Type and structure of timelike singularities in the general theory of relativity: from the gamma metric to the general solution

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A method is proposed that makes it possible to determine whether a timelike singularity corresponds to a point, linear, or other type of gravitational field source. It is shown that in the general theory of relativity it is also possible to have sources of a quite different type with no analogs in a space of finite curvature. An analysis is made of some well-known solutions containing timelike singularities whose type varies depending on the signs of the functions that occur in the solutions. The form of the solution near simple linear sources [W. Israel, Phys. Rev. **D15**, 935 (1977)] and generalized anisotropic solutions [S. L. Parnovsky, Physica (Utrecht) **104A**, 210 (1980); E. M. Lifshitz and I. M. Khalatnikov, Sov. Phys. Usp. **6**, 359 (1963)] is determined more accurately; the space-time described by the γ metric (3) is completely investigated; and the form of the metric near the ends and singular points of linear Weyl singularities is found.

1. INTRODUCTION

In classical Newtonian gravitational theory, one considers not only distributed sources of the gravitational field with finite volume mass density but also compact sources of zero volume, these being divided into surface, linear, and point sources. In the general theory of relativity, the former correspond to a gravitational field with source described by the energy-momentum tensor of the matter on the righthand side of Einstein's equations. If in this case the space-time is everywhere regular, its study does not contain fundamental difficulties. To a lesser degree, this applies to gravitational fields without sources, for example, to gravitational waves, and to sources such as black holes or other spacelike singularities such as the initial cosmological singularity, which do not have analogs in Newtonian theory. These objects have been fairly fully investigated, and we have a reasonable idea of their nature.

It seems that the least studied case is that of a source in the form of a bare timelike singularity of space-time, corresponding to zero-volume sources in Newtonian theory. To this case the present paper is devoted. In it we show which of the timelike singularities can be identified as point or linear sources of the gravitational field and for which the concept of a type cannot be introduced. It turns out that in the general theory of relativity there may exist a new type of source, different from point, linear, and surface sources and without analogs in a space of finite curvature. All forms of sources can be described by the same solutions with different values of the functions or parameters that occur in them. In this paper, we investigate these solutions, beginning with the special case of the γ metric (3) and ending with the most general solution with sources, which have the type (36).

We define in more detail the object of our investigations. It is a space-time containing singularities. The hypersurfaces infinitesimally close to them are timelike, and this means that the singularities are timelike. We shall consider only singularities for which the curvature invariants of the space-time diverge as they are approached. Thereby we exclude sources of δ-function type such as conical singularities,[1,2] and also fictitious and "pathological" singularities.

A timelike singularity may be present within a black hole, as occurs in the Reissner-Nordström solution. Then its structure does not affect the gravitational field of this hole. Therefore, of greater physical interest are the so-called naked singularities, i.e., timelike singularities not hidden by an event horizon from a distant observer. In this paper, we shall mainly be concerned with such singularities, although the greater part of our results also apply to timelike singularities within black holes.

It is not yet known whether naked singularities can arise by collapse and exist as real astronomical objects. There is the "cosmic censorship" hypothesis, which denies this possibility. However, it is necessary to investigate naked singularities for a number of reasons. They arise in many well-known solutions of Einstein's equations, and without a determination of their type and structure it is impossible to analyze these solutions. Without this, it is also impossible to study correctly distributed sources of small size with strong gravitational field. Finally, without an investigation of naked singularities it is not possible to clarify the question of the validity of the cosmic censorship hypothesis.

Because the space-time curvature tends to infinity near the singularities, we do not in the majority of cases know how the system of coordinates in which the form of the space-time metric tensor is expressed behaves in this region. Therefore, it is often quite difficult to distinguish between the different types of naked singularities.

In order to establish whether we are dealing with a point, linear, or some other source, we propose to use diagrams that reflect the simplest invariant properties of the space-time near the singularities. The method used to construct them is explained in Sec. 2. We shall use these diagrams in investigating the sources in the solutions considered in the following sections.

We begin with the simplest case of a static and axisymmetric space described by the Weyl metric

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$$ds^2=e^{\nu(\rho,z)}dt^2-e^{-\nu}[e^{\gamma(\rho,z)}(d\rho^2+dz^2)+\rho^2d\varphi^2],$$
$$\nu_{,zz}+\rho^{-1}(\rho\nu_{,\rho})_{,\rho}=0,\quad \gamma_{,\rho}=2^{-1}\rho(\nu_{,\rho}{}^2-\nu_{,z}{}^2),\quad \gamma_{,z}=\rho\nu_{,\rho}\nu_{,z}. \tag{1}$$

If we consider an auxiliary flat space with cylindrical coordinates ρ, φ, z, which we shall also call the coordinate space, the function $\nu(\rho, z)$ will satisfy in it the Poisson equation $\Delta\nu = 0$, naturally, without the angular part. Now the function $\gamma(\rho, z)$ can be determined from known $\nu(\rho, z)$ up to an additive constant, the value of which for an isolated source of finite size can be found from the condition of absence of conical singularities.[1,2] Far from such a source $\nu, \gamma \to 0$, and the coordinates ρ, φ, z become cylindrical coordinates.

A solution of the equation $\Delta\nu = 0$ with such a boundary condition must have sources if it is not the trivial $\nu = 0$. If they are volume or surface sources in the coordinate space, then the space-time (1) will be regular. Singularities arise only if $\nu(\rho, z)$ has in the coordinate space point or linear sources, which, naturally, must be axisymmetric. In the real curved space-time, the source of the gravitational field has the same coordinates but may have an entirely different nature.

An example is a source that is a point source in the coordinate space. Going over to spherical coordinates and expanding ν in multipoles,

$$\nu=\sum_{n=0}^{N}a_nP_n(\cos\theta)r^{-1-n},\quad \rho=r\sin\theta,\quad z=r\cos\theta, \tag{2}$$

we see that for $1\leqslant N<\infty$ the space-time (1) has at $r = 0$ a so-called directional singularity. As the singularity is approached along certain directions (in certain ranges of variation of θ), its curvature invariants diverge, and, having traversed a finite distance, we arrive at the singularity. But if we approach $r = 0$ along different directions, we find that these invariants tend to zero, and the distance to the "point" $r = 0$ is infinite, i.e., we move to a point infinitely distant in space. For $N = 0$, a singularity with positive mass has such a structure,[3] while for $N = \infty$ Weierstrass's essential singularity theorem guarantees the presence of such a singularity.

Therefore, the point and linear singularities in (1) in which we are interested are described by a function $\nu(\rho, z)$ with a source that is linear in the coordinate space. Suppose it is an interval of the straight line $\rho = 0$ characterized by its length L and distribution along it of a linear mass density μ, these quantities being measured in the coordinate space. In other words, $\nu(\rho, z)$ is the Newtonian gravitational potential of an infinitely thin rod of length L with linear mass density $\mu(z)$.

In the simplest case $\mu(z) = \mu =$ const, we obtain the γ solution (its name is due to the fact that the quantity μ is frequently denoted by the letter γ):

$$ds^2=\operatorname{th}^{2\mu}\frac{v}{2}dt^2-\frac{L^2}{4}\operatorname{th}^{-2\mu}\frac{v}{2}\operatorname{sh}^2v\left[\left(1+\frac{\cos^2u}{\operatorname{sh}^2v}\right)^{1-\mu^2}(du^2+dv^2)+\cos^2u\,d\varphi^2\right]. \tag{3}$$

During the last decade, this solution has been the subject of studies by a group of authors that include Witten, Esposito, Papadopoulos, Stewart, and others. Among the large number of papers they have published, we mention Refs. 4 and 5.

