Quantum corrections to kinetic coefficients of pure conductors

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The quantum corrections to the kinetic coefficients of conductors are discussed. The case of a pure conductor, when the quantum corrections are necessitated by interference produced in quasielastic electron scattering by phonons, is investigated in detail. An explicit expression is obtained for the cooperon with allowance for the phase relaxation due to the inelasticity. If only one quasielastic mechanism is effective, quantum corrections can be obtained only if $\overline{\omega}\tau \ll 1$, where $\overline{\omega}$ is the characteristic energy transferred to the electron by a collision and τ is the relaxation time. If the scattering is due to the simultaneous action of elastic and quasielastic mechanisms, this condition is not necessary for interference to set in. Calculation shows, however, that the phase-relaxation time depends on the ratio of $\overline{\omega}$ to the inelastic-relaxation time τ_{in}.

1. INTRODUCTION

Our purpose here is to develop a theory of the quantum corrections that must be introduced in the kinetic coefficients of pure conductors to allow for interference between an electronic state and a time-reversed state. We have in mind conductors in which there is no structural disorder, and the interference of interest to us is due to quasielastic electron-phonon interaction.

The conductors heretofore considered from this viewpoint have been dirty ones in which the principal momentum-relaxation mechanism is elastic scattering by defects, and it is the latter which determine the mean free path l. In the lowest order in the parameter $(pb)^{-1}$, where p is the momentum ($\hbar = 1$) and is assumed to be small, the quantum corrections are determined by the sum of the so-called "fan diagrams" (by the cooperon)—see the papers by Gor'kov, Larkin, and Khmel'nitskiĭ[1] and of Abrahams and Ramakrishnan.[2]

In the present paper we investigate the case of relatively low densities and high temperatures, when the electron gas can be either degenerate or nondegenerate, i.e., it obeys Boltzmann statistics. We consider the investigation of the quantum corrections in the latter situation as the most interesting, for in this case it is possible to achieve an electron density such that the electron-electron interaction plays no role. As shown by Al'tshuler and Aronov,[3] single-particle interference corrections and corrections for the electron-electron interaction coexist in a degenerate electron gas and it becomes necessary to separate them in the experiment. Under Boltzmann statistics, on the other hand, conditions can be created for the study of the interference phenomena in pure form.

The necessary (but, as we shall show, far from sufficient) condition for the existence of quantum interference phenomena in pure conductors is quasielasticity of the electron-phonon collisions. For electron scattering by acoustic phonons this condition is known to be satisfied in semiconductors (in the case of Boltzmann statistics) at practically any temperature, in semimetals at sufficiently low temperatures, and in metals below the Debye temperature. It is satisfied for scattering by optical phonons at temperatures higher than the maximum phonon frequency.

The random field produced by the phonons is in this case practically static. An electron can move in such a field without relaxation of the wave-function phase for a relatively long time, so that interference is also possible. If the conductor is pure enough, this interference-formation process should predominate at relatively high temperatures.

Our task is to develop an analytic theory of quantum corrections, i.e., to calculate the cooperon in this situation. To this end, we sum the "fan" diagrams that describe quasielastic scattering of electrons by phonons. The result is an integral equation for the cooperon. Its solution is analyzed for the case

$$\overline{\omega} \ll T, \tag{1}$$

which is the requirement that the characteristic frequency transfer be small compared with the temperature T. The analysis result depends substantially on the relation between $\overline{\omega}$ and the relaxation time τ. It turns out that the fan diagrams are pronounced only if

$$\overline{\omega}\tau \ll 1. \tag{2}$$

An explicit expression for the cooperon [Eq. (7)] is obtained for this case and differs from that obtained when pure elastic scattering dominates [Eq. (5)]. This difference can play a major role, for example, in the analysis of the kinetic coefficients as functions of the magnetic field.

Under the inverse condition

$$\overline{\omega}\tau \gg 1, \tag{3}$$

however, the phase relaxes completely during the time τ between two collisions. As a result the "fan" diagrams are not distinguished from other cross diagrams and the quantum corrections of interest to us do not occur.

These corrections can, however, appear also at $\overline{\omega}\tau \gg 1$ if, for example, two scattering mechanisms act simultaneously. One of them, elastic, is mainly the one that determines the relaxation time, while the other, quasielastic, is characterized by a time $\tau_{in} \gg \tau$ between collisions. This situation is also

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analyzed in this article.

Let us examine how the inelasticity of the scattering can influence a cooperon. In the case of elastic scattering the sum of the fan diagrams is of the form[1]

$$C=\int(dq)\,(Dq^2-i\Omega)^{-1}, \tag{4}$$

where Ω is the frequency of the external field, D is the electron diffusion coefficient, $(dq)=d^u q/(2\pi)^u$, $u=2$ or 3 is the dimensionality of the space that characterizes the electron problem. In the case of greatest interest, that of a two-dimensional electron gas ($u=2$) this expression diverges in the static limit as $\Omega\to 0$. Therefore the magnitude of the quantum correction is determined in fact by the mechanism that eliminates the divergence in (4). Such a mechanism can be, for example, scattering by magnetic impurities, combined with spin-orbit scattering; this scattering was investigated for this problem by Larkin, Hikami, and Nagaoka[4] and by Lee.[5] Such a mechanism may turn out to be the most important in "dirty" conductors. (In "pure" semiconductors, transitions with spin flip can occur also in quasielastic scattering by phonons.) The divergence is cut off also by the inelasticity of the electron-phonon scattering, and this mechanism can prevail in the case of interest to us, of relatively high temperatures and pure materials. The influence of inelastic scattering is usually taken into account phenomenologically, by adding to the denominator of (4) a term τ_φ^{-1}, where τ_φ is the so-called phase-relaxation time of the electron wave function, i.e., by writing C in the form

$$C=\int(dq)\,(Dq^2-i\Omega+\tau_\varphi^{-1})^{-1}. \tag{5}$$

In the case of essentially inelastic scattering by phonons $(\bar\omega\gtrsim T)$ τ_φ equals the departure time τ_{in} for the inelastic processes.[7,8]

If, however, the electron-phonon scattering is quasielastic, i.e., if the inequality (1) holds, it may turn out that one scattering act changes the electron phase little and the phase-memory loss is produced by many such acts.

