# Dispersion properties of the conductivity of a binary medium

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(Submitted 28 April 1984; resubmitted 14 December 1984)
Zh. Eksp. Teor. Fiz. **88**, 1664–1675 (May 1985)

The analytic properties of the static electrical conductivity of a binary medium in the complex plane of one of its arguments are discussed. The dispersion of the conductivity of such a medium in a quasisteady electric field is analyzed. This analysis is related to the properties of a discrete $LC$ model, a binary lattice of inductive and capacitive reactances. Low-frequency dynamic methods for studying binary media can reveal detailed information on metal-insulator phase transitions.

## 1. INTRODUCTION

The experimental discovery[1,2] of an anomalous increase in the dielectric permeability near the point of a metal-insulator phase transition[3,4] has attracted interest to the more general problem of the low-frequency dispersion of the conductivity of such systems. Analysis of this problem on the basis of a scaling hypothesis[5] and in terms of the theory of an effective medium[6] has revealed several important aspects of this phenomenon and has demonstrated the importance of studying the conductivity dispersion for achieving a more profound understanding the metal-insulator phase transition. Dubrov *et al.*[6] also carried out an interesting model experiment which confirmed the basic theoretical conclusins. Vinogradov *et al.*[7] determined the critical indices of a percolation theory[5,8] in a study of the conductivity dispersion by numerical methods.

In the present paper we take a more general approach to the problem of the low-frequency dispersion of the conductivity than that in Refs. 5 and 6; our approach imposes no restriction on the range of applicability of the scaling hypothesis or of the effective-medium theory. To take this approach we need to know the properties of the function $f(p, h)$ ($p$ is the concentration, and $h$ of the ratio of the conductivities of the two components), which describes the effective static electrical conductivity of a binary system. We will examine the analytic properties of $f$ in the complex $h$ plane. We find that the function $f$ is analytic throughout this plane except on the negative real semiaxis. We derive a dispersion relation from which we can determine $f(p, h)$ at any point in the $h$ plane in terms of the values of $f$ on this semiaxis. According to the standard scaling hypothesis,[5] the singularity of $f$ nearest the origin is a branch point (on the negative real semiaxis) with an exponent which is equal to the critical index $s$. The distance from this branch point to the origin is determined by the proximity to the point of the phase transition as a function of concentration. We present arguments to support the assertion that at a concentration other than the critical concentration, $h = 0$ is a "weak" singular point of the function $f(p, h)$. By studying the singularities of the function $f$ we can thus find a complete description of the conductivity dispersion of a binary medium and develop a new approach to the problem of metal-insulator phase transitions.

We show in this paper that this question, which might seem to be a purely formal mathematical question, can be assigned a physical content. It turns out that the behavior of the function $f$ on the negative real semiaxis in the $h$ plane is directly related to the properties of a lattice model whose connections are either inductive or capacitive reactances (an $LC$ model). By measuring the impedance of the $LC$ model one can find the function $f$ at $h<0$ and thereby study its singularities. The significance of the function $f$ extends beyond the problem of the electrical conductivity of a binary medium. As was shown in Refs. 9 and 10, knowledge of the function $f(p, h)$ allows a complete description of the galvanomagnetic properties of $2D$ binary media and of the thermoelectric properties of both $2D$ and $3D$ binary media. A comprehensive study of the function $f(p, h)$ is therefore a fundamental problem of percolation theory.

## 2. BASIC PROPERTIES OF THE FUNCTION $f(p, h)$

We write the effective static electrical conductivity $\sigma_e$ of an isotropic binary system in the form

$$\sigma_e=\sigma_e(p; \sigma_1, \sigma_2)\equiv\sigma_1 f(p, h), \quad h=\sigma_2/\sigma_1. \tag{1}$$

Here $p$ is the concentration of the first component, and $\sigma_i$ ($i = 1,2$) is the conductivity of component $i$. The function $f$ in (1) depends on the particular structure of the system.