The analysis of the solution (3) made in Sec. 3 of the present paper shows that its nature varies essentially when μ is varied. For $0<\mu<1$, the singularity $v = 0$ will be linear, while for $\mu = 1$ the metric (3) goes over into the Schwarzschild solution with horizon $v = 0$. For $\mu < 0$, the space-time contains a point singularity with negative mass. This last circumstance is not fortuitous, since in the vacuum an uncharged nonrotating source of point type with positive mass can be only a black hole and not a timelike singularity. For $\mu > 1$, the singularity is one of the new type of sources mentioned above. For $\mu\geqslant 2$, one further effect arises—at the points $v = 0$, $u = \pm\pi/2$ there appear two directional singularities, these corresponding to two infinitely separated points connected by the singularity $v = 0$. With allowance for the $v \to \infty$ region, the space-time then contains three different spatial infinities. This treatment of directional singularities is quite different from the one proposed in Ref. 5.

In Sec. 4, it is shown that the Weyl singularities obtained in the more general case $\mu(z) \neq$ const have the same nature as for the γ metric. In Sec. 4 we also find and investigate all possible forms of space-time near the ends of a linear Weyl singularity.

In Sec. 5 of the paper simple linear sources are investigated. This more general class of singularities was identified by Israel in Ref. 1. Such a source is situated in vacuum, does not rotate, is asymptotically axisymmetric near the singularity, and is not the source of any nongravitational field. According to Israel, the metric near it but far from its ends must have the approximate form

$$ds^2\approx-d\rho^2+A(z,t)\rho^{2p_1(z,t)}dt^2-B(z,t)\rho^{2p_2(z,t)}d\varphi^2-C(z,t)\rho^{2p_3(z,t)}dz^2,$$
$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1. \tag{4}$$

In the present paper, it will be shown that the results of Ref. 1 must be augmented. This also applies to the expression (4), to which it is necessary to add the small nondiagonal term (25), and the interpretation of the source of this gravitational field, which will indeed be a linear source for $p_3<0$ and a point source with negative mass for $p_1<0$, will also include the same type as the γ metric for $\mu > 1$ when $p_2 < 0$. But the metric near the ends of simple linear sources cannot be obtained by generalizing the corresponding solutions for Weyl singularities.

An even more general form of singularity is considered in Sec. 6. This is the generalized spatially anisotropic solution (36), obtained in Ref. 6. It contains in the vacuum three physically arbitrary functions of three variables. Since its generalization, which contains four such functions, i.e., a spatially oscillatory solution,[6] cannot be associated with any of the source types, the considered solution is evidently the most general form of point and linear sources and singularities of the γ-metric type for $\mu > 1$. In the present paper, we determine more precisely the form of the generalized spatially anisotropic solution and the Lifshitz-Khalatnikov solution[7] near a spacelike singularity and determine the type of the source of the gravitational field in the first of them. We also discuss the influence of matter and nongravitational fields on the form of this solution. Their presence does not change the nature of the solution near a nonpoint source. In this case, the only restriction is the absence of rotation of this source.

2. DIAGRAMS CHARACTERIZING THE SOURCE TYPE

Through the investigated space-time we describe two hypersurfaces (sections). One of them is timelike and near the singularity. Except for singular cases to be discussed below, we shall consider the hypersurface $x = \text{const}$ in a semigeodesic coordinate system (in Ref. 6, this was called pseudosynchronous) with interval

$$ds^2 = -dx^2 + \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha = (0,\ 2,\ 3),$$
$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{5}$$

the singularity corresponding to $x = 0$. The coordinate x orthogonal to the singularity is determined uniquely, while the three remaining coordinates x^α admit three arbitrary transformations into each other that do not affect the form of the diagrams.

As the second section through the space-time, we take the spacelike hypersurface $t = \text{const}$. The diagram we construct is also independent of its choice. The only section through the space-time that can be an exception to this is the case of a section asymptotically isotropic in the limit $x \to 0$, and this case we shall avoid.

After describing the sections $x, t = \text{const}$, we obtain the plane yz, a surface that covers the singularity and tends to it as $x \to 0$. The distances on it are determined by the two-dimensional metric tensor

$$l_{ab} = -\gamma_{ab} + \gamma_{0a}\gamma_{0b}/\gamma_{00}. \tag{6}$$

As $x \to 0$, the length of any curve joining two distinct points in the yz plane may tend to zero. Then these points tend to each other as $x \to 0$, i.e., correspond to the same singularity point. It is on the basis of this that we construct the diagram. We take a square in the yz plane, its sides being along the y and z axes, and we then join by a continuous line the points of the square between which the distances tend to zero as $x \to 0$. In practice, to construct the diagram it is sufficient to reduce the metric near the investigated singularity point to the form (5), at least approximately, by means of the procedure described in Appendix G of Ref. 7 and then, having constructed l_{ab} using (6), investigate its limit as $x \to 0$.

We shall return to the question of the existence of this limit and its possible form. Here, we construct some diagrams for the simplest case of flat space-time. There being no singularities in it, we construct the diagrams near the regular plane $x = 0$ in a Cartesian coordinate system, the line $\rho = 0$ in a cylindrical one, and the point $r = 0$ in a spherical one (Figs. 1a–1c). In the first case, the square will be white, in the third black, and in the second the diagram will be hatched with horizontal lines $\varphi = \text{const}$. But, on the other hand, from the form of the diagrams 1a–1c we can determine whether we are dealing with a point, line, or surface.

What is changed when we construct the diagrams for timelike singularities? One can show that in the general theory of relativity such a singularity cannot be a surface, and a diagram of type 1a corresponds to a fictitious singularity of the event-horizon type. But diagrams of the form 1b and 1c are entirely possible, except that the distances between any points in Fig. 1b that do not lie on one line $z = \text{const}$ diverge as $x \to 0$.

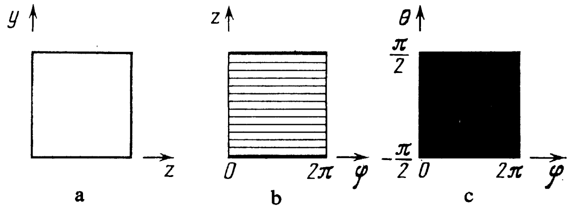

FIG. 1.

The infinite curvature near a singularity allows the existence of new types of singularities. One of the possibilities is a directional singularity, in which the singularity is glued to an infinitely distant point, for example, the solution (1), (2). In this case, we cannot, when constructing the diagrams, use the coordinate system (5); we shall consider the solution in the Weyl form. We shall encounter such a singularity in the following section.

When the surface surrounding a directional singularity, including spatial infinity, is contracted, its area diverges, whereas for the remaining forms of singularity it tends to zero. In the latter case, the determinant of l_{ab} in (6) tends to zero as $x \to 0$, and with it also the product of the eigenvalues of l_{ab}. If there exists a limit for the eigenvectors of l_{ab}, then if both eigenvalues tend to zero we have a point singularity with a diagram of the type 1c. But if one of them tends to zero and the other to infinity, then besides the linear source with diagram 1b one can also have the case of a diagram with vertical hatching. This new type of source will be considered in the following section.

But if the eigenvectors of l_{ab} do not have a limit as $x \to 0$, the source cannot be associated with one of the known types. Moreover, the very concept of a type does not apparently apply to it. As an example, we mention the spatially oscillatory solution of Ref. 6.

Finally, we note that the type of the singularity may be different at different parts of it or at different times

3. ANALYSIS OF THE GAMMA SOLUTION

To investigate the properties of the γ metric, it is also convenient to use a different form of it, which is obtained from (3) by making the transformation $r = L\cosh^2(v/2)$:

$$ds^2 = \left(1 - \frac{L}{r}\right)^\mu dt^2 - \left(1 + \frac{L^2\cos^2 u}{4r(r-L)}\right)^{1-\mu^2} \left(1 - \frac{L}{r}\right)^{1-\mu} \times \left[\frac{dr^2}{1 - L/r} + r^2(du^2 + \cos^2 u\, d\varphi^2)\right]. \tag{7}$$

The singularity $v = 0$ corresponds to $r = L$. Its mass $M = \mu L/2$ can be readily found from the form of g_{00} at large distances from the source. The ratio M/L is half μ, the linear density in the coordinate space.