A situation in which the phase relaxation is due to quasielastic scattering while the quantum interference is produced in scattering from elastic impurities was discussed by Al'tshuler, Aronov, and Khmel'nitskiĭ.[9] They presented for τ_φ an estimate that they interpreted in the following manner.

Let an energy $\bar\omega$ be imparted to the electron in one scattering act. The energy change $\delta\varepsilon$ during the time τ_φ of interest to us is then of order

$$(\delta\varepsilon)^2=(\bar\omega^2/\tau_{in})\,\tau_\varphi$$

(it is assumed that the energy changes diffusely). On the other hand, the electron state can interfere with time-reversed states if the condition $\delta\varepsilon\tau_\varphi\lesssim 1$ is satisfied. This leads to the estimate

$$\tau_\varphi\sim\tau_{in}^{1/3}\,\bar\omega^{-2/3}. \tag{6}$$

The foregoing arguments invoke neither a specific cooperon-formation mechanism nor a quasielastic scattering mechanism.

The present calculation of the quantum corrections for the case of quasielastic scattering of electrons by phonons leads to the following results. When condition (2) is satisfied, the cooperon is described by the expression

$$C=\int(dq)\int_0^\infty dt\,\exp\left[-Dq^2t+i\Omega t-\frac{t^3}{\tau_\varphi^3}\right], \tag{7}$$

and it is found here that τ_φ agrees with the estimate (6). If, however, the interference is due to pure elastic scattering by impurities and the phase relaxation is determined by quasielastic electron-phonon scattering, the cooperon takes the form (7) if the condition (2), in which τ is replaced by τ_{in}, is met and the expression for the time τ_φ also agrees with the estimate (6). This result coincides with the estimate formulated in Refs. 8 and 9. We wish, however, to discuss the foregoing physical arguments in somewhat greater detail, since the condition for their validity remain unclear to us. This uncertainty was one of the stimuli for a consistent calculation of the cooperon C.

If arguments of the type advanced above are to be valid in the inelastic-scattering case of interest to us, the electron energies before and after the collision should, in our opinion, to be determined with accurately, namely, condition (3) and the ensuing inequality $\bar\omega\tau_\varphi\gg 1$ must be satisfied. On the other hand, the estimate (6) can be rewritten in the form

$$\tau/\tau_\varphi\sim(\bar\omega\tau)^{2/3}.$$

Since, however, the onset of interference requires satisfaction of the condition $\tau\ll\tau_\varphi$, the inequality (2), the inverse of (3), is automatically satisfied. This is precisely why the physical interpretation given in Ref. (8) for the estimate (6) is not quite clear.

If, however, the inequality (3) holds, no interference takes place at all, owing to the substantial phase relaxation during each inelastic collision, provided only that the quasielastic mechanism plays the principal role in the formation of the Cooper channel. If, on the other hand, the principal interference-formation mechanism is scattering by elastic impurities, which determines the time τ against the background of which a weak phonon increment appears, then a logarithmic correction to the kinetic coefficient appears in the case $\bar\omega\tau_{in}\gg 1$, as indicated in Ref. 9. But the characteristic time under the logarithm sign is in this case not equal to (6) even in quasielastic scattering of electrons by phonons, and it is simply of the order of the time τ_{in} between two electron-phonon scattering events. Thus, the estimates of τ_φ is found in this case to be the same as in essentially inelastic scattering. This means that when the difference between the electron energies before and after the collision is well defined $(\bar\omega\tau_{in}\gg 1)$ the interaction with the phonons only disrupts the interference.[1)] We point out that the concept of electron diffusion in energy is certainly applicable in precisely this situation. The degree of its applicability in the case of the inverse inequality, as already noted above, remains unclear to us.

By way of illustration we present here the specific form of the inequality (1) for the case of a Boltzmann gas having an isotropic quadratic spectrum $\varepsilon_p=p^2/2m$ and interacting with acoustic phonons. For a sample of thickness d much larger than the electron de Broglie wavelength p^{-1}, we have

$\overline{\omega}\approx(Tmw^2)^{1/2}$, where w is the speed of sound. Inequality (1) can therefore be rewritten in the form $T \gg mw^2$. If, however, the electrons are concentrated in a thin surface layer of a thick sample, so that the thickness $d \ll p^{-1}$, we have $\overline{\omega}\approx w/d$. We obtain then $w/d \ll T$ in place of (1).

2. ANALYTIC CONTINUATION OF THE EXPRESSION FOR THE COOPERON

We begin for the sake of argument with the case of electron scattering by acoustic phonons and use the Matsubara technique. The kinetic coefficients of interest to us are expressed in terms of Matsubara loop diagrams having two vector vertices and continued analytically into the half-plane $\mathrm{Im}\Omega>0$. In the expressions for the quantum corrections these loops have each an insert comprising the sum of fan diagrams[1,2] of the type shown in Fig. 1a. We shall represent this insert graphically as shown in Fig. 1b. If the electron-scattering amplitude depends on the scattering angle, contributions to the calculation of the response of the current are made also by the diagrams of the type shown in Fig. 1c. All these diagrams contain a block P_M (Fig. 1b) which we now proceed to analyze.

