Rotational invariance and magnetoflexural oscillations of ferromagnetic plates and rods

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A theory of magnetobending oscillations is developed for thin, free ferromagnetic rods and plates, and it is shown that transverse rigidity arises if a magnetic field $\mathbf{H}_0$ is applied parallel to $\mathbf{m}_0$ and normal to $\mathbf{n}$, where $\mathbf{m}_0$ is the magnetic moment and $\mathbf{n}$ is the unit normal to the surface of the rod or plate. This rigidity leads to a replacement of the bending dispersion law by a sound law at low frequencies. A novel mechanism is proposed for low-temperature magnetostriction in elastically one- and two-dimensional magnets which is based on damping of the bending wave amplitude by the external field.

1. This paper is concerned with the dynamics of magnetoelastic interactions in ferromagnetic dielectrics whose elastic properties are highly anisotropic along one or two crystallographic directions. Such crystals may be regarded as quasi-one or quasi-two-dimensional systems with various distinctive properties. In particular, the dispersion law for the elastic oscillations is unusual and in the extreme case of noninteracting layers or chains reduces to the dispersion law for bending waves (cf., e.g., Ref. 1) in thin films, plates (membranes), and rods (chains).

The natural oscillations in magnetically ordered materials are mixed, or magnetoelastic.[2–5] However, the magnetoelastic coupling constants are generally quite small,[5] so that the elastic and spin waves are appreciably mixed only near magnetoacoustic resonance, i.e., for $\widetilde{\omega}_{\rm ph}(\mathbf{q})\approx\widetilde{\omega}_s(\mathbf{q})$, where $\widetilde{\omega}_{\rm ph}(\mathbf{q})$ and $\widetilde{\omega}_s(\mathbf{q})$ are the energies of the interacting phonon and magnon, and $\mathbf{q}$ is the wave vector. We will see below that a radically different situation arises when spin waves interact with bending waves in thin films (rods) or quasi-$2D$ (quasi-$1D$) systems; in this case, in addition to the usual repulsion between the bending and spin-wave branches near magnetoacoustic resonance,[2,4] the magnetic subsystem can appreciably alter the elastic properties even away from resonance. We will show that this interaction generates an effective transverse spin rigidity in free plates or rods and in quasi-$2D$ and quasi-$1D$ systems.

We note that in most magnetoelastic calculations one assumes that the elastic and magnetoelastic energies depend on a symmetric small-deformation tensor $\hat{u}$. Although translational invariance is satisfied, this approach violates rotational invariance (i.e., the energy should be independent of rotations of the system as a whole); this was first pointed out for magnetically ordered materials by Vlasov,[6,7] and somewhat later by Tierston[8] and Brown[9] (cf. also Ref. 5). A rigorous theory (invariant under both rotations and translations) must directly allow for the fact that local deformations are accompanied by rotation as well as displacement. If the system has an order parameter (e.g., magnetization) that is "attached" to the local crystallographic axes and rotates with them, the noncentral (dipole-dipole, spin-orbit) forces in the material will generate an additional elastic energy which is described by an antisymmetric tensor for small rotations ω. However, this fact has been neglected in the physics of magnetoelastic interactions, because the complete theory has not been needed to accommodate the available experimental data. The first observation requiring the use of a rotationally invariant thermodynamic potential was the finding in Ref. 10 that the velocities of transverse sound differ along and normal to the easy axis in $\mathrm{MnF_2}$. In what follows we will analyze another consequence of rotational invariance which is characteristic for bending waves propagating in low-dimensional magnets.

2. We will consider the simplest case of bending oscillations in a thin ferromagnetic uniaxial plate (film) in a magnetic field $\mathbf{H}_0$ directed perpendicular to the normal $\mathbf{n}$ to the plate ($\mathbf{n}$ is parallel to the y axis). We assume that the magnetic moment $\mathbf{m}_0$ points along $\mathbf{H}_0\|z$ and write the rotationally invariant free-energy density of the plate as the sum