To study the metric (3), it is also helpful to use the solution that in the coordinate space has a source in the form of an infinite filament $\rho = 0$ with constant linear mass density μ. It

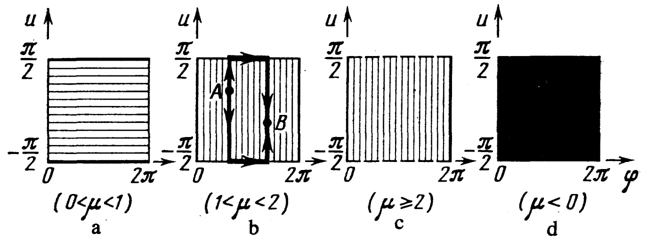

FIG. 2.

is given in Weyl form by

$$ds^2=\rho^{2\mu}dt^2-\rho^{2(\mu^2-\mu)}(d\rho^2+dz^2)-a_1^2\rho^{2-2\mu}d\varphi^2,$$
$$a_1=\text{const},\quad 0\leqslant\varphi<2\pi \tag{8}$$

and after the transformation $x=\rho^{\mu^2-\mu+1}$ and changes of scale along the t,ρ, and z axes goes over into the spatial Kasner metric

$$ds^2=-dx^2+x^{2p_1}dt^2-a_2^2x^{2p_2}d\varphi^2-x^{2p_3}dz^2,$$
$$a_2=\text{const},\quad 0\leqslant\varphi<2\pi, \tag{9}$$

$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1, \tag{10}$$

$$p_1=\frac{\mu}{1-\mu+\mu^2},\quad p_2=\frac{1-\mu}{1-\mu+\mu^2},\quad p_3=\frac{\mu^2-\mu}{1-\mu+\mu^2}. \tag{11}$$

In analyzing the form of the space-time near the ends of the singularity (3)—at the points $v=0$, $u=\pm\pi/2$—we use the fact that its asymptotic form is identical to another exact solution which in the coordinate space has a source in the form of a semi-infinite filament with $\mu=$ const:

$$ds^2=v^{2\mu}dt^2-v^{2-2\mu}[(u^2/v^2+1)^{1-\mu^2}(du^2+dv^2)+u^2d\varphi^2]. \tag{12}$$

After the coordinate transformation $2r=u^2+v^2$, $\theta=2\tan^{-1}(v/u)$, we obtain

$$ds^2=(2r)^{\mu}\sin^{2\mu}(\theta/2)\,dt^2-(2r)^{-\mu}\sin^{-2\mu}(\theta/2)$$
$$\times\ [\sin^{2\mu^2}(\theta/2)\,(dr^2+r^2d\theta^2)+r^2\sin^2\theta d\varphi^2]. \tag{13}$$

The singularity $v=0$ or $\theta=0$ is noneliminable for $\mu\neq 0, 1$, like $\rho=0$ in (8).

We now investigate all these solutions. For $\mu=0$, we have flat space-time in prolate elliptical (3), parabolic (12), spherical (13), and cylindrical (8), (9) rotational coordinates, with, in the last case for $a_1=a_2\neq 1$ a conical singularity on the axis $\rho=0$.[2,1]

For $0<\mu<1$, the singularity in (3) will be linear. The diagram corresponding to it (Fig. 2a) has the same form as the one shown in Fig. 1b. In the space (3) we consider the lengths of the circles v, $u=$ const, which we denote by $L_\varphi(v,u)$, and also the length $L_u(v)$ of the curve v, $\varphi=$ const and S, the area of the surface $v=$ const. In the metric (7), L_φ corresponds to the lengths of the "equator" and "parallels," L_u to the length of the "meridians," and S to the area of the surface $r=$ const. Expressions for these quantities are found in Ref. 4, though the $v\to 0$ limit in which we are interested can be readily found without knowing them. For $0<\mu<1$, we have S, $L_\varphi\to 0$, $L_u\to\infty$ as $v\to 0$, and this ensures a vanishing distance between points with equal values of u in diagram 2a and an infinite distance between points with different u. This last result has the consequence that although the linear, for $0<\mu<1$, singularity in (3) is finite (it can be surrounded by a surface of finite size), direct measurement of its length is impossible. From measurements, one can determine only the value of the parameter L.

The solutions (8) and (12) also have linear sources with diagrams and properties analogous to the singularity in (3), but they are not finite.

For $\mu=1$, the solution (7) goes over into the Schwarzschild metric with $v=0$ corresponding to the horizon. In this case, L_u, L_φ, and S tend as $v\to 0$ to finite limits, and the diagram is identical to 1a. The solutions (8) with $a_1=1$ and (12) then describe part of flat space-time.

For $1<\mu<2$, the space-time (3) has a source of new type, neither point, nor linear, nor surface, nor volume. Its properties are rather unusual. As $v\to 0$, we have L_u, $S\to 0$, $L_\varphi\to\infty$. This last means that the length of the circle v, $u=$ const diverges as its radius tends to zero. This alone shows that the source cannot be linear. We now consider the rotation axis, on which the norm of the rotational Killing vector $\xi^i=(0,0,1,0)$ vanishes. For $0<\mu<1$, it consists of the two half-axes $u=\pm\pi/2$ connected by the singularity $v=0$. For $1<\mu<2$, it consists only of the two half-axes $u=\pm\pi/2$. Since $L_u\to 0$, the distance between the points $v=0$, $u=\pm\pi/2$ is zero and these half-axes are connected to each other. But as the singularity is approached, the norm $\xi^i\xi_i$ first decreases as v decreases, reaches a minimum at $\cosh v=\mu$, and then increases and diverges as $v\to 0$. This also indicates that the singularity is not linear. We note that it cannot be a torus, as is shown by the value of the Gauss-Bonnet invariant for the surface $v=$ const, or a flat disk.

We shall study the properties of the source of the gravitational field (3) for $1<\mu<2$ by means of its diagram, shown in Fig. 2b. It is a square with vertical hatching and two heavy horizontal straight lines $u=\pm\pi/2$, which correspond to the "ends" of the singularity in (3) or its "poles" in (7). Any two points A and B in the diagram can be connected by two lines of zero length, which are shown in Fig. 2b and pass through one of the "poles" $u=\pm\pi/2$. In this the source is similar to the point source. However, in contrast to it, the lengths of all other paths connecting A and B tend to infinity. This makes the properties of the source in the γ metric for $1<\mu<2$ different from the properties of all the sources with which we are familiar in a space of finite curvature.

The source in the solution (8) for $\mu>1$ has the same diagram with vertical hatching but without the horizontal lines, since it is not finite. In this case, two points with different values of φ cannot be connected by a line of zero length. Since this diagram differs from the one shown in Fig. 1b only by the substitution $z\rightleftarrows\varphi$, it is rather natural to regard z in (8) as an angular coordinate and φ as a linear coordinate, as was proposed by Khalatnikov and the present author in Ref. 8. Support for such an interpretation is also to be seen in the fact that the shift transformation by a finite interval with respect to the z coordinate has a family of fixed points on the axis $\rho=0$, as is characteristic for rotation, while the shift transformation with respect to φ does not have fixed points in the entire space. Regarding φ and z as linear and angular coordinates, respectively, we obtain a linear source with diagram 1b.

For the γ solution with $1<\mu<2$, the shift transformations with respect to u and with respect to φ behave near the singularity similarly. However, the nature of these coordinates is determined by their behavior far from it. Thus, the shift with respect to φ has fixed points at $u=\pm\pi/2$, indicating that this is a rotation. Therefore, it is impossible to reduce the source for the γ solution with $0<\mu<2$ to an ordinary type by changing the interpretation of the coordinates employed.