In the case of inelastic scattering the object P_M of interest to us depends on three Matsubara frequencies: ε_n of the electron, Ω_n of the external field, and ω of the energy transfer; the sequence of the variables is clear from Fig. 1b. To calculate the kinetic coefficient in the case of isotropic scattering we need only the sum at $\mathbf{p}=-\mathbf{p}_1$.

The quantity P_M, as a function of the three complex variables ε, Ω, and ω, has seven cuts determined by the vanishing of the imaginary parts of ε(I), ω(II), Ω(III), as well as of the sums $\varepsilon+\omega$(IV), $\varepsilon+\Omega$(V), $\varepsilon+\omega+\Omega$(VI) and $2\varepsilon+\Omega+\omega$(VII) (see Ref. 10). Figure 2 shows the pattern of the cuts on the $\mathrm{Im}\varepsilon\otimes\mathrm{Im}\omega$ plane at $\mathrm{Im}\Omega>0$ (this is precisely the object we need in order to calculate the retarded response of the current).

The regions corresponding to different analytic branches of P_M, marked by Arabic numerals, and the branches themselves, will be designated P^R_μ, where μ is the number of the region and R means that the continuation was made into the region $\mathrm{Im}\Omega>0$. The unnumbered regions make no contribution in the principal approximation, since they contain only products of R- or A-electron Green's functions. (As usual, in our problem we calculate the response of the current to a vector potential that depends only on the frequency Ω.)

Let us ascertain which combinations P^M_μ enter in the cooperon corrections to the kinetic coefficients. Their calculation gives rise to a sum of the form

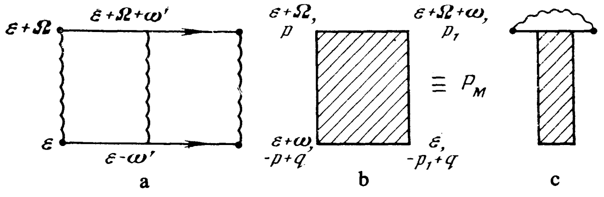

FIG. 1.

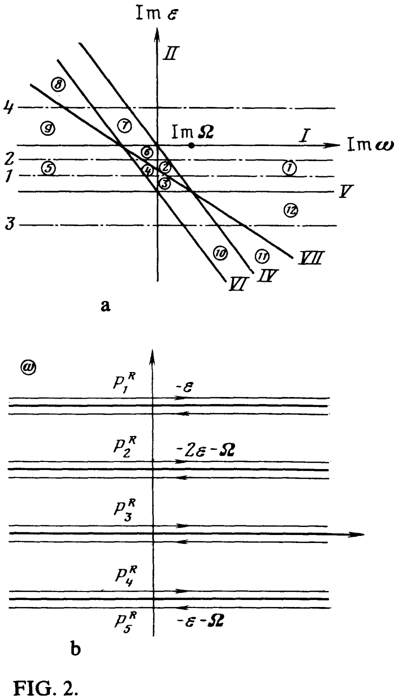

FIG. 2.

$$K_M(\varepsilon,\Omega)=T\sum_{\omega}G_M(\varepsilon+\Omega+\omega)G_M(\varepsilon+\omega)P_M, \tag{8}$$

where the subscripts M label Matsubara functions; for brevity we omitted the discrete-frequency subscripts and the momentum arguments. It is convenient to sum in (8) by transforming to contour integrals in the complex ω plane. We use here the method described by Maleev.[10] The ensuing configuration of the cuts on the complex ω plane depends on the relation between ε and Ω, i.e., on which branch P^R_μ makes the contribution; in all, four cut configurations are possible.

By way of example, Fig. 2b shows the integration contour for one of the configurations corresponding to the intersection of Fig. 2a and the dash-dot line 1. We denote the contribution of this configuration by $K^R_1(\varepsilon\ \Omega)$; the subscripts of K are the numbers of the dash-dot intersection on Fig. 2a. Thus K^R_M consists of four branches K^R_i, each determined by its own configuration of cuts on the plane. We now sum the quantities $TK^R_M(\varepsilon,\Omega)G_M(\varepsilon)G_M(\varepsilon+\Omega)$ over ε. This is likewise best done by changing from summation to contour integration. The complex ω plane has three cuts passing through the points $\varepsilon=-\Omega/2$, $-\Omega$, 0. Let us illustrate briefly the summation procedure.

The cut configuration shown in Fig. 2b is possible only if $-\Omega_m<\varepsilon_n<-Q_m/2$. This means that in this complex ε plane the function $K^R_M(\varepsilon_n,\Omega_m)$ coincides with the function $K^R_1(\varepsilon,\Omega)$ defined for all ε and Ω. In this case the sum over the discrete frequencies can be expressed in the usual manner in terms of an integral along the edges of the cuts passing through $\varepsilon=-\Omega_m$, $-\Omega_m/2$. The summation in all the remaining regions is similar. This reduces the entire integration to integrals along the real axis, after which the continuation in the variable Ω is standard. The expression obtained for $K^R(\Omega)$ is valid in any inelasticity approximation. We, however, are interested in the first approximation: we shall retain the inelasticity only to the extent that it leads to cutoff of the C divergence. We therefore leave out of the expression for the loop the terms that obviously tend to zero as $\omega \to 0$. In consequence,

$$K^R(\Omega) = -\Omega \int \frac{d\varepsilon\, d\omega}{(2\pi i)^2} \frac{\partial n}{\partial \varepsilon} G^R(\varepsilon+\Omega) G^R(\varepsilon+\omega+\Omega)$$

$$\times G^A(\varepsilon) G^A(\varepsilon+\omega)$$

$$\times \{ n(\varepsilon+\omega)[P_2^R - P_4^R] + N(\omega) P + N(\omega+2\varepsilon) P^- \}. \quad (9)$$

Here $n(\varepsilon)$ is the Fermi function, $N(\omega)$ the Planck function;

$$2P = P_2^R + P_3^R - P_4^R - P_6^R, \quad (10)$$

$$2P^- = P_2^R + P_6^R - P_3^R - P_4^R. \quad (11)$$

We have again left out of (9) the momentum arguments of the Green's function. The function $K^R(\Omega)$ contains all the singularities of the interference contribution to the kinetic coefficients. Their calculation reduces to substitution of (9) in the corresponding expression, to multiplication by the vertices, and to integration over the momenta. We shall not perform this calculation here, and focus our attention on the calculation of τ_φ. To this end we must first determine the functions P_μ^R contained in (9).