Let us examine the basic properties of $f(p, h)$ for positive real $h$ and introduce the necessary notation. We will be restricting the analysis to media with randomly distributed components (although this restriction is not of fundamental importance). For such systems, the simultaneous replacements $p \to 1-p$ and $\sigma_1 \rightleftarrows \sigma_2$ do not alter the macroscopic properties, so that we find from (1)

$$f(p, h)=hf(1-p, 1/h). \tag{2}$$

In the $2D$ case the function $f$ satisfies the reciprocity relation[11,12]

$$f(p, h)f(p, 1/h)=1, \tag{3}$$

which can be written in the form

$$f(p, h)f(1-p, h)=h \tag{4}$$

for a randomly inhomogeneous medium.

If $h = 0$, a metal-insulator phase transition will occur in the system at $p=p_c$, where $p_c$ is the critical concentration.[5,8] In the limit $h \to 0$ the function $f(p, h)$ takes the form

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$$p>p_c:\ f(p,h)=f_d(p)+hf_d'(p)+\frac{1}{2}h^2f_d''(p)+\dots, \tag{5a}$$

$$p=p_c:\ f(p,h)\approx\text{const}\cdot h^s, \tag{5b}$$

$$p<p_c:\ f(p,h)=hf_s(p)+\frac{1}{2}h^2f_s'(p)+\dots. \tag{5c}$$

In percolation theory it is assumed that the series in (5a) and (5c) converge. To simplify expressions (5) we have introduced some new notation,

$$\begin{aligned} &p>p_c:\ f_d(p)=f(p,0),\ f_d'(p)=\left[\frac{\partial f(p,h)}{\partial h}\right]_{h=0},\\ &f_d''(p)=\left[\frac{\partial^2 f(p,h)}{\partial h^2}\right]_{h=0},\dots,\\ &p<p_c:\ f_s(p)=\left[\frac{\partial f(p,h)}{\partial h}\right]_{h=0},\\ &f_s'(p)=\left[\frac{\partial^2 f(p,h)}{\partial h^2}\right]_{h=0},\dots, \end{aligned} \tag{6}$$

and we have made use of the condition $f(p,0)=0$ at $p<p_c$. In (5b), $s$ is the critical index,[5,8] and in (5a) and (5c) the subscripts $d$ and $s$ on the function $f$ specify systems with insulating and ideally conducting inclusions, respectively. We note that we have $f_d'(p)>0$, since allowing a nonvanishing conductivity of the "insulating" inclusions increases the conductivity of the system, while on the other hand we have $f_s'(p)>0$, since allowing the "ideally conducting" inclusions to have a finite conductivity reduces $\sigma_e$.

According to the scaling hypothesis, the function $f$ has the following form[5] in the region $|\tau|\ll 1 (\tau=(p-p_c)/p_c)$, $h\ll 1$:

$$\tau>0,\ \Delta\ll\tau\ll 1:\ f=\tau^t\{A_0+A_1(h/\tau^{t/s})+\dots\}, \tag{7a}$$

$$|\tau|\ll\Delta:\ f=h^s\{a_0+a_1(\tau/h^{s/t})+\dots\}, \tag{7b}$$

$$\tau<0,\ \Delta\ll|\tau|\ll 1:\ f=\frac{h}{(-\tau)^q}\left\{B_0+B_1\frac{h}{(-\tau)^{t/s}}+\dots\right\}. \tag{7c}$$

Here $\Delta=h^{s/t}$ is the size of the smearing region,[5] and the critical indices $t$, $s$, and $q$ are related by[5] $q=t(1-s)/s$. The constants $A_0$, $B_0$, and $\alpha_0$ are obviously positive. In accordance with the discussion above, we have $A_1>0$ and $B_1<0$; it is also simple to see that we have $\alpha_1>0$.

In the $3D$ case we have $t\approx 1.6$, $s\approx 0.62$, and $q\approx 1$ (Ref. 5); the critical concentration $p_c$ depends on the particular structure of the system. In the $2D$ case we have[5] $t\approx 1.3$. For a randomly inhomogeneous $2D$ medium we have[11] $p_c=1/2$, and in this case we find[11] from (4)

$$f(^1/_2,h)=h^{1/2}. \tag{8}$$

Comparing (8) with (5b), we conclude that $s=1/2$. It then follows from the relation among the critical indices that in the $2D$ case we have[5] $q=t$.