But if $\mu\geqslant 2$, we encounter a new phenomenon—the appearance at the points $v=0$, $u=\pm\pi/2$ of directional singularities. Near them, the solution (3) is asymptotically identical to (12), (13) at small r. In (13) with $\mu<2$ points with finite value of r not lying on the singularity $\theta=0$ are at a finite distance from the point $r=0$ and an infinite one from $r=\infty$. For $\mu>2$, conversely, they are at a finite distance from the point $r=\infty$ and at an infinite distance from the infinitely distant point $r=0$. We introduce the more convenient coordinate $R=r^{1-\mu/2}$ and obtain from (13)

$$ds^2=2^\mu R^{2\mu/(2-\mu)}\sin^{2\mu}\frac{\theta}{2}\,dt^2-2^{-\mu}\times\sin^{-2\mu}\frac{\theta}{2}\left[\sin^{2\mu^2}\frac{\theta}{2}\left(\frac{dR^2}{(1-\mu/2)^2}+R^2d\theta^2\right)+R^2\sin^2\theta d\varphi^2\right]. \quad (14)$$

From the form of g_{00} it can be concluded that at the point $R=0$ there is a source with negative mass. However, we are more interested in the region of small r, i.e., $R\to\infty$ for $\mu>2$. Moving away from the singularities $\theta=0$ and $R=0$, we move to an asymptotically flat region $R=\infty$, $\theta\neq 0$. But one can also choose a path to $R=\infty$ that approaches the singularity $\theta=0$. Following it, we find that the curvature does not decrease but increases as $R\to\infty$.

On the transition to the previous coordinate r, all these paths lead to $r=0$, where there is a directional singularity. The curvature invariants, for example,

$$R_{iklm}R^{iklm}\propto r^{2\mu-4}\sin^{-4}(\theta/2), \quad (15)$$

tend to zero as $r\to 0$, $\theta\neq 0$ and may diverge if as $r\to 0$ we also have $\theta\to 0$, for example, $\theta\propto r$. The length of paths with $\theta\neq 0$ is infinite. Proceeding along them, we find that among the components of the curvature tensor some are divergent, but this is due solely to the unfortunate choice of the coordinate system. After transition to the coordinates $t, r, r^\alpha\theta, r^\beta\varphi$, where $\alpha<-1-\mu/2, \beta<-\mu/2$, all the components of R_{iklm} tend to zero as $r\to 0$. The length of paths with $\theta\to 0$ leading to the point $r=0$ may be finite.

For $\mu=2$, the space (13), like (8), has the Petrov type D, although the γ solution itself is not so degenerate and is type I. Points with finite r are at finite distance from both $r=0$ and $r=\infty$, and the curvature invariants do not depend on r. As the singularity is approached along the direction $r=0$, the space-time for $\mu=2$ does not become asymptotically flat, although it does include the infinitely distant point $r=0$, $\theta\neq 0$.

The solution (3) for $\mu\geqslant 2$ therefore contains two infinitely distant points $v=0$, $u=\pm\pi/2$, these being connected by a singularity $v=0$ of the same type as the γ metric for $1<\mu<2$. This means that in the limit $v\to 0$ we have L_u L_φ, $S\to\infty$, although the length of the curve $v,\varphi=$ const joining the points $u_1=-\pi/2+\varepsilon$ and $u_2=\pi/2-\varepsilon$ tends in the limit $v\to 0$ to zero for any arbitrarily small ε.

The space-time (3) contains three different spatial infinities—two infinitely distant points $v=0$, $u=\pm\pi/2$ and the region $v\to\infty$.

The diagram of the source, shown in Fig. 2c, in this case also has vertical hatching and differs from Fig. 2b in that the two horizontals $u=\pm\pi/2$ correspond to directional singularities. We have denoted them nominally by dashed lines. Two points with different φ cannot be connected by a line of zero length when $\mu\geqslant 2$.

For $\mu<0$, the diagram of the singularity, Fig. 2d, is a black square. This is a point source, as is also indicated by the fact that in the limit $v\to 0$ we have L_u, L_φ, $S\to 0$. In the special case $\mu=-1$, after the transformation $r=L\sinh^2(v/2)$, the solution (3) goes over into the Schwarzschild metric with negative mass $M=-L/2$.

4. WEYL SINGULARITIES

We now consider the space-time with Weyl metric (1) in which the function $\nu(\rho,z)$ is equal to the Newtonian potential of an infinitely thin rod with linear mass density $\mu(z)$ in the nominal flat coordinate space. It can be readily expressed in the form of an integral, but near the singularity, at small ρ, its asymptotic expression is $\nu(\rho,z)=2\mu(z)\ln\rho+O(\rho^2\ln\rho)$. Constructing the diagram for the source, we find that, as for the γ, solution, for $0<\mu(z)<1$ the singularity is linear, for $\mu(z)<0$ point, and for $\mu(z)>1$ of the same type as the γ metric for $1<\mu<2$. Since the function $\mu(z)$ can pass through $\mu=0$ and $\mu=1$, different parts of the singularity can have different types.

We investigate the form of the space-time near the end of the linear Weyl singularity. Suppose that at the point $r=0$ there is situated the end of the singularity $\theta=0$ with linear density $\mu(r)$ in the coordinate space. If $\mu(0)=\mu\neq 0$, then the asymptotic form of the metric for $r\ll r_0$ will be (13). Here, r_0 is the characteristic scale of variation of $\mu(r)$. But if for $r\lesssim r_0$ we have $\mu(r)\approx Kr^\lambda$, $\lambda>0$, then the required metric for $r\ll r_0$ will have the form (1) or

$$ds^2=e^\nu dt^2-e^{-\nu}[e^\gamma(dr^2+r^2d\theta^2)+r^2\sin^2\theta d\varphi^2], \quad (16)$$

where the function $\nu(r,\theta)$ has a singularity at $\theta=0$ of the form $\nu\to 2Kr^\lambda\ln\theta$ as $\theta\to 0$, is regular at $\theta=\pi$, and satisfies, at least approximately, the equation $\Delta\nu=0$, while the function $\gamma(r,\theta)$ is determined from ν in accordance with (1).

For $\lambda\neq 1, 2, 3\ldots$, all these conditions are satisfied by the function

$$\nu_{K,\lambda}(r,\theta)=\frac{K\pi}{\sin(K\lambda)}r^\lambda P_\lambda(-\cos\theta),$$

$$K=\text{const},\quad \lambda=\text{const}\neq 1,2,3,\ldots,\quad \lambda>0, \quad (17)$$

$$\gamma_{K,\lambda}(r,\theta)=\frac{K^2\pi^2}{4\lambda\sin^2(K\lambda)}r^{2\lambda}\sin^2\theta[\lambda^2P_\lambda^2(-\cos\theta)-P_\lambda'^2(-\cos\theta)\sin^2\theta-2\lambda P_\lambda(-\cos\theta)P_\lambda'(-\cos\theta)\cos\theta],$$

where $P_\lambda(\xi)$ and $P'_\lambda(\xi)$ are a Legendre function and its derivative.

For integral values of λ, this solution is not suitable. To construct the solution for this case, we note that $\nu = Q(r, \theta) \ln (r^{1/2} \sin(\theta/2))$, where Q is a regular function, has at $\theta = 0$ a linear singularity with $\mu(r) = Q(r, 0)$, and the condition $\Delta\nu = 0$ gives in the first approximation $\Delta Q = 0$. Choosing $Q = 2Kr^{\lambda} P_{\lambda}(\cos\theta)$, we obtain for $\lambda = 0$ the solution (13), and for $\lambda = 1, 2, 3 \ldots$ We find the form of $\mu(r)$ that we require. By adding to this function a regular term of the form $Kr^{\lambda} N_{\lambda}(\cos\theta)$, where $N_{\lambda}(\xi)$ is a polynomial of degree $\lambda - 1$, we can achieve exact fulfillment of the equation $\Delta\nu = 0$. As a result, for $r \ll r_0$ we obtain the approximate solution