3. EQUATION FOR THE FUNCTIONS P_μ^R

The integral equation for the Matsubara function P_M is shown in Fig. 3.

Analytic continuation transforms this equation into a set of equations that connect the functions P_μ^R with the functions P_ν^R and P_α^A (this connection is due to the cut (III)$\mathrm{Im}(\Omega + \omega - \omega') = 0$). The fact that the R- and A-functions do not separate in the equations complicates considerably the task of analytically continuing the equation for P_μ^R. Besides the picture of the cuts for the functions P_μ^R (Fig. 2a) we must consider the analogous picture for the functions P_μ^A. This picture can be obtained from Fig. 2a by reversing the signs of all the energies. We shall also number the regions P_μ^A correspondingly. Our next task is to transfer the grid of analytic branches for $P_{R(A)}$ to the integral term of the equation, since summation over ω' entangles partially these branches. We shall carry out this summation using contour integration in the ω' plane. The system of cuts for P_M in the $\mathrm{Im}\,\varepsilon \otimes \mathrm{Im}\,\omega'$ plane is shown in Fig. 4 (for the case $\mathrm{Im}\Omega_0 \equiv \mathrm{Im}(\Omega + \omega) > 0$); the cuts are numbered as in Fig. 2a. As for the numbering of the regions (branches), it must be recognized that to the left of cut III we have here the R functions (the numbering is as in Fig. 2a), and to the right the A functions, and in this region the numbering corresponds to that for the P_μ^A branches (see above).

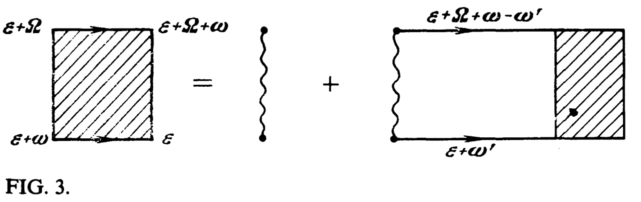

FIG. 3.

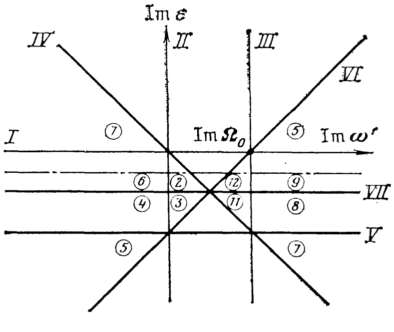

FIG. 4.

To obtain an equation for, say, P_μ^R we must thus cut Fig. 4 by a horizontal line that passes through the region numbered μ on the left of the cut III. All the points of intersection of this line with cuts I to VII will determine cuts in the complex ω' plane and those branches P_μ^R (on the left of cut III) or P_μ^A (on the right of this cut) which must be substituted in the integral term. The corresponding branches of the Green's functions are determined directly. Figure 4 shows as an example the dash-dot line that corresponds to the equation for the functions P_6^R, P_2^R, P_{12}^R and P_9^A. The equations for the A-functions are derived similarly.

We begin with the equation for the difference $P_2^R - P_6^R$. The right-hand side of the equation (Fig. 3) contains besides $G_M(\varepsilon+\omega')G_M(\varepsilon+\Omega+\omega-\omega')P_M(\omega')$ also the phonon function $\mathscr{D}_M(\omega-\omega')$. On going from summation over the discrete points ω' to a contour integral it is important to take into account (besides the cuts that determine the interchange of the P_M branches) the cut produced at $\omega' = \omega$ and due to the $\mathscr{D}$ function. Taking all the foregoing into account, it is easy to determine the analytic branches of the Matsubara functions in different regions of the ω' plane complex. The analytic structures of the right-hand sides of the equations for P_2^R and P_6^R respectively are shown in Figs. 5a and 5b. It can be seen that these figures differ only in the location of the cut $\omega = \omega'$. Therefore only the cuts with $\omega' = 0$ and $\omega' = \omega$ contribute to the equation for the difference $P_2^R - P_6^R$; the contributions of the remaining cuts cancel out. Ultimately the difference of interest to us in the integral term does not become linked with other combina-

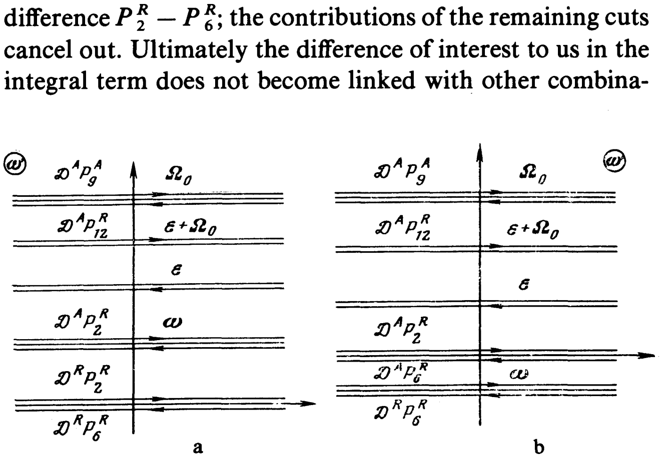

FIG. 5.

tions of the branches $P_{\mu}^{R(A)}$. This equation is analytically continued in standard fashion. The situation is the same for the difference $P_3^R - P_4^R$.