We will need to know the properties of the function $f$ in the limit $h\to\infty$. Using (5), we find from (2)

$$p>1-p_c:\ f(p,h)=f_s(1-p)+\frac{1}{2h}f_s'(1-p)+\dots, \tag{9a}$$

$$p=1-p_c:\ f(p,h)\sim h^{1-s}, \tag{9b}$$

$p<1-p_c$:

$$f(p,h)=hf_d(1-p)+f_d'(1-p)+\frac{1}{2h}f_d''(1-p)+\dots. \tag{9c}$$

The results in (9) have a simple meaning. In the $3D$ case there are two critical concentrations (two percolation thresholds).[8] As the concentration $p$ increases, a percolation with respect to the first component sets in at $p=p_{c1}\equiv p_c$, while the percolation with respect to the second component disappears at $p=p_{c2}>p_{c1}$. For randomly inhomogeneous media, because of the symmetry under the replacements $p\to 1-p$ and $\sigma_1\rightleftarrows\sigma_2$, we have $p_{c2}=1-p_c$. When we go from $p\lesssim p_c$ to $p\gtrsim p_c$ in the case $h\gg 1$ (i.e., $\sigma_1\ll\sigma_2$), the onset of percolation in terms of the poorly conducting (first) component has no important effect on the conductivity of the system, and the function $f$ is given by expression (9c) both below and above the transition point $p_c$. In contrast, near the second critical concentration, $p_{c2}=1-p_c$, where the percolation in terms of the well-conducting (second) component disappears (or appears), the function $f(p,h)$ exhibits critical properties in the limit $h\to\infty$. In the $2D$ case there is a single critical concentration, $p_c=p_{c1}=p_{c2}$, and there is no intermediate region $p_{c1}<p<p_{c2}$. For a randomly inhomogeneous medium we have $p_c=1/2$; in this case, we have an exact equality in (9b) with $s=1/2$ [see (8)].

We wish to emphasize that the assumption that the function $f(p,h)$ can be expanded around $h=0$ (if $p\neq p_c$) is introduced in the theory without justification. In fact, there is reason to believe (see §6) that $h=0$ is a singularity for $f(p,h)$ even at a concentration different from the critical concentration. In this case, the series in (5a), (5c) and (7a), (7c) [and thus in (9a), (9c)] do not converge.

### 3. ANALYTIC PROPERTIES OF THE FUNCTION $f(p,z)$

In a low-frequency (quasisteady; §58 in Ref. 13) electric field, the expression for the effective conductivity still has the form (1), but $\sigma_e$ and $\sigma_i$ are complex functions of the frequency $\omega$. In this case the argument $h$ of the function $f(p,h)$ is also complex. In order to use expression (1) for $\omega\neq 0$ we must therefore know the properties of $f(p,z)$ in the plane of the complex variable

$$z=h(\omega)=\sigma_2(\omega)/\sigma_1(\omega). \tag{10}$$

The conductivity (like the dielectric permeability), as a function of the complex frequency $\omega$, is analytic in the upper half-plane, $\text{Im}\,\omega>0$ (§82 in Ref. 13). Furthermore, according to §82 in Ref. 13, the conductivity has no zeros at $\text{Im}\,\omega>0$ and at finite $\omega$. The function $f=\sigma_e(\omega)/\sigma_1(\omega)$ is therefore analytic in the upper $\omega$ half-plane.

To determine the analytic properties of $f(p,z)$ as a function of the complex variable $z$ we need to know the region in the $z$ plane into which the upper half-plane $\text{Im}\,\omega>0$ is mapped by transformation (10). Analysis of specific transformations $Z=h(\omega)$ [see (20), (21), and, especially, (28)] shows that $\text{Im}\,\omega>0$ is mapped, generally speaking, onto the entrie $z$ plane, except for the negative real semiaxis. Consequently, in this region of the $z$ plane the function $f(p,z)$ is analytic. The point at infinate $z=\infty$ requires a special analysis. If the concentration $p$ is not equal to the critical value ($p\neq p_c=1/2$ in the $2D$ case or $p\neq 1-p_c$ in the $3D$ case), the function $f(p,z)$ at $z=\infty$ will either be finite or have a simple pole, according to (9).