$$\Phi=\Phi_m+\Phi_e+\Phi_{me}, \tag{1}$$

where

$$\Phi_m=\frac{1}{2}\alpha_1\left[\left(\frac{\partial\mathbf{M}}{\partial x}\right)^2+\left(\frac{\partial\mathbf{M}}{\partial y}\right)^2\right]+\frac{1}{2}\alpha_3\left(\frac{\partial\mathbf{M}}{\partial z}\right)^2-\frac{1}{2}KM_z^2-H_0m_z-\frac{1}{2}\mathbf{H}_m\mathbf{M}, \tag{2}$$

$$\Phi_e={}^1/_2c_{11}(E_{xx}+E_{yy})^2+{}^1/_2c_{33}E_{zz}^2+c_{13}(E_{xx}+E_{yy})E_{zz}+2c_{44}(E_{xz}+E_{yz})^2+{}^1/_2c_{66}(E_{xx}^2+E_{yy}^2+2E_{xy}^2), \tag{3}$$

$$\Phi_{me}={}^1/_2B_{11}(M_x^2+M_y^2)(E_{xx}+E_{yy})+{}^1/_2B_{13}M_z^2(E_{xx}+E_{yy})+{}^1/_2B_{31}(M_x^2+M_y^2)E_{zz}+{}^1/_2B_{33}M_z^2E_{zz}+{}^1/_2B_{44}(M_xM_zE_{xz}+M_yM_zE_{yz})+{}^1/_2B_{66}(M_x^2E_{xx}+M_y^2E_{yy}+2M_xM_yE_{xy}). \tag{4}$$

Here Φ_m and Φ_e are the energies of the magnetic and elastic subsystem; Φ_{me} is the magnetoelastic interaction energy; K is the anisotropy constant, $H_m=-4\pi\mathbf{n}(\mathbf{Mn})$ is the demagnetizing field, and the α_i are the inhomogeneous exchange constants; c_{ij} and B_{ij} are the elastic and magnetoelastic moduli. In order to satisfy the rotational invariance, we have written $\mathbf{M}$ for the magnetization in the intrinsic (local) coordinate system whose axes rotate together with the corresponding local region during the inhomogeneous deformations; for finite deformations we have also used the tensor $\hat{E}$, which is known to be invariant under rotations.[5,8,9] If we go from the intrinsic coordinate systems to the laboratory frame, the quantities in (1) transform as

$$\mathbf{M}=\hat{R}^{-1}\mathbf{m},\quad \hat{R}=\hat{I}+\hat{\omega}+{}^{1}/{}_{2}\hat{\omega}^{2}-{}^{1}/{}_{2}(\hat{u}\hat{\omega}+\hat{\omega}\hat{u}),$$
$$\hat{E}=\hat{u}+{}^{1}/{}_{2}(\hat{u}-\hat{\omega})(\hat{u}+\hat{\omega}),\tag{5}$$

where $\hat{I}$ is the identity operator, $\hat{u}$ and $\hat{\omega}$ are the tensors for infinitesimal deformations and rotations, and $\hat{R}$ is the orthogonal (length-preserving) operator for finite rotations, which we have expanded out to second order in the small tensors $\hat{u}$ and $\hat{\omega}$. The above expression for the operator $\hat{E}$ in terms of $\hat{u}$ and $\hat{\omega}$ is thus complete. After some straightforward algebra, we can write the components of the free energy (1) in the form

$$\Phi_{m}=\frac{1}{2}\alpha_{1}\left[\left(\frac{\partial\mathbf{m}}{\partial x}\right)^{2}+\left(\frac{\partial\mathbf{m}}{\partial y}\right)^{2}\right]+\frac{1}{2}\alpha_{3}\left(\frac{\partial\mathbf{m}}{\partial z}\right)^{2}$$
$$+\frac{1}{2}\left(K_{eff}+\frac{H_{0}}{m_{0}}\right)m_{x}^{2}+\frac{1}{2}\left(K_{eff}+\frac{H_{0}}{m_{0}}+4\pi\right)m_{y}^{2},\tag{6}$$