$$\nu_{K,\lambda}(r,\theta) = Kr^{\lambda}\left[P_{\lambda}(\cos\theta)\ln\left(r\sin^2\frac{\theta}{2}\right) + N_{\lambda}(\cos\theta)\right],$$
$$\lambda = 1, 2, 3, \ldots, \quad K = \text{const},$$
$$\gamma_{K,\lambda}(r,\theta) = K^2 r^{2\lambda}\left[\cos^2\frac{\theta}{2}\left(\mathrm{in}\left(r\sin^2\frac{\theta}{2}\right) + C_{\lambda}\right)\right.$$
$$+\sin^2\theta\left(a_{\lambda}(\cos\theta)\ln^2\left(r^{1/2}\sin\frac{\theta}{2}\right)\right.$$
$$\left.\left.+b_{\lambda}(\cos\theta)\ln\left(r^{1/2}\sin\frac{\theta}{2}\right) + d_{\lambda}(\cos\theta)\right)\right],$$
$$N_{\lambda}(\xi) = 2\sum_{k=0}^{\lambda-1}\frac{2k+1}{(\lambda-k)(\lambda+k+1)}P_k(\xi),$$
$$C_{\lambda} = N_{\lambda}(1) - \frac{1}{2\lambda} = 2\lambda\sum_{k=1}^{\lambda}\frac{1}{k(k+\lambda)} - \frac{1}{2\lambda},$$
$$a_{\lambda}(\xi) = \lambda\frac{P_{\lambda}^2(\xi) - P_{\lambda-1}(\xi)}{1-\xi^2}, \tag{18}$$
$$b_{\lambda}(\xi) = \frac{P_{\lambda-1}^2(\xi) - P_{\lambda}^2(\xi)}{2(1-\xi^2)} + \frac{1-P_{\lambda}^2(\xi)}{\xi-1}$$
$$+\lambda N_{\lambda}(\xi)\frac{P_{\lambda}(\xi) - \xi P_{\lambda-1}(\xi)}{1-\xi^2}$$
$$+\frac{\xi^3 P_{\lambda}(\xi) - P_{\lambda-1}(\xi)}{1-\xi^2}N_{\lambda}'(\xi),$$
$$d_{\lambda}(\xi) = -\frac{b_{\lambda}(\xi)}{4\lambda} + 2P_{\lambda}^2(\xi) + 2\lambda N_{\lambda}(\xi)P_{\lambda}(\xi)$$
$$-N_{\lambda}'^2(\xi) + (\lambda N_{\lambda}(\xi) - \xi N_{\lambda}'(\xi))^2$$
$$+\frac{N_{\lambda}(1) - N_{\lambda}(\xi)P_{\lambda}(\xi)}{2(\xi-1)}$$
$$N_{\lambda}'(\xi) = dN_{\lambda}(\xi)/d\xi.$$

The terms with a_{λ}, b_{λ}, C_{λ} d_{λ} in $\gamma_{K,\lambda}(r,\theta)$ are regular for $\theta = 0$. Calculating the curvature invariants for the solutions (16)–(18), we find, for example, that

$$I \equiv R_{iklm}R^{iklm} \xrightarrow[\theta\to 0]{} \theta^{-4}, \quad I \xrightarrow[\substack{r\to 0\\ \theta\neq 0}]{} r^{2\lambda-4} \ (\lambda \neq 1, 2, 3, \ldots),$$
$$I \xrightarrow[\substack{r\to 0\\ \theta\neq 0}]{} r^{2\lambda-4}\ln^2 r \ (\lambda = 1, 2, 3, \ldots). \tag{19}$$

For $0 < \lambda < 2$ and $\lambda = 2$ we have, respectively, power-law and logarithmic divergences of this and the other invariants as $r \to 0$. For $\lambda > 2$, there is at the point $r = 0$ a directional singularity, but in this case this is not an infinitely distant point.

It is also easy to find the approximate form of the metric near the singular points of the function $\mu(r)$—its discontinuities or its vanishing in accordance with some law, which may be different as the singular point is approached from different sides. Thus, if $\mu(r) \approx Kz^{\lambda}$ for $z > 0$ and $\mu(z) \approx M(-z)^{\eta}$ for $z < 0$, then as $\nu(r, \theta)$ it is necessary to take a sum of $\nu_{K,\lambda}(r, \theta)$ and $\nu_{M,\eta}(r, \theta + \pi)$ of the form (17), (18) or (13) for $\eta = 0$. In the case of a jump in the linear density from μ_- as $z \to -0$ to μ_+ as $z \to +0$, we obtain the solution (16) with

$$\nu = 2\mu_+ \ln v + 2\mu_- \ln u$$
$$= (\mu_+ + \mu_-)\ln(2r) + 2\mu_+ \ln\sin(\theta/2) + 2\mu_- \ln\cos(\theta/2), \tag{20}$$
$$\gamma = 2\mu_+^2 \ln v + 2\mu_-^2 \ln u - (\mu_+ - \mu_-)^2 \ln(u^2+v^2) + \text{const}$$
$$= 2\mu_+\mu_- \ln r + 2\mu_+^2 \ln\sin(\theta/2) + 2\mu_-^2 \ln\cos(\theta/2) + \text{const},$$

where u and v are parabolic coordinates. This exact solution goes over into (12), (13) for $\mu_- = 0$ and into (8) for $\mu_+ = \mu_-$ and serves as an approximate solution near any discontinuity of $\mu(z)$. The curvature invariants

$$I \equiv R_{iklm}R^{iklm} \xrightarrow[r\to 0]{} r^{2\mu_+ + 2\mu_- - 4\mu_+\mu_- - 4} \ (\theta \neq 0),$$
$$I \longrightarrow \sin^{-4}\theta \tag{21}$$

and the others show that for $\mu_+ + \mu_- - 2\mu_+\mu_- \geqslant 2$ at $r = 0$ there is a directional singularity, this taking the form of an infinitely distant point from which two singularities emanate. It is easy to show that their types must be different, each of them being either a linear source, or a source of the type of the γ metric for $\mu > 1$, or having $\mu < 0$. In the last case, the singularity will not be a point singularity but will have infinite length. Its diagram will be a black square of the type shown in Fig. 2d but bounded on one side by a broken line, which represents the infinitely distant point.

5. SIMPLE LINEAR SOURCES

Simple linear sources are a natural generalization of Weyl singularities. The concept was introduced by Israel in Ref. 1. He there also formulated conditions necessary for a space-time to contain such a source at $\rho = 0$, where ρ is the coordinate orthogonal to the singularity in a system of coordinates $x^i = (t, \rho, \varphi, z)$ that in the absence of the singularity become cylindrical. In the limit $\rho \to 0$, the space-time must be asymptotically axisymmetric, and the intrinsic curvature of the hyperspace $\rho =$ const must not diverge faster than ρ^{-2}, this eliminating Weyl singularities of the type (2) and their generalizations. In addition, the source must not rotate or be the source of any nongravitational field.

The simple linear sources being nonstatic, their sizes and distribution $\mu(z, t)$ can vary in time. We shall determine the latter quantity from the form of the metric near the singularity (4) by analogy with the Weyl case (11). The static nature of the Weyl linear singularities requires compensation of the force of attraction between its different parts by an internal "pressure" force, which tends to increase its length. We find a similar situation in the Curzon solution, in which the force of attraction of two singularities is compensated by the pressure of a conical singularity.[2] But in the general case, such compensation will not occur and the singularity will evolve from the initial Weyl form, changing its size and $\mu(z,t)$. There is the curious possibility that a linear singularity will in this process contract until $\mu = 1$ is reached, after which it is transformed into a black hole. In such a case, if the principle of cosmic censorship were violated, the violation would be temporary.

The form of the metric of a complete space containing a simple linear source cannot be determined, but near the source, as $\rho \to 0$, it has the form (4), as Israel pointed out.[1] Constructing the diagram for the singularity $\rho = 0$ in this solution, we find that its type is the same as for the metric (8), a generalization of which it is. For $p_1 < 0$, we are dealing with a point singularity of negative mass, for $p_2 < 0$ with an actually linear source, and for $p_3 < 0$ with a singularity of the same type as in the γ metric for $\mu > 1$. Thus, (4) by no means always describes a linear source, as was assumed in Ref. 1.