With the properties of the function $f$ which have now been established, we are in a position to write a dispersion relation. For definiteness we consider the concentration interval corresponding to an expansion of $f$ as in (9a) in the limit $h \to \infty$. By the standard method[14] we then find (assuming that the $z$ plane is cut along the negative real semiaxis)

$$f(p,z)=f_s(1-p)+\frac{1}{2\pi i}\int_{C_0}\frac{f(p,z')}{z'-z}dz', \tag{11}$$

where the contour $C_0$ runs from $-\infty$ to 0 along the upper edge of the cut, circumvents the point $z'=0$ on the right, and then returns to $-\infty$ along the lower edge of the cut. If the function $f$ is known on the upper and lower edges of the cut $(-\infty,0)$, we can then find $f(p,z)$ at any point in the $z$ plane from dispersion relation (11).

The function $f(p,z)$ has a useful symmetry property. According to §82 in Ref. 13, we have $\sigma(-\omega^*)=\sigma^*(\omega)$, where the asterisk means the complex conjugate. It follows that we have $f(p,\ h(-\omega^*))=f^*(p,\ h(\omega))$, with $h(-\omega^*)=h^*(\omega)$, so that we have

$$f(p,z^*)=f^*(p,z). \tag{12}$$

Writing the function $f(p,z)$ in the form $(z=x+iy)$

$$f(p,z)=f_1(p;x,y)+if_2(p;x,y),$$

we find the parity properties of the real and imaginary parts of $f$ from (12):

$$f_1(p;\ x,\ -y)=f_1(p;\ x,\ y),\ f_2(p;\ x,\ -y)=-f_2(p;\ x,\ y). \tag{13}$$

Using the parity properties (13), we can rewrite dispersion relation (11) as

$$f(p,z)=f_s(1-p)-\frac{1}{\pi}\int_0^\infty\frac{\operatorname{Im}f_+(p,-t)}{t+z}dt. \tag{14}$$

Here $f_+(p,\ -t)=f(p,\ -t+i\delta)$, where $\delta\to+0$, is the value of the function $f(0,z)$ on the upper bank of the cut. We can thus determine $f(p,z)$ throughout the $z$ plane if we know simply the imaginary part of $f$ on the upper edge of the cut (or on the lower edge by virtue of the relation $\operatorname{Im}f_+=-\operatorname{Im}f_-$).

The derivation above was based exclusively on general properties of the conductivity as a function of the complex frequency $\omega$, so that we cannot draw conclusions about the particular nature of the singularities of $f(p,z)$ on the semiaxis $\operatorname{Im}z=0$, $\operatorname{Re}z\leqslant 0$. From what we know about the properties of $f$ at positive real $z=h$ we can draw some inferences about the nature of these singularities.

We first consider a $2D$ system at the critical concentration, $p=1/2$. An analytic continuation of the function $f$, defined in accordance with (8), into the complex $z$ plane is described by

$$f(1/2,z)=z^{1/2}. \tag{15}$$

It is assumed that in expression (15), which holds for arbitrary $z$ (in the case of a randomly inhomogeneous $2D$ system), we choose that branch of the root for which we have $\operatorname{Re}f>0$ at $\operatorname{Im}z=0$, $\operatorname{Re}z>0$. In accordance with the general properties of $f$ discussed above, the function (15) is analytic at finite $z$ in the entire plane, with a cut along the negative real semiaxis. At $z=0$ and at infinity, the function $f$ has branch points with an exponent of 1/2.

In the $3D$ case $(p=p_c)$ the analytic continuation of expression (5b) [where we are using (7b)] which holds at $|z|\ll 1$ is

$$f(p_c,z)\approx a_0z^s \tag{16}$$

with the same choice of branch as in (15). According to (16), the function $f$ has a branch point (with an exponent $s$) at $z=0$.