$$\Phi_{e}={}^{1}/{}_{2}c_{11}(u_{xx}+u_{yy})^{2}+{}^{1}/{}_{2}c_{33}u_{zz}^{2}+c_{13}(u_{xx}+u_{yy})u_{zz}+2c_{44}(u_{xz}^{2}$$
$$+u_{yz}^{2})+{}^{1}/{}_{2}c_{66}(u_{xx}^{2}+u_{yy}^{2}+2u_{xy}^{2})+{}^{1}/{}_{2}K_{eff}m_{0}^{2}\omega_{xz}^{2}$$
$$+{}^{1}/{}_{2}(K_{eff}+4\pi)m_{0}^{2}\omega_{yz}^{2}-{}^{1}/{}_{2}B_{44}m_{0}^{2}(u_{xz}\omega_{xz}+u_{yz}\omega_{yz}),\tag{7}$$

$$\Phi_{me}={}^{1}/{}_{2}B_{44}m_{0}(m_{x}u_{xz}+m_{y}u_{yz})$$
$$-K_{eff}m_{0}m_{x}\omega_{xz}-(K_{eff}+4\pi)m_{0}m_{y}\omega_{yz}.\tag{8}$$

Here we have neglected terms quadratic in m_x, $m_y \ll m_0$, u_{ij}, and ω_{ij} and written

$$K_{eff}=K+(B_{11}+B_{66}-B_{13})(E_{xx}^{(0)}+E_{yy}^{(0)})-(B_{33}-B_{31})E_{zz}^{(0)}$$

for the anisotropy constant, where we include the deformation

$$E_{xx}^{(0)}=E_{yy}^{(0)}=-\frac{1}{2}\frac{c_{33}B_{13}-c_{13}B_{33}}{(2c_{11}+c_{66})c_{33}-2c_{13}^{2}}m_{0}^{2},$$
$$E_{zz}^{(0)}=-\frac{1}{2}\frac{(2c_{11}+c_{66})B_{33}-2c_{13}B_{13}}{(2c_{11}+c_{66})c_{33}-2c_{13}^{2}}m_{0}^{2}$$

of the ground state. In principle, the quantities $E_{ii}^{(0)}$ should depend on the external field, which can flip the moment $\mathbf{m}_0$ or change the value of m_0^2 near the paramagnetic transition. These changes in $E_{ii}^{(0)}$ (or equivalently, in the lattice constant) correspond to the familiar magnetostriction in ferromagnets, which is generally quite small.

Expressions (7) and (8) show that the anisotropic interactions give rise to new terms in the components Φ_e and Φ_{me} of the free energy. These terms are a consequence of rotational invariance and contain the antisymmetric tensor $\hat{\omega}$, in contrast to the ordinary theory.

The expressions

$$\sigma_{xx}=(c_{11}+c_{66})u_{xx}+c_{11}u_{yy}+c_{13}u_{zz},\qquad \sigma_{xy}=\sigma_{yx}=c_{66}u_{xy},$$
$$\underset{zx}{\sigma_{xz}}=2c_{44}u_{xz}\pm{}^{1}/{}_{2}K_{eff}m_{0}^{2}\omega_{xz}\mp{}^{1}/{}_{4}B_{44}m_{0}^{2}(u_{xz}\pm\omega_{xz}),$$
$$\sigma_{yy}=c_{11}u_{xx}+(c_{11}+c_{66})u_{yy}+c_{13}u_{zz},\qquad \sigma_{zz}=c_{33}u_{zz}+c_{13}(u_{xx}+u_{yy}),$$
$$\underset{zy}{\sigma_{yz}}=2c_{44}u_{yz}\pm{}^{1}/{}_{2}(K_{eff}+4\pi)m_{0}^{2}\omega_{yz}\mp{}^{1}/{}_{4}B_{44}m_{0}^{2}(u_{yz}\pm\omega_{yz}),$$

for the stress tensor components follow readily from (6)–(8); in this case $\hat{\sigma}$ is no longer symmetric because of the presence of the tensor $\hat{\omega}$. The terms associated with the magnetoelastic interaction Φ_{me} (8) contribute little to $\hat{\sigma}$ and have been neglected in the above expressions for the σ_{ij}. Since $\sigma_{ij}=0$ for a free plate,[1] we get the expressions