In the upshot, the combination P satisfies the equation

$$P(\mathbf{p},\mathbf{p}_1,\omega)=I(\omega,\mathbf{p}-\mathbf{p}_1)$$

$$-\int\frac{d\omega'(dp')}{(2\pi i)}P(\mathbf{p}',\mathbf{p}_1,\omega')I(\omega-\omega',\mathbf{p}'-\mathbf{p})$$

$$\times G^R(\varepsilon+\Omega+\omega-\omega',\mathbf{p}')G^A(\varepsilon+\omega',\mathbf{q}-\mathbf{p}')$$

$$\times[N(\omega')-N(\omega'-\omega)], \tag{12}$$

where

$$I(\omega,\mathbf{k})=\mathscr{D}^R(\omega,\mathbf{k})-\mathscr{D}^A(\omega,\mathbf{k}), \tag{13}$$

and $\mathscr{D}_{R(A)}$ are the phonon Green's functions. We have not written out in (12) the arguments ε, Ω, and q on which P depends as on parameters. It is similarly possible to derive a system of equations for terms 1 and 3 of Eq. (9). Analysis has shown that the first and third terms of (9) can be neglected in the principal approximation in the inelasticity.

4. SOLUTION OF INTEGRAL EQUATION FOR P IN DIFFERENT CASES. CALCULATION OF τ_φ

To be specific, we consider in detail the case of electron interaction with longitudinal acoustic phonons via a deformation potential, when

$$I(\omega,\mathbf{k})=-i\pi g^2\omega_k[\delta(\omega-\omega_k)-\delta(\omega+\omega_k)], \tag{14}$$

where $\omega_k = kw$ is the frequency of a phonon with wave vector $\mathbf{k}$.

It is convenient to integrate in (12) first with respect to $\varepsilon_{p'} = p'^2/2m$. It suffices then to take into account only the dependence of the product of the Green's functions on $|\mathbf{p}'|$; everywhere else we can put $|\mathbf{p}| = |\mathbf{p}'|$. At low values of the parameters $\Omega\tau$ and $qv\tau$ this integration yields the factor

$$\nu\int d\varepsilon_{p'}G^RG^A=\frac{2\pi\nu\tau}{1-i\tau(\omega-2\omega')}(1-i\Omega\tau-iqv\tau\cos\theta')^{-1}, \tag{15}$$

where $\nu(\varepsilon)$ is the density of the electronic states, θ' is the angle between the vectors $\mathbf{q}$ and $\mathbf{p}'$, $v = p/m$ is the electron velocity, and τ is the relaxation departure time:

$$\frac{1}{2\tau}=2\int\frac{d\omega(dk)}{2\pi}\operatorname{Im}\mathscr{D}^A(\omega,\mathbf{k})\operatorname{Im}G^A(\varepsilon-\omega,\mathbf{p}-\mathbf{k})N(\omega)$$

$$=\pi\nu g^2T. \tag{16}$$

We note next that expression (9) for $K_R(\Omega)$ contains not the function ω, but the combination $N(\omega)P \approx TP/\omega$. It is convenient therefore to analyze the equation not for P but for a new function

$$\mathscr{F}(\omega)=\frac{P(\omega)}{\omega}(1-i\Omega\tau-iqv\tau\cos\theta)^{-1}, \tag{17}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$. In the principal approximation in the small parameters $\Omega\tau$, $qv\tau$, and ω/T Eq. (12) is transformed into

$$(1-i\Omega\tau-iqv\tau\cos\theta)\mathscr{F}(\omega,\mathbf{p},\mathbf{p}_1)=\Phi(\omega,\mathbf{p}-\mathbf{p}_1)$$

$$-\nu T\int\frac{d\omega'}{\omega-2\omega'+i/\tau}\langle\mathscr{F}(\omega',\mathbf{p}',\mathbf{p}_1)\Phi(\omega-\omega',\mathbf{p}-\mathbf{p}')\rangle, \tag{18}$$

where $\Phi = I/\omega$, and the angle brackets denote averaging over the directions of the vector $\mathbf{p}'$.

It is now necessary to integrate over the directions of the vector $\mathbf{p}'$. To this end we note that $\mathscr{F}(\mathbf{p},\mathbf{p}_1,\mathbf{q})$ depends on the angle between $\mathbf{p}$ and $\mathbf{q}$, but not on the angle between $\mathbf{p}-\mathbf{p}_1$ and $\mathbf{q}$. This can be easily verified by examining the perturbation-theory series for $\langle\mathscr{F}(\omega)\rangle$.

In the lowest-order approximation in the indicated small parameters we need retain only the first two terms in the expansion of $\mathscr{F}(\omega,\theta)$ in Legendre polynomials, and can neglect the integral term in the equation for the term proportional to $\cos\theta$ (this term vanishes in the elastic approximation, and introduces a correction to the diffusion coefficient in the next higher orders). We ultimately arrive at the following expression for $\langle\mathscr{F}(\omega)\rangle$:

$$(1-i\Omega\tau+D(\varepsilon)q^2\tau)\langle\mathscr{F}(\omega,\mathbf{p}-\mathbf{p}_1)\rangle=\langle\Phi(\omega,\mathbf{p}-\mathbf{p}_1)\rangle$$

$$-\nu T\int\frac{d\omega'}{\omega-2\omega'+i/\tau}\langle\mathscr{F}(\omega',\mathbf{p}'-\mathbf{p}_1)\rangle\langle\Phi(\omega-\omega',\mathbf{p}-\mathbf{p}')\rangle, \tag{19}$$

where $D(\varepsilon) = v^2\tau/u$ is the diffusion coefficient of an electron with energy ε. We note that in the quasi-elastic limit the time τ coincides with the transport time.