From the assumption that the function $f(p,h)$ can be expanded in a converging series around $h=0$ if $p\neq p_c$ (see the preceding section) it follows that the analytic continuation of this function, $f(p,z)$, is regular in a certain neighborhood of the point $z=0$. In this case the branch point is separated from the origin by a gap and is at $z=-h_1$ $(h_1>0)$, where $h_1\to 0$ as $p\to p_c$. We can estimate $h_1$ in order of magnitude on the basis of the scaling hypothesis. From (7) we conclude that

$$h_1\sim|\tau|^{t/s}. \tag{17}$$

Generally speaking, the exponent of the branch point may depend on the concentration, but it would be natural to assume that this exponent would be the same as the exponent $s$ in the critical region, $|\tau|\ll 1$. Near the branch point the nonanalytic part of the function $f$ would then be of the form

$$\delta f(p,z)\propto(z+h_1)^s \tag{18}$$

with $h_1$ from (17).

We thus reach the conclusion that, according to the standard scaling hypothesis, the singularity of the function $f(p,z)$ nearest the origin is a branch point at $z=-h_1$ with $h_1$ given by (17). From the mathematical standpoint, the approach of the system to the metal-insulator transition corresponds to an approach of the singularity of the function $f$—the branch point—to the origin in the $z$ plane. Analogous conclusions follow from the expression for $f(p,h)$ given by the effective-medium theory.[15]

The conclusion that there is a gap leans heavily on the assumption that the function $f(p,h)$ $(p\neq p_c)$ can be expanded aroung $h=0$. There is reason to believe (§6), however, that $h=0$ is a singularity for the function $f(p,h)$ at all concentrations, so that the corresponding series do not converge. This conclusion means that the imaginary part of $f$ is generally nonzero for all $z<0$ and that the gap mentioned above does not exist. Nevertheless, according to the arguments in §6, $\operatorname{Im}f$ is small in the region $-h_1\lesssim z<0$, so that the singularity of $f$ at the point $z=0$ is "weak." On the other hand, the point $z=-h_1$ is special because as we go from $z\gtrsim -h_1$ to $z\lesssim -h_1$ the imaginary part of $f$ increases rapidly. It would thus be natural to call the singularity of the function $f(p,z)$ at the point $z=-h_1$ "strong" (although this point is not singular in the strict sense of the term). The assumption that series (5a), (5c); (7a), (7c); and (9a), (9c) converge corresponds to neglecting the weak singularity of the function $f$ at the point $z=0$.

## 4. DISPERSION OF THE CONDUCTIVITY

We assume that an inhomogeneous sample is in a low-frequency (quasisteady; §58 in Ref. 13) periodic external

electric field. One of the conditions for a quasisteady situation (§58 in Ref. 13) is that the dimension of the sample ($l$) be small in comparison with the wavelength $\lambda \sim c/\omega$ ($c$ is the velocity of light) corresponding to the field frequency $\omega \ll c/l$. Under this condition we may assume that the sample is in a uniform field $\mathbf{E}(t) = \mathbf{E}(\exp(-i\omega t)$. On the other hand, we assume that $l$ is large in comparison with the dimensions of the inhomogeneities, so that we can speak in terms of average (or effective) characteristics of the sample.

For "poor" conductors (e.g., semiconductors) we can introduce both an electrical conductivity $\sigma$ and a dielectric constant $\varepsilon$ (§58 in Ref. 13). This is conveniently done by means of the complex conductivity

$$\sigma(\omega)=\sigma-i\omega\varepsilon/4\pi. \tag{19}$$

Yet another condition for a quasisteady situation is that $\sigma$ and $\varepsilon$ must be independent of the frequency (§58 in Ref. 13), and we assume that this condition holds. In this case, $\sigma$ and $\varepsilon$ in (19) should be understood as the static values of the electrical conductivity and of the dielectric constant.