$$u_{xx}=-y\frac{\partial^{2}\eta}{\partial x^{2}},\quad u_{xy}=0,\quad u_{xz}=-y\left(1-\frac{\lambda}{2}\right)\frac{\partial^{2}\eta}{\partial x\,\partial z},$$
$$u_{yz}=\frac{\lambda}{2}\frac{\partial\eta}{\partial z},$$
$$u_{yy}=\frac{y}{c_{11}+c_{66}}\left[c_{11}\frac{\partial^{2}\eta}{\partial x^{2}}+(1-\lambda)c_{13}\frac{\partial^{2}\eta}{\partial z^{2}}\right],$$
$$u_{zz}=-y(1-\lambda)\frac{\partial^{2}\eta}{\partial z^{2}},\tag{9}$$
$$\omega_{xy}=-\frac{\partial\eta}{\partial x},\quad \omega_{xz}=-y\frac{\lambda}{2}\frac{\partial^{2}\eta}{\partial x\,\partial z},\quad \omega_{yz}=\left(1-\frac{\lambda}{2}\right)\frac{\partial\eta}{\partial z},$$

where $\eta\equiv\eta(x,z)$ is the displacement of points on a neutral (undeformed) surface, and

$$\lambda=(2K_{eff}+B_{44}+8\pi)m_{0}^{2}[4c_{44}+(K_{eff}+B_{44}+4\pi)m_{0}^{2}]^{-1}.$$

After substituting (9) into (6)–(8) and averaging over the thickness h of the plate, we find a system of equations that describes the elastic oscillations of the plate, for which the Kittel boundary conditions (cf. Ref. 5) are identically satisfied. For simplicity, we will state the equation for the bending waves in an isotropic elastic plate only, for which

$$c_{11}+c_{66}=c_{33}=E\frac{1-\sigma}{(1+\sigma)(1-2\sigma)},\quad c_{11}=c_{13}=E\frac{\sigma}{(1+\sigma)(1-2\sigma)},$$
$$2c_{44}=c_{66}=E\frac{1}{1+\sigma},$$

where E is the Young's modulus and σ is the Poisson coefficient. The equation reads

$$\rho\ddot{\eta}+\frac{h^{2}}{12}\frac{E}{1-\sigma^{2}}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial z^{2}}\right)^{2}$$
$$\times\eta-\left[(K_{eff}+4\pi)-\lambda\left(K_{eff}+4\pi+\frac{1}{2}B_{44}\right)\right]m_{0}^{2}\frac{\partial^{2}\eta}{\partial z^{2}}$$
$$+\left[(K_{eff}+4\pi)-\frac{\lambda}{2}\left(K_{eff}+4\pi+\frac{1}{2}B_{44}\right)\right]m_{0}\frac{\partial m_{y}}{\partial z}=0,\tag{10}$$

where ρ is the density. The closed system also contains the Landau-Lifshitz equations, which after linearization yield

$$\dot{m}_{x}=\gamma H_{eff}^{y}m_{0},\quad \dot{m}_{y}=-\gamma H_{eff}^{x}m_{0},\quad \dot{m}_{z}=0,$$
$$H_{eff}^{x}=-\left(K_{eff}+\frac{H_{0}}{m_{0}}\right)m_{x}+\alpha_{1}\frac{\partial^{2}m_{x}}{\partial x^{2}}+\alpha_{3}\frac{\partial^{2}m_{x}}{\partial z^{2}},\tag{11}$$
$$H_{eff}^{y}=-\left(K_{eff}+\frac{H_{0}}{m_{0}}+4\pi\right)m_{y}+\alpha_{1}\frac{\partial^{2}m_{y}}{\partial x^{2}}+\alpha_{3}\frac{\partial^{2}m_{y}}{\partial z^{2}}$$
$$+\left[(K_{eff}+4\pi)-\frac{\lambda}{2}\left(K_{eff}+4\pi+\frac{1}{2}B_{44}\right)\right]m_{0}\frac{\partial\eta}{\partial z},$$
$$H_{eff}=0$$

