the scattering of waves by irregular boundaries calculated by the perturbation method (Bragg scattering) can be used, under the assumption that the slopes of these boundaries are small, to construct an approximation which holds for an arbitrary

height of the irregularities. Specifically, this height need not be small in comparison with the length of the incident wave. This is done by introducing in the expression for the scattering amplitude a factor which imparts certain transformation properties to S. This factor is of essentially purely geometric origin; it reflects the circumstance that different rays in the incident wave travel unequal paths because of the boundary irregularities, and an originally plane phase front of the wave becomes distorted. It turns out that in a first approximation in the slope of irregularities the expression for the scattering amplitude can be factored into this quantity of geometric origin and a kernel which describes Bragg scattering, the resulting approximation [in particular, expression (11)] substantially solves the problem of describing the scattering by irregular surfaces in an approach which combines perturbation theory with the Kirchhoff approximation. The assumption of small slopes need not be a substantial restriction, since in the case of a surface of a general type, with slopes on the order of unity, a repeated reflection of waves will necessarily arise in the high-frequency limit, and the Kirchhoff approximation in its standard formulation will become inapplicable.

In particular, we find from these results the following experimental condition under which the slopes of the irregularities at the rough surface being probed are small [see (13), (14)]: The ratios of the second moments for the scattering amplitudes for waves of different types or of different polarizations, under the condition that the wave vectors of the incident and scattered waves are equal, must not depend on the spectrum of irregularities, and they must agree with the results calculated in first-order perturbation theory [i.e., in accordance with (14)].

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