According to the discussion above, in the quasisteady approximation the problem of determining the effective characteristics of the medium differs from the static case only in that the conductivity $\sigma$ in (1) is replaced by the complex conductivity $\sigma(\omega)$ from (19):

$$\begin{aligned}\sigma_e-i\frac{\omega\varepsilon_e}{4\pi}&=\left(\sigma_1-i\frac{\omega\varepsilon_1}{4\pi}\right)f(p,h(\omega)),\\ h(\omega)&=\left(\sigma_2-i\frac{\omega\varepsilon_2}{4\pi}\right)\Big/\left(\sigma_1-i\frac{\omega\varepsilon_1}{4\pi}\right).\end{aligned} \tag{20}$$

Let us examine in slightly more detail a system with insulating inclusions ($\sigma_s = 0$), for which we have (under the condition $\sigma_1 \gg \omega\varepsilon_1/4\pi$)

$$\sigma_e-i\frac{\omega\varepsilon_e}{4\pi}=\sigma_1 f(p,h(\omega)),\quad h(\omega)=-i\frac{\omega\varepsilon_2}{4\pi\sigma_1}. \tag{21}$$

In this case we need to know the function $f(p, z)$ on the imaginary axis in the $z$ plane in order to determine $\sigma_e$ and $\varepsilon_e$. Let us assume that at $p \neq p_c$ the derivatives $f'$ and $f''$ exist at the point $z = 0$. In the low-frequency limit, to within terms $\sim\omega^2$, inclusively, we then find from (20) and (21)

$$\sigma_e=\sigma_1\left[f(p,0)-\frac{1}{2}\left(\frac{\omega\varepsilon_2}{4\pi\sigma_1}\right)^2 f''(p,0)\right], \tag{22}$$

$$\varepsilon_e=\varepsilon_1 f(p,0)+\varepsilon_2 f'(p,0), \tag{23}$$

where the prime means differentiation of $f(p, z)$ with respect to its argument $z$. In (23) we have also written a term with $\varepsilon_1$, which was omitted from (21).

At $p > p_c$, according to (6), we have $f(p, 0) = f_d(p)$, $f'(p,0) = f_d'(p)$, etc. In this case the second term in (22) is a correction $\sim\omega^2$ to the main term $\sigma_1 f_d(p)$. According to the scaling hypothesis, we find from (7a)

$$f_d(p)=A_0\tau^t,\ f_d'(p)=A_1/\tau^q,\ \ldots;$$

expressions (22) and (23) in this case are the same as the corresponding results in Ref. 5. The condition under which the term $\sim\omega^2$ in $\sigma_e$ is small can be written as

$$\tau \gg \Delta(\omega)=|h(\omega)|^{s/t} \tag{24}$$

with $h(\omega)$ from (21). The quantity $\Delta(\omega)$ serves as the dimension of the smearing region.

At $p < p_c$, according to (5) and (6), we have $f(p, 0) = 0$, $f'(p,0) = f_s(p)$, and $f''(p,0) = f_s'(p)$, so that

$$\sigma_e=-\frac{1}{2}\sigma_1\left(\frac{\omega\varepsilon_2}{4\pi\sigma_1}\right)^2 f_s'(p),\quad \varepsilon_e=\varepsilon_2 f_s(p). \tag{25}$$

Expression (25) for $\varepsilon_e$ can also be derived in the static approach to the problem (Ref. 16, for example). As was mentioned in §2, we have $f_s'(p) < 0$; i.e., the conductivity $\sigma_e$ is positive. According to the scaling hypothesis we find from (7c)

$$f_s(p)=B_0|\tau|^{-q},\ f_s'(p)=2B_1|\tau|^{-(sq+t)/s},$$

and expressions (25) become the corresponding expressions of Ref. 5, which hold outside the smearing region, $|\tau| \ll \Delta(\omega)$.

At the critical point in the limit $z \to 0$ the function $f$ is given by (16), so that we find from (21)

$$\sigma_e\approx\sigma_1\left(\frac{\omega\varepsilon_2}{4\pi\sigma_1}\right)^s a_0\cos\frac{\pi s}{2},\quad \varepsilon_e\approx\varepsilon_2\left(\frac{4\pi\sigma_1}{\omega\varepsilon_2}\right)^{1-s} a_0\sin\frac{\pi s}{2}, \tag{26}$$

again in agreement with Ref. 5. Expressions (26) also hold $p \neq p_c$ if $|\tau| \ll \Delta(\omega)$. It follows from (26) that the critical index $s$ can be determined by measuring the loss angle and the frequency dependence of $\sigma_e$ and $\varepsilon_e$ in the region $|\tau| \ll \Delta(\omega)$.