We obtain one further correction to Israel's results by noting that in deriving the solution (4) he used only the diagonal components of the Einstein equations, which this solution satistfies in the leading terms. The conditions $R_{02} = R_{12} = R_{23} = 0$ are satisfied because of the asymptotic axial symmetry. In the equations $R_{01} = R_{13} = 0$, the terms of order $\rho^{-1}\ln\rho$ cancel automatically, and those of order ρ^{-1} if

$$2p_3' = \alpha'(p_1-p_3)+\beta'(p_2-p_3),\quad 2\dot p_1 = \dot\beta(p_2-p_1)+\dot\gamma(p_3-p_1). \tag{22}$$

In these conditions, the prime denotes the derivative with respect to z, the dot the derivative with respect to time, and $\alpha = \ln A, \beta = \ln B, \gamma = \ln C$. Complexities arise in the study of the equation

$$R_{03} \approx C_1 \ln^2\rho + C_2\ln\rho + C_3,\quad C_1 = -2\dot p_2 p_2',$$

$$C_2 = \frac{1}{2}[\dot\beta(p_1-p_3)' + \beta'(\dot p_3 - \dot p_2) + \dot p_2\alpha' + p_2'\dot\gamma] - \dot p_2'$$

$$= \frac{p_3'\dot p_2 - \dot p_1 p_2'}{p_1-p_3} - p_2'\dot\beta - \dot p_2\beta', \tag{23}$$

$$4C_3 = \dot\beta\alpha' + \beta'\dot\gamma - \dot\beta\beta' - 2\dot\beta' = 2\left(\frac{p_3'\dot\beta - \dot p_1\beta'}{p_1-p_3} - \frac{\dot B'}{B}\right).$$

Here, in the expressions for C_2 and C_3 we have eliminated $\dot\alpha'$ and γ by means of (22). The condition $R_{03} = 0$ for the metric (4) is equivalent to $C_1 = C_2 = C_3 = 0$. It is easy to show that this can be satisfied only in three cases: Either the metric does not depend on the time, i.e., reduces to the Weyl metric, or it does not depend on z, which is impossible for finite singularities, or (p_1, p_2, p_3) depends neither on t nor z and the solution reduces to

$$B = f(t)+q(z),\quad A = a(t)B^{(p_2-p_3)/(p_1-p_3)},\quad C = c(z)B^{(p_2-p_1)/(p_1-p_3)}, \tag{24}$$

where f, q, a, c are arbitrary functions. This very particular solution can be simplified still further after transition to the coordinates $T = f(t), Z = q(z)$ and, for $f = \text{const}$ or $q = \text{const}$ to $T = a(t)$ or $Z = c(z)$.

If Israel's metric (4) is nevertheless to satisfy all the Einstein equations, without losing its generality, it must be augmented by a small nondiagonal term:

$$ds^2 \approx ds_{(0)}{}^2 + 2\rho^2(\lambda_1\ln^2\rho + \lambda_2\ln\rho + \lambda_3)\,dt\,dz, \tag{25}$$

$$\lambda_1 = -\frac{C_1}{2p_2^2},\quad \lambda_2 = -\frac{C_2}{2p_2^2} + \frac{C_1(1+p_2)}{2p_2^4},$$

$$\lambda_3 = -\frac{C_3}{2p_2^2} + \frac{C_2(1+p_2)}{4p_2^4} - \frac{C_1(1+2p_2)}{4p_2^6}, \tag{26}$$

where $ds^2_{(0)}$ denotes the metric (4), and C_1, C_2, and C_3 are given in (23). The functions that occur in this solution must also satisfy the conditions (22).

The solution (25) does not depend on φ. Therefore, by Cotton's theorem,[9] the three-dimensional metric tensor obtained from it for $d\varphi = 0$, and with it the complete solution, can be diagonalized. However, the transformations needed for this will evidently contain at $\rho = 0$ an essential singularity. For if we restrict ourselves to transformations that can be expanded in series in powers of ρ, then, as can be shown, the only transformations that can affect the form of the nondiagonal term in (25) will be

$$\rho = f(t,z)\,\tilde\rho^{q(t,z)},\quad t = \tilde t + \psi(t,z)\rho^{2-2p_1},\quad z = \tilde z + \Phi(t,z)\rho^{2-2p_3}, \tag{27}$$

which contain four arbitrary functions of t and z. This is not sufficient to annihilate the three functions $\lambda_1, \lambda_2, \lambda_3$ without violating the two conditions $g_{01} = g_{13} = 0$. In particular, we can annihilate λ_1 only for $q(t,z) = p_2^{-1}$. Further analysis shows that the condition of annihilation of λ_2 determines our function $f(t,z)$ up to multiplication by an arbitrary function μ. But the freedom in the choice of this last arbitrary function remaining in the transformations (27) (ψ and Φ are completely determined by the conditions $g_{01} = g_{13} = 0$) is not sufficient for the elimination of λ_3 as well. Therefore, the metric (25) cannot be diagonalized. Note that we obtained this result by regarding $\mu(z,t)$, $A(z,t)$, $B(z,t)$, and $C(z,t)$ in (4) and (25) as arbitrary functions subject only to the conditions (22), although in the evolution of the singularity they must vary in an interconnected manner because of the conservation laws at the least. It is readily seen that the solution (25) is obtained by generalizing the exact solution (9) by replacing μ by $\mu(z,t)$, multiplying the diagonal terms by functions of z and t, and adding specially chosen nondiagonal terms that ensure fulfillment of the nondiagonal components of the gravitational equations. These terms must be small, i.e., must satisfy near the singularity the condition

$$g_{ik}{}^2 \ll |g_{ii}|\,|g_{kk}| \tag{28}$$

(here, no summation over repeated indices). Then they will not affect the determinant of the metric tensor, and the diagonal components of Einstein's equations will be satisfied automatically.

We attempt in this manner to obtain the approximate form of the metric near the end of a simple linear souce, i.e., at small r. We restrict ourselves to the leading term in the expansion of $\mu(r,t)$ in r, and consider the case $\mu(0,t) = \mu(t) \neq 0$, which generalizes (13), and $\mu(r,t) \approx K(t)r^{\lambda(t)}$, which generalizes the solutions (16)–(18). We seek the metric in the form

$$ds^2 = A\,dt^2 - B\,dr^2 - C\,d\varphi^2 - D\,d\theta^2 + 2a\,dt\,dr + 2b\,dt\,d\theta + 2c\,dr\,d\theta, \tag{29}$$

where the diagonal components A, B, C, D and the small nondiagonal terms a, b, c are functions of r, θ, t. Generalizing the metric (13), we set

$$A=r^{\mu}\sin^{2\mu}\frac{\theta}{2}e^{2\alpha},\quad B=r^{-\mu}\sin^{2(\mu^2-\mu)}\frac{\theta}{2}e^{2\beta_1}$$
$$C=r^{2-\mu}\sin^2\theta\sin^{-\mu}(\theta/2)e^{2\beta_2},\quad D=r^{2-\mu}\sin^{2(\mu^2-\mu)}(\theta/2)e^{2\beta_3}. \tag{30}$$

Generalizing the solutions (16)–(18), we set

$$A=\exp(\nu+2\alpha),\ B=\exp(\gamma-\nu+2\beta_1),$$
$$C=r^2\sin^2\theta\exp(2\beta_2-\nu),$$
$$D=r^2\exp(\gamma-\nu+2\beta_3), \tag{31}$$

where we take the functions ν and γ from (17) and (18), setting in them $K=K(t)$ and for (17) $\lambda=\lambda(t)$. The factors $\exp 2\alpha$ and $\exp 2\beta_i$ can depend only on t, since if they did depend essentially on r or θ this would lead to nonfulfillment of the diagonal components of Einstein's equations. The same would happen if $\beta_1\neq\beta_2$. In addition, it is only for $\beta_2=\beta_3$ that there is no conical singularity at $\theta=\pi$. Therefore, in what follows we shall denote $\beta_1=\beta_2=\beta_3$ by β.