To solve this equation it is convenient to transform to the time domain with respect to the variable ω. The transformed equation is

$$\hat{L}F\equiv(1+\Delta)F(t)-\int_{-\infty}^{t}\frac{dt'}{\tau}e^{-(t-t')\tau}F(t')\Psi(2t-t')$$

$$=-ig^2\cos(\omega_{\mathbf{p}-\mathbf{p}_1}t). \tag{20}$$

Here $F(t)$ is the Fourier transform of $\langle\mathscr{F}(\omega)\rangle$; $\Delta = D(\varepsilon)q^2\tau - i\Omega\tau$, $|\Delta|\ll 1$, $\Psi(t) = \langle\cos(\omega_{p-p'}t)\rangle$. To determine $\Psi(t)$ we must thus average over the directions of $\mathbf{p}'$ with $\mathbf{p}$ given ($|\mathbf{p}| = |\mathbf{p}'|$ to the accuracy of interest to us). If $d\gg p^{-1}$, the electrons and phonons can be regarded in this averaging as three-dimensional, and $\omega_{p-p'} = \omega_m \sin(\mathbf{p},\hat{\mathbf{p}}'/2)$. (Note that if it turns out here that $d < (D\tau_\varphi)^{1/2}$, the integration with respect to q in (7) is effectively two-dimensional and the correction is proportional to $\ln(\tau_\varphi/\tau)$.) In addition, it will be seen that an important role is played in the problem by values of $2t - t'$ such that the product $\omega_m(2t-t')\ll 1$. We can therefore put

$$\Psi(t)=1-\frac{\omega_m^2t^2}{2}\langle\sin^2(\widehat{\mathbf{p},\mathbf{p}'}/2)\rangle=1-\frac{\omega_m^2t^2}{4}. \tag{21}$$

The situation is somewhat different when the electrons are two-dimensional but the phonons are three-dimensional (a film making good acoustic contact with a substrate). At $d\ll p^{-1}$ the transferred phonon frequency is practically independent of the angle between $\mathbf{p}$ and $\mathbf{p}_1$, but is determined by the value $q_\perp$ of the phonon wave vector perpendicular to the surface. In this case $\omega_{\mathbf{p}-\mathbf{p}'} \approx \omega_{q\perp}$, and in lieu of $\langle\Phi(\omega-\omega', \mathbf{p}-\mathbf{p}_1)\rangle$ we get in the right-hand side of (19)

$$\int dq_\perp |M(q_\perp)|^2 \Phi(\omega-\omega', q_\perp) \Big/ \int dq_\perp |M(q_\perp)|^2, \qquad (22)$$

where $M(q_\perp)$ is a diagonal matrix element of the operator $\exp(iq_\perp z)$ in the electronic state $\varphi(z)$. For $\Psi(t)$ we obtain in this case

$$\Psi(t) = \int dz |\varphi(z)|^2 |\varphi(z-wt)|^2 \Big/ \int dz |\varphi(z)|^4. \qquad (23)$$

The important values in this integral are $z \approx d$. We are interested in small enough values $t \sim \tau_\varphi$, much smaller than $d/w \equiv \omega_c^{-1}$ (otherwise the condition $\tau_\varphi \gg \tau$ cannot be satisfied). We therefore obtain for $\Psi(t)$ an expression similar to (21), but with $b\omega_c^2$ in place of $\omega_m^2/4$, where

$$b = \frac{d^2}{2} \int dz (\nabla_z |\varphi|^2)^2 \Big/ \int dz |\varphi|^4 \approx 1.$$

Thus, in all cases $\Psi(t)$ can be represented in the form

$$\Psi(t) = 1 - \alpha(t/\tau)^2, \qquad (24)$$

where the dimensionless parameter α is equal to $(\omega_m \tau/2)^2$ or $b(\omega_c \tau)^2$. The condition $\tau \ll \tau_\varphi$ forces us to regard it as small.

Equation (20) is easily solved by Evans's method.[11,12] This method is based on a relation between the kernel of the integral equation (20):

$$K\left(\frac{t}{\tau}, \frac{t'}{\tau}\right) = -(1-\Delta) e^{-(t-t')/\tau} \left[1 - \alpha\left(\frac{2t}{\tau} - \frac{t'}{\tau}\right)^2 \right]$$

and its resolvent $R(t, t')$

$$\int_{t/\tau}^{t'/\tau} d\xi K\left(\frac{t}{\tau}, \xi\right) R\left(\xi, \frac{t'}{\tau}\right) = K\left(\frac{t}{\tau}, \frac{t'}{\tau}\right) + R\left(\frac{t}{\tau}, \frac{t'}{\tau}\right). \qquad (25)$$

Noting that the kernel K satisfies the differential equation

$$\tau^3 K_{ttt} + 3\tau^2 K_{tt} + 3\tau K_t + K = 0, \qquad (26)$$

we differentiate (25) one, two, and three times, multiply by the coefficients contained in (26), and add up all these expressions. The result is a differential equation for the resolvent with respect to the first variable, while the second is fixed. We have thus in lieu of three integration constants three unknown functions of the variable t'. This functional leeway is eliminated by using Eq. (25) and the required number of its derivatives with respect to the first argument at an arbitrary point $t = t'$. Retaining for each derivative only the terms of highest order in the large parameter t/τ, we have

$$\tau^3 R_{ttt} + \tau^2[2+\Delta+\alpha(t/\tau)^2] R_{tt} + \tau[1+2\Delta+2\alpha(t/\tau)^2] R_t + [\Delta+\alpha(t/\tau)^2] R = 0. \qquad (27)$$