A study of the low-frequency dispersion of the conductivity can thus furnish rather detailed information on the metal-insulator phase transition. At the same time we should stress that the specific expressions for $\sigma_e$ and $\varepsilon_e$ outside the smearing region (like the results of Ref. 5, which follow from them) are applicable if the derivatives $f'(p,0)$ and $f''(p,0)$ exist. Otherwise, the conductivity and the dielectric constant will not be analytic functions of the frequency even as $p \neq p_c$.

Lattice models offer far greater opportunities (in the sense of flexibility in the functional dependence of the complex conductivity on the frequency $\omega$). Let us assume that we are given a square lattice (or a simple cubic lattice in $3D$) whose connections are elements with a complex resistance (impedance) $Z_k(\omega)$. The impedance $Z_k$ takes on the value $Z_1$ with a probability of $p$ and the value $Z_2$ with a probability of $1 - p$. Since $Z^{-1}$ plays the role of the electrical conductivity in this case, we can find an expression for the effective impedance of the system from (1) by using the replacement $\sigma \to Z^{-1}$. By combining resistances, capacitances, and inductances, we can construct essentially any functional dependence of the impedance of the connection, $Z_k$, on the frequency $\omega$. Dubrov *et al.*[6] studied a square lattice each of whose connections is an element consisting of a resistance $R$ and a capacitance $C$ in parallel. The impedance ($Z_k$) of such an element can be found easily ($k = 1,2$):

$$Z_k^{-1}(\omega)=R_k^{-1}-i\omega C_k.$$

Comparison with (19) shows that a lattice model with such couplings is equivalent to a binary continuous medium with a complex conductivity like that in (19). The results of (20)–(26) can thus be transferred directly to the case of binary lattices. We simply note that Dubrov *et al.*[6] analyzed the problem of nodes,[8] for which we cannot use the exact result

found by Dykhne,[11] $Z_e = (Z_1 Z_2)^{1/2}$, which applies to connections on a square lattice with $p = 1/2$. We also note that the expressions given for the conductivity $\sigma_e$ and the dielectric constant $\varepsilon_e$ in Ref. 6 can be found from general expressions (20) by using the function $f(p, h)$ from Ref. 15.

Of primary interest from the standpoint of learning about the singularities of the function $f$ is a lattice $LC$ model whose connections are either purely capacitive or purely inductive reactances. In view of the importance of this case, we will examine the $LC$ model in slightly more detail.

## 5. LATTICE *LC* MODEL

We consider a binary lattice whose connections have, with a probability of $p$, an inductive reactance (an $L$ coupling) and, with a probability of $1-p$, a capacitive reactance (a $C$ coupling) (§62 in Ref. 13):

$$Z_1 = -i\frac{\omega L}{c^2}, \quad Z_2 = \frac{i}{\omega C}. \tag{27}$$

Here $L$ is the inductance, and $C$ the capacitance. If expressions (27) are to be analytic, we must understand the frequency $\omega$ to be the quantity $\omega + i\delta$, where $\delta \to +0$. In this case the function

$$z = h(\omega) = Z_1(\omega)/Z_2(\omega), \tag{28}$$

with $Z_1$ and $Z_2$ from (27), maps the upper half-plane $\mathrm{Im}\,\omega > 0$ onto the entire $z$ plane except for the negative real semiaxis. The effective impedance of the $LC$ model is given by the following expression ($\omega > 0$) according to (1) and (27):

$$Z_e^{-1}(\omega) = i\frac{c^2}{\omega L} f_-\left(p, -\frac{\omega^2}{\Omega^2}\right), \quad \Omega = \frac{c}{\sqrt{LC}}, \tag{29}$$

where $p$ is the density of $L$ connections. In accordance with the rule $\omega \to \omega + i\delta$, we have taken into account the circumstance that along the cut we should take the value of the function $f$ on the lower edge. The effective impedance of the $LC$ system is thus determined by the properties of the function $f(p, z)$ on the negative real semiaxis of the complex frequency $z$.