The metric (29), (30) satisfies all the gravitational equations except $R_{01}=R_{03}=0$ for arbitrary small a, b, c. The same conditions give in the first approximation

$$\mu a+a''+a'\left(\operatorname{ctg}\theta-\mu\operatorname{ctg}\frac{\theta}{2}\right)+\mu r^{-1}\left(b'+\mu b\operatorname{ctg}\frac{\theta}{2}\right)-b_{,r}{}'$$
$$-b_{,r}\left(\operatorname{ctg}\theta+(\mu-\mu^2)\operatorname{ctg}\frac{\theta}{2}\right)=-2e^{2\beta}r^{1-\mu}\sin^{2(\mu^2-\mu)}\frac{\theta}{2}$$
$$\times\left[\mu\dot\mu\left(\ln\sin\frac{\theta}{2}-\frac{1}{2}\ln r\right)-\dot\mu-\mu\dot\beta\right],$$
$$r^2b_{,rr}+(\mu^2-\mu)b+r\left(\mu^2a\operatorname{ctg}\frac{\theta}{2}-\mu a'\right)$$
$$+r^2\left(\mu a_{,r}\operatorname{ctg}\frac{\theta}{2}-a_{,r}{}'\right)$$
$$=-2e^{2\beta}r^{2-\mu}\sin^{2(\mu^2-\mu)}\frac{\theta}{2}\left[\operatorname{ctg}\frac{\theta}{2}\left(\dot\mu(1-\mu)-\frac{\mu\dot\mu}{2}\ln r+\mu\dot\beta\right)\right.$$
$$\left.-\mu\dot\mu\operatorname{tg}\frac{\theta}{2}\ln\sin\frac{\theta}{2}\right]. \tag{32}$$

Here, the prime denotes the derivative with respect to θ, the dot the derivative with respect to t, and the index r after the semicolon the derivative with respect to r. The dependence of a and b on r has the form

$$a=r^{2-\mu}[H(\theta,t)+W(\theta,t)\ln r],$$
$$b=r^{2-\mu}[E(\theta,t)+V(\theta,t)\ln r] \tag{33}$$

and satisfies the condition of smallness (28). At the same time

$$\mu W+W''+W'\left(\operatorname{ctg}\theta-\mu^2\operatorname{ctg}\frac{\theta}{2}\right)$$
$$+2(\mu-1)V'+V\left[(\mu-2)\operatorname{ctg}\theta\right.$$
$$\left.+\mu(4\mu-2-\mu^2)\operatorname{ctg}\frac{\theta}{2}\right]=\mu\dot\mu e^{2\beta}\sin^{2(\mu^2-\mu)}\frac{\theta}{2}$$
$$2(1-\mu)V+W\mu\operatorname{ctg}\frac{\theta}{2}-W'=\mu\dot\mu e^{2\beta}\sin^{2\mu^2-2\mu-1}\frac{\theta}{2}\cos\frac{\theta}{2}. \tag{34}$$

As $\theta\to 0$, we obtain from here

$$W\propto|\theta|^{2\mu^2-2\mu},\quad V\propto|\theta|^{2\mu^2-2\mu-1}.$$

This contradicts the condition (28). We obtain the same violation of this condition as $\theta\to 0$ for the solution (29), (31), where the conditions $R_{01}=R_{03}=0$ give us

$$[(a'-b_{,r}-\dot c)\sin\theta]'=-2r^2\sin\theta e^{\gamma-\nu}\dot\nu_{,r},$$
$$\nu\to Kr^{\lambda}\ln\sin\frac{\theta}{2}\quad(\theta\to 0),$$
$$b_{,rr}-a_{,r}{}'-\dot c_{,r}-2\dot c r^{-1}=-(\dot\nu\operatorname{ctg}\theta+2\dot\nu')e^{\gamma-\nu}. \tag{35}$$

Since near the singularity $\theta=0$ the smallness condition is violated, we cannot generalize the solutions (13), (16)–(18). If the nondiagonal components in (29) are not small, Einstein's equations become so complicated that they cannot be solved. If at the same time we take the functions A, B, C, D from (30) or (31), they do not, as can be shown, have solutions. In the general case we not only cannot find solutions but we do not even know whether it is even possible for there to be a nonstatic generalization of the metrics near the end of a Weyl singularity or not. The latter possibility would mean that near the ends of the singularity $\mu(r)$ does not change and the possibility of transforming a linear naked singularity by evolution into a black hole is not realized.

6. MOST GENERAL FORM OF LINEAR, POINT, AND OTHER SOURCES

The most general solution of the gravitational equations near timelike singularities containing in vacuum four physically arbitrary functions of three variables and keeping the same form in the presence of matter is the spatially oscillatory solution of Ref. 6. However, as was pointed out at the end of the second section of this paper, it cannot be classified by one of the source types. The most general solution near singularities of the types we have considered will evidently be a generalized spatially anisotropic solution, containing in vacuum three physically arbitrary functions of three variables $x^{\alpha}=(t, y, z)$.[6] Near the singularity $x=0$ it has the form (5) with

$$\gamma_{\alpha\beta}=x^{2p_l}l_{\alpha}l_{\beta}-x^{2p_m}m_{\alpha}m_{\beta}-x^{2p_n}n_{\alpha}n_{\beta}. \tag{36}$$

The exponents in it and the nine quantities $l_{\alpha}, m_{\alpha}, n_{\alpha}$, combined into the components of the three three-dimensional vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$, are arbitrary functions of x^{α} connected by a number of relations needed for the solution (36) to satisfy in the leading terms as $x\to 0$ the Einstein equations. The Kasner exponents (p_l, p_m, p_n) are related in the same way as (p_1, p_2, p_3) in (10). This ensures fulfillment of the condition $R_{11}=0$. From $R_{1\alpha}=0$ we obtain the three relations

$$[\mathbf{mn}]_{\alpha}\partial p_l/\partial x^{\alpha}=(p_l-p_n)\mathbf{m}\operatorname{rot}\mathbf{n}+(p_m-p_l)\mathbf{n}\operatorname{rot}\mathbf{m},$$
$$[\mathbf{nl}]_{\alpha}\partial p_m/\partial x^{\alpha}=(p_m-p_l)\mathbf{n}\operatorname{rot}\mathbf{l}+(p_n-p_m)\mathbf{l}\operatorname{rot}\mathbf{n},$$
$$[\mathbf{lm}]_{\alpha}\partial p_n/\partial x^{\alpha}=(p_n-p_m)\mathbf{l}\operatorname{rot}\mathbf{m}+(p_l-p_n)\mathbf{m}\operatorname{rot}\mathbf{l}, \tag{37}$$

where the vector operations are to be carried out formally, as in flat Euclidean space. We write the remaining components of Einstein's equations in covariant components:

$$0=R_{\alpha\beta}=P_{\alpha\beta}+Q_{\alpha\beta},$$

$$2Q_{\alpha\beta}=\frac{\partial \varkappa_{\alpha\beta}}{\partial r}+\frac{1}{2}\varkappa_{\alpha\beta}\varkappa_\gamma{}^\gamma-\varkappa_\alpha{}^\gamma\varkappa_{\beta\gamma},\quad \varkappa_{\alpha\beta}=\frac{\partial \gamma_{\alpha\beta}}{\partial x}. \tag{38}$$

Here, $P_{\alpha\beta}$ is the three-dimensional curvature tensor constructed from $\gamma_{\alpha\beta}$. Since all terms in the expression for $Q_{\alpha\beta}$ cancel each other, for approximate fulfillment of the gravitational equations it is necessary that among the terms in $P_{\alpha\beta}$ there be none with higher order in x^{-1} than in $Q_{\alpha\beta}$. This leads to one further equation,

$$\mathbf{k}\,\mathrm{rot}\,\mathbf{k}=0, \tag{39}$$

where **k** is that one of the vectors **l**, **m**, **n** that corresponds to the negative exponent $p_k<0$. If the condition (39) is not satisfied, the solution (36) goes over into a more general oscillatory solution.[6]

The coordinate system (5) admits three arbitrary transformations of the coordinates x^α into each other. They can be used to annihilate three of the nine functions l_α, m_α, n_α. In order to annihilate simultaneously two of the three components of the vector **k**, it must have the form $\mathbf{k}=\psi(x^\alpha)\cdot\nabla\chi(x^\alpha)$; then, choosing $\chi(x^\alpha)$ as coordinates, we obtain $k_\alpha=(0,0,\psi)$. But a necessary and sufficient condition for this is (39). Similarly, for the simultaneous annihilation of the same components of two vectors, say l_2 and m_2, we require fulfillment of the condition (39) for the vector $\mathbf{k}=\mathbf{l}\times\mathbf{m}$. Therefore, for an oscillatory solution, in which this condition is not satisfied by any of the vectors **l**, **m**, **n**, $\mathbf{l}\times\mathbf{m}$, $\mathbf{m}\times\mathbf{n}$, $\mathbf{n}\times\mathbf{l}$, it is possible to annihilate, for example, l_0, m_2, n_3 or any other set of nonidentical components, one in each of **l**, **m**, **n**.