(γ is the gyromagnetic ratio). If we assume plane waves in (10) and (11), we get the dispersion equation

$$\omega^{4}-[\tilde{\omega}_{s}^{2}(\mathbf{q})+\tilde{\omega}_{ph}^{2}(\mathbf{q})+v^{2}q_{z}^{2}]\omega^{2}+\tilde{\omega}_{s}^{2}(\mathbf{q})\tilde{\omega}_{ph}^{2}(\mathbf{q})$$
$$+\gamma^{2}[H_{0}(H_{0}+H_{a})+m_{0}(2H_{0}+H_{a})\alpha(\mathbf{q})]v^{2}q_{z}^{2}=0,\tag{12}$$

for the magnetobending eigenoscillations, where

$$\tilde{\omega}_{ph}^2(\mathbf{q})=\frac{h^2E}{12\rho(1-\sigma^2)}q^4,\quad \tilde{\omega}_s^2(\mathbf{q})=\Omega_1\Omega_2,$$

$$\Omega_1=\gamma[H_0+H_a+\alpha(\mathbf{q})m_0],$$

$$\Omega_2=\Omega_1+4\pi\gamma m_0,\quad \alpha(\mathbf{q})=\alpha_1q_x^2+\alpha_3q_z^2,\quad H_a=K_{eff}m_0,$$

$$q^2=q_x^2+q_z^2,\quad \rho v^2=m_0\left[(1-\lambda)(H_a+4\pi m_0)-\frac{1}{2}\lambda B_{44}m_0\right].$$

Equation (12) describes the dispersion of magnetobending waves for arbitrary wave vectors; for frequencies $\tilde{\omega}_{\rm ph}(\mathbf{q})=\tilde{\omega}_s(\mathbf{q})$ it gives a condition for magnetoacoustic (here, magnetobending) resonance which differs from the usual condition.[2,4,5] Apart from the different character of the dispersion and the polarization anisotropy of the elastic wave, (12) has the additional interesting consequence that the dispersion of the bending wave changes for frequencies $\tilde{\omega}_{\rm ph}(\mathbf{q})\ll\tilde{\omega}_s(0)$ when $\mathbf{H}_0\neq 0$. Indeed, (12) yields

$$\omega_{ph}^2(\mathbf{q})=\frac{H_0}{H_0+H_a+4\pi m_0}v^2q_z^2+\tilde{\omega}_{ph}^2(\mathbf{q})\tag{13}$$

in the quasistatic approximation, which is valid for $\omega\ll\tilde{\omega}_s(0)$.

In other words, a transverse rigidity arises when the external field is parallel to the magnetic moment; as a consequence, the bending dispersion law is replaced by a sound law for low-frequency waves traveling at angles $<\pi/2$ relative to the field. A similar expression also holds for a ferromagentic rod in a longitudinal magnetic field $\mathbf{H}_0$. Expression (13) can clearly also be derived for crystals consisting of layers or chains if we include the standard corrections (cf. Ref. 11). The results have a straightforward interpretation—if $\mathbf{H}_0=0$, the bending oscillations induce oscillations of the magnetic moments of the ions, which because of the magnetoelastic coupling follow the instantaneous local direction of the axis of anisotropy. If $\mathbf{H}_0\neq 0$ the spins become aligned and hinder the oscillations, which is reflected in an additional transverse rigidity for the bending waves. If we assume that $K_{\rm eff}m_0^2\sim(10^6\text{–}10^8)$ erg/cm^3 and $\rho\sim 5$ g/cm^3, we get the estimate $v\approx(10^3\text{–}10^4)$ cm/s for the rigidity if $H_0\gg K_{\rm eff}m_0$. This is $\sim 1\text{–}10\%$ of the speed of sound in most materials and is thus comparable to the coefficient of $\mathbf{q}^2$ in the original form of the dispersion law (neglecting rotational invariance) for elastic waves in layered or chain-like materials.

3. The dispersion law (13) indicates that a fundamentally new mechanism of magnetostriction in elastically one- and two-dimensional magnets may occur. Indeed, because these materials have a negative thermal expansion coefficient,[11] the bending waves can cause contraction in the plane of the layers (or along the chains). Because the external magnetic field favors a linear (sound) dispersion, it damps the oscillations and the original dimensions are restored. We note that this magnetostriction mechanism should become more important as the bending waves increased in amplitude (i.e., it should be enhanced with heating); moreover, the magnetostriction itself may be comparatively large, because the field-induced expansion is independent of changes in the interatomic distance. This phenomenon should be observable in magnets with a layered or chain-like structure that favors the propagation of bending waves (e.g., chalcogenides of the palladium group or organic chain magnets). Neutron-diffraction studies of the dispersion curves for such systems in magnetic fields would also be of interest.

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