Since we are interested only in $t \gg \tau$, we seek a quasiclassical solution $R = \exp[\varphi(t)/\tau]$. In the lowest approximation in τ the equation for $\varphi(t)$ can be written in the form

$$[\tau\varphi'(t)+1]^2 \left[\tau\varphi'(t) + \Delta + \alpha\left(\frac{t}{\tau}\right)^2\right] = 0, \qquad (28)$$

and its unique quasiclassical solution is[2]

$$R(t, t') = \exp\left[-\Delta \frac{t-t'}{\tau} - \frac{\alpha}{3} \frac{t^3 - t'^3}{\tau^3}\right]. \qquad (29)$$

The answer of interest to us contains the integral of $\mathscr{F}(\omega)$ over all the frequencies ω, i.e., $F(0)$. The principal (diffusion) contribution to this quantity

$$F(0) = -ig^2 \int_{-\infty}^{0} dx \exp\left(\Delta x + \frac{\alpha}{3} x^3\right) \qquad (30)$$

cannot be reduced to the form $-ig^2(\Delta + \tau/\tau_\varphi)^{-1}$. The kinetic coefficients contain the integral of $F(0)$ with respect to q. In the two dimensional case we have $dq = d\Delta/4\pi D(\varepsilon)$, whence

$$\int_0^{\Delta_m} d\Delta F(0) \propto \int_0^\infty \frac{dx}{x} [1 - e^{-\Delta_m x}] e^{-\alpha x^3/3} \approx \ln\left(\frac{3}{\alpha \Delta_m^3}\right)^{1/3}, \qquad (31)$$

where $\Delta_m \sim 1$. If the argument of the logarithm is presented in the form $\tau_\varphi/\Delta_m \tau$, then $\tau_\varphi \approx \tau^{1/3} \omega_m^{-2/3}$, which coincides with the estimate (4). The temperature dependence of τ_φ is determined by the dependences of the relaxation time τ and of the characteristic frequency transfer. In the case of Boltzmann statistics we have for two-dimensional electrons and two-dimensional phonons $\omega_m \propto T^{1/2}$ and $\tau \propto T^{-1}$, therefore $\tau_\varphi \propto T^{-3/2}$; in the case of three-dimensional phonons, ω_c does not depend on T, and $\tau_\varphi \propto T^{-1/2}$. For three-dimensional electrons and phonons $\tau \sim T^{-3/2}$ and $\omega_m \propto T^{1/2}$, so that $\tau_\varphi \propto T^{-5/6}$.

Let us recall the validity limits of the approximation employed, when account is taken of one "fan" diagram (at $p^{-1} \ll d \ll (D\tau_\varphi)^{1/2}$):

$$\left(\frac{T}{T_c}\right)^{1/2} \ln \frac{T^{2/3}}{(T_c T_0)^{1/3}} \ll 1, \qquad (32)$$

where T_c is defined by $2\pi\nu(T_c) g^2 = 1$, and $T_0 = mw^2/2$. As the temperature is raised this parameter ceases to be small and we find it necessary to sum all the multi-cooperon diagrams.

We call attention to the following curious circumstance, which is typical of the present case. Namely, the quantum corrections become more and more important with rising temperature. At sufficiently high temperatures the contribution of the "fan" diagram to the kinetic coefficients becomes formally of the order of their classical values. This means in fact that the one-cooperon approximation is no longer valid. Since the sign of this contribution is the opposite of that of the classical value, this should lead, say, for mobility, to a faster decrease with rise of temperature (compared with the $T^{-5/2}$ law). The interesting problem of determining the law that governs this decrease has not yet been solved.

The results of calculations for scattering by acoustic phonons may create the impression that τ_φ always includes the departure relaxation time. To look into this question, we consider the case of piezoelectric scattering of electrons by acoustic phonons, when the transport and departure times differ parametrically. In piezoelectric scattering the interaction constant depends substantially on the magnitude and direction of the phonon wave vector: $g^2 \to g_0^2(\mathbf{p}, \mathbf{p}')|\mathbf{p}|^2/\times|\mathbf{p}-\mathbf{p}'|^2$ (in the quasielastic approximation we can put

$|\mathbf{p}| = |\mathbf{p}'|$) and g_0 depends only on $s = \sin(\mathbf{p}, \mathbf{p}'/2)$.

The calculation is similar to that in the case of strain-phonon scattering. The difference are the following:

1) The constant g^2 in the expression for $1/\tau$ is replaced by

$$\left\langle g_0{}^2 \frac{p^2}{|\mathbf{p}-\mathbf{p}'|^2} \right\rangle = \langle g_0{}^2/4s^2 \rangle;$$

in the three-dimensional case this quantity is equal to

$$\int\limits_{s_0}^{1} \frac{ds}{s} g_0{}^2(s),$$

and in the two-dimensional case

$$\int\limits_{s_0}^{1} ds g_0{}^2(s)/s^2(1-s^2)^{1/2}.$$

The lower limit of integration s_0 in this expression is determined by the mechanism that screens the piezoelectric fields, and can be quite small.

2) The integral term in Eq. (18) for the first spherical harmonic F_1 does not vanish in the elastic approximation. Allowance for this term causes the departure time in the expression for the diffusion coefficient D to be replaced by the transport time τ_{tr} that is proportional to

$$\left\langle g_0^2 \frac{p^2}{|\mathbf{p}-\mathbf{p}'|^2} (1 - \cos \widehat{\mathbf{p}, \mathbf{p}'}) \right\rangle.$$

In the three-dimensional case we have the ratio

$$a \equiv \frac{\tau}{\tau_{\mathrm{tr}}} = 2 \int\limits_{s_0}^{1} s ds g_0{}^2(s) \Big/ \int\limits_{s_0}^{1} ds \frac{g_0{}^2(s)}{s}, \tag{33}$$

and in the two-dimensional case the integrals in the numerators and denominators acquire additional factors $s^{-1}(1 - s^2)^{-1/2}$. We note that a can be much smaller than 1.