In our case, the condition (39) is satisfied by one of the vectors, for example, **n**, and together with it $\mathbf{l}\times\mathbf{n}$ and $\mathbf{m}\times\mathbf{n}$ also. Directing along them the axes x^α, we can achieve that $n_0=n_2=l_2=m_0=0$. But at the same time we also annihilate the component Q_{02} in (38), and it becomes necessary to ensure fulfillment of the equation $R_{02}=0$. For the oscillatory solution, all the components of $Q_{\alpha\beta}$ are nonzero and no additional problems arise.

If in accordance with (38) we require vanishing of

$$\begin{aligned}P_{02}=&-\dot p_{n,y}\ln x-(\ln n_3)_{,y}{}^{\boldsymbol\cdot}\\&+(p_l\ln x+\ln l_0)_{,y}(p_n\ln x+\ln n_3)\\&+(p_n\ln x+\ln n_3)_{,y}(p_m\ln x+\ln m_2)^{\boldsymbol\cdot}\\&-(p_n\ln x+\ln n_3)^{\boldsymbol\cdot}(p_n\ln x+\ln n_3)_{,y}\\&=C_1\ln^2 x+C_2\ln x+C_3,\end{aligned} \tag{40}$$

then this leads to very strong restrictions on the functions that occur in (36). We have a situation analogous to that which we encountered in the case of simple linear sources; moreover, (40) and (23) are almost identical. Therefore, here too we add a small nondiagonal term in order to satisfy the condition $R_{02}=0$. The solution (36) takes the form

$$\begin{aligned}ds^2=&-dx^2+x^{2p_l}(l_0dt+l_3dz)^2-x^{2p_m}(m_2dy+m_3dz)^2-x^{2p_n}n_3{}^2dz^2\\&+2x^2(\lambda_1\ln^2x+\lambda_2\ln x+\lambda_3)\,dtdy,\quad p_n<0,\end{aligned} \tag{41}$$

where λ_i can be expressed in terms of C_i from (40) in accordance with (26) with p_2 replaced by p_3. At the same time, the conditions (37) go over into

$$\begin{aligned}&\dot p_l=(p_n-p_l)(\ln n_3)^{\boldsymbol\cdot}+(p_m-p_l)(\ln m_2)^{\boldsymbol\cdot},\\&p_{m,y}=(p_l-p_m)(\ln l_0)_{,y}+(p_n-p_m)(\ln n_3)_{,y},\\&l_0m_2p_{n,z}-l_3m_2\dot p_n-l_0m_3p_{n,y}=(p_n-p_m)[l_3\dot m_2+l_0(m_{3,y}-m_{2,z})]\\&\qquad+(p_l-p_n)[m_2(l_{0,z}-\dot l_3)-m_3l_{0,y}].\end{aligned} \tag{42}$$

The first two of them are analogous to (22). Note that if $l_3=0$, then to the solution (41) it is necessary to add a small nondiagonal term g_{03} to satisfy the condition $R_{03}=0$, and for $m_3=0$ to satisfy $R_{23}=0$ it is necessary to add a small term g_{23}. The form of these terms is analogous to that of g_{02} in (41). Thus, in the special case of simple linear sources we obtain the solution (25) directly with the additional term. A small nondiagonal term must also be added to the generalized anisotropic solution of Lifshitz and Khalatnikov,[7] which is related to (5), (36) by the transformation $t\rightleftarrows x$ and a change of the signature.

We consider the nature of the source for the field (41). Its diagram will be hatched parallel to the y axis. Since the space posseses no symmetry, we cannot distinguish the diagrams for a linear singularity and a source of the type of the γ metric for $\mu>1$, it being unclear which of the coordinates y or z will be the generalization of the rotational coordinate φ for the Weyl and simple linear sources. We can determine this only if we know the metric of the complete space which goes over in the limit of small x into (41). The generalization of φ will then be the coordinate y or z with respect to which the shift transformation has fixed points outside the singularity on a line that either joins the two ends of the singularity or consists of two lines joining these ends to the spatial infinity. In the case of axial symmetry, these lines go over into the rotation axis. If the generalization of φ is y, then the solution (41) has a linear source; but if it is z, then the source is of the type of the γ metric for $\mu>1$.

If a negative exponent p_k corresponds to the time coordinate and **k** coincides with **l**, we obtain a solution from (41), (42) by the transformation $t\rightleftarrows z$ and a change of the signature. This case corresponds to a point source of negative mass.

We now investigate the effect on the investigated solutions of matter and nongravitational fields. It is shown in Ref. 6 that hydrodynamically moving matter does not change the form of the solution (36). The influence of fields on it was considered in Ref. 10. If the singularity is the source of a scalar field, then the form of the relations (10) is changed. At the same time, all the exponents p_l, p_m, p_n can be positive, this corresponding to a point source with positive mass. The electromagnetic field does not affect the metric near the singularity for $p_l>0$,[10] i.e., in the case of a linear singularity or a source of the type of the γ solution for $\mu>1$. But near a charged point singularity the form of the solution is strongly changed.

7. CONCLUSIONS

We have considered a number of solutions with timelike singularities, beginning with the γ metric and ending with the generalized spatially anisotropic solution (36). Strictly speaking, they are all particular cases of the last solution. And whereas in them the function μ, which is related to the exponents by (11), depends on three variables, for the simple linear sources it depends on t and z, for the Weyl singularities on the single coordinate z, and for the γ metric it is constant.

However, we can learn more about the properties of these particular solutions than about the general case (36). For the γ metric (3) we know the form of the entire space-time and can investigate it in detail. Weyl singularities also admit investigation not only near the singularity but also near its ends. But for simple linear sources this last is no longer possible. We do not even know whether at their ends there can be directional singularities, either containing an infinitely distant point (of the type (13) for $\mu \geqslant 2$) or without it (of the type (16)–(18) for $\lambda > 2$). However, the asymptotic axial symmetry of such sources makes it possible to determine the rotational coordinate z, and for the analysis of their diagrams there is no need to know the form of the space-time far from the singularity.

As was already pointed out in Sec. 2, all sources of the gravitational field of zero volume, besides those not having a type or containing an infinitely distant point, can be either point sources, or linear sources, or have the same type as the γ metric for $\mu > 1$. We see that all these source types are described by the same solutions, differing only in the signs of the exponents. We consider them in order, beginning with the point sources, which in the absence of nongravitational fields always have negative mass. This rules out the possibility of their being formed by collapse. But if a point singularity is the source of a scalar field, the total mass of it and of this field can be positive. Point singularities have internal structure. This can be seen from the fact that their gravitational field does not possess symmetry and depends on three physically arbitrary functions of the time and two nonradial coordinates.

Linear sources also possess internal structure in the plane orthogonal to them. In the process of their evolution, not only this structure but also the mutual disposition of the different parts of the singularity change. Their mass is always positive.

The third type of source is described by the diagram shown in Fig. 2b. Its properties differ from those of the point and linear sources. Their investigation has shown that we are here dealing with a new type of source, one impossible in a space with finite curvature. Since it must have a large mass, its formation by collapse is impossible, as in the case of directional singularities including an infinitely distant point. Therefore, to investigate the validity of the cosmic censorship hypothesis it is necessary to clarify the possibility or impossibility of the formation by collapse of linear naked singularities and sources that do not have a type. None of the other naked singularities can be formed in such a way.

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