3) The function $\Psi(t)$ in the kernel of the integral equation takes the form

$$\Psi(t) = \frac{\langle g_0{}^2 \cos(\omega_{\mathbf{p}-\mathbf{p}'} t)/s^2 \rangle}{\langle g_0{}^2/s^2 \rangle} \approx 1 - \frac{t^2}{2} \frac{\langle \omega_{\mathbf{p}-\mathbf{p}'}^2 g_0{}^2/s^2 \rangle}{\langle g_0{}^2/s^2 \rangle}. \tag{34}$$

The value of the parameter α is thus changed. If the electrons and phonons are of the same dimensionality, then the coefficient of $t^2/2$ becomes $\omega_m{}^2\tau/2\tau_{tr}$. The parameter α is thus given by

$$\alpha = \omega_m{}^2 \tau^3 / 4\tau_{tr}. \tag{35}$$

The results are modified in a very interesting manner in the case of quasielastic scattering by optical phonons ($\omega_0 \ll T$). This case can be important in a real experimental situation, since the contribution of the quantum-mechanical corrections in scattering by acoustic phonons increases with increasing T. Scattering by optical phonons therefore becomes very important at high temperatures. In this case $T\tau_{\mathrm{tr}} \propto \mathrm{T}^{1/2}$ and the scattering of electrons by optical phonons causes condition (32) to be satisfied.

If optical-phonon dispersion is neglected, the function $\Psi(t) = \cos \omega_0 t$ becomes equal to unity not only as $t \to 0$ but also at the instants $\omega_0 t = 2\pi k$ (k are integers). Equation (20) can nonetheless be solved only by Evans's method. It is found that the corrections to $F(0)$ for the instants $2\pi k/\omega_0$ ($k \neq 0$) are exponentially small and should be discarded.

5. CASE OF TWO QUASIELASTIC SCATTERING MECHANISMS

We analyze now the case when there are two quasielastic scattering mechanisms characterized by energy transfers $\omega^{(1)}_{\mathbf{p}-\mathbf{p}'}$ and $\omega^{(2)}_{\mathbf{p}-\mathbf{p}'}$ and by departure relaxation times τ_1 and τ_2. This analysis will enable us to describe both the case of simultaneous scattering by acoustic and optical phonons (which occurs at high temperatures) and the case, important at low temperatures, of simultaneous action of impurity and phonon scattering. Of greatest theoretical importance here is the fact that the parameters τ/τ_φ and $\bar{\omega}\tau$ are independent, so that the case $\bar{\omega}\tau \gg 1$ can be studied when the concept of electron diffusion in energy might not be applicable.

In this case the phonon propagator $\mathscr{D}$ in the initial and integral terms is replaced by the propagator sum $\mathscr{D}^{(1)} + \mathscr{D}^{(2)}$. The end result is Eq. (20), in which τ is replaced by $(1/\tau_1 + 1/\tau_2)^{-1}$, and the function $\Psi(t)$ takes the form

$$\Psi(t) = \frac{\tau}{\tau_1} \langle \cos(\omega^{(1)}_{\mathbf{p}-\mathbf{p}'} t) \rangle + \frac{\tau}{\tau_2} \langle \cos(\omega^{(2)}_{\mathbf{p}-\mathbf{p}'} t) \rangle. \tag{36}$$

To simplify the final expressions we assume that $\tau_1 \ll \tau_2$. In addition, we neglect the inelasticity of the first scattering mechanism. Then

$$\Psi(t) = 1 - \frac{\tau_1}{\tau_2} \langle 1 - \cos(\omega^{(2)}_{\mathbf{p}-\mathbf{p}'} t) \rangle. \tag{37}$$

The simplest to analyze is the limiting case $\omega^{(2)}\tau \ll 1$. We demonstrate it for a situation in which the second mechanism is scattering by optical phonons and $\langle \cos(\omega^{(2)}_{\mathbf{p}-\mathbf{p}'} t) \rangle = \cos \omega_0 t$. The calculation here is completed analogous to the one above and yields

$$\ln \frac{\tau_\varphi}{\tau} = \int\limits_{\omega_0\tau}^{\infty} \frac{dx}{x} \exp\left[-\frac{1}{\omega_0 \tau_2} (x - \sin x) \right]. \tag{38}$$

This leads for $\omega_0 \tau_2 \ll 1$ an answer that coincides with the estimate (4). If, however, $\omega_0 \tau_2 \gg 1$, then $\tau_\varphi \sim \tau_2 = \tau_{in}$. The case $\omega_0 \tau \gg 1$ can also be analyzed by Evans's method and as a result $\sin x$ in the argument of the exponential acquires a factor 3/4. The estimate $\tau_\varphi \sim \tau_2 = \tau_{in}$ thus remains valid. It can be directly verified that this conclusion in fact does not alter the actual form of the quasielastic scattering.

It can be deduced from our results that the phase-relaxation mechanism depends on the dimensionless parameter $\bar{\omega}\tau_{in}$, and not on the quasielasticity parameter $\bar{\omega}/\varepsilon$. In particular, inelastic interaction in which the electron acts as a classical particle with a well defined energy ($\bar{\omega}\tau_{in} \gg 1$) can only disrupt the quantum interference between the electronic state and the time-reversed state.

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[1] Only diagrams with inelastic increments to the self-energy parts "work" in this case.

[2] Obviously, the solutions that decrease at $t \sim \tau$ cannot become diffusive and must therefore be discarded. This can be verified formally by using the initial conditions from (25) or by taking the limit as $\omega_m \rightarrow 0$ in the final expression for $F(t)$.

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