For connections on a square lattice with $p = 1/2$ the function $f$ is given by expression (15); hence $f_-(1/2, z) = -i|z|^{1/2}$ (for $z < 0$). As a result we find from (29)

$$Z_e = c^{-1}(L/C)^{1/2} \equiv Z_0. \tag{30}$$

This result, which was derived in Ref. 11, seems at first glance to be paradoxical: An energy dissipation occurs in a lattice consisting of purely reactive impedances. Expression (30) with $p = p_c$ and $\omega \ll \Omega$ is valid in order of magnitude for plane lattices of all types.

In the $3D$ case with $p = p_c$, for which the function $f$ is given by (16), we find, in an analogous way,

$$Z_e \approx a_0 Z_0 (\Omega/\omega)^{2s-1} \exp\left[i(s-1/2)\pi\right], \tag{31}$$

where $Z_0$ is the same as in (30). There are two important distinctions between expressions (31) and (30). First, the effective impedance in (31) depends on the frequency (at $s \approx 0.62$ we have $2s - 1 \approx 0.24$; i.e., $Z_e$ varies with the frequency nearly in accordance with $\omega^{-1/4}$, for which we would have $s = 5/8$). Second, the imaginary part of $Z_e$ is nonzero, and we have

$$\mathrm{Im}\,Z_e/\mathrm{Re}\,Z_e = \mathrm{tg}\left[(s-1/2)\pi\right].$$

Both of these results stem from the circumstance $s \neq 1/2$. Consequently, we can determine the extent to which $s$ deviates from $1/2$ by measuring the loss angle and frequency dependence of the impedance.

If the "weak" singularity of the function $f(p, z)$ is ignored (§§3 and 6), the real part of $Z_e$ vanishes at $p \neq p_c$ if $0 < \omega < \omega_\tau$. According to (17), we have, in order of magnitude, $\omega_\tau \sim \Omega|\tau|^{t/2s}$. At $\omega \gtrsim \omega_\tau$, according to (18), $\mathrm{Re}\,Z_e$ varies in accordance with

$$\mathrm{Re}\,Z_e \propto (\omega - \omega_\tau)^s, \quad \omega_\tau \sim \Omega|\tau|^{t/2s}; \tag{32}$$

i.e., there is a threshold in the functional dependence of $\mathrm{Re}\,Z_e$ on $\omega$. In the region $\omega_\tau \ll \omega \ll \Omega$ the quantity $\mathrm{Re}\,Z_e$ is given by expression (31) [or expression (30) in the $2D$ case]. In this approximation, the frequency dependence of the real part of the effective impedance of the $LC$ model under the conditions $|\tau| \ll 1$ and $\omega \ll \Omega$ is as shown schematically by the solid lines in Figs. 1a ($s = 1/2$) and 1b ($s > 1/2$). When the weak singularity of the function $f(p, z)$ is taken into account, the real part of $Z_e$ is nonzero in the region $0 < \omega < \omega_\tau$ also. The dotted lines show the $\omega$ dependence of $\mathrm{Re}\,Z_e$ in this frequency region.

An analysis of the $LC$ model thus makes it possible to determine the basic characteristics of systems with a metal-insulator phase transition: the percolation threshold and the critical indices. Furthermore, as follows from (29), measurements of the $\omega$ dependence of $Z_e$ can reveal the function $f(p, z)$ on the entire negative real semiaxis of the $z$ plane. Another topic of considerable interest is the generalized $LC$ model consisting of a binary lattice whose connections consist of an inductance and a capacitance in series. A study of the impedance of such a lattice makes it possible to study the properties of the function $f(p, z)$ on the entire real axis in the $z$ plane.

## 6. LOCAL OSCILLATIONS IN THE *LC* MODEL

In the preceding section we used the known properties of the function $f(p, z)$ to study the $LC$ model. The inverse problem—of analyzing the $LC$ model in order to determine the properties of the function $f$—is extremely complicated and requires a separate treatment. We will accordingly offer only a few comments on the inverse problem here. It is pertinent to note that a study of the properties of the $LC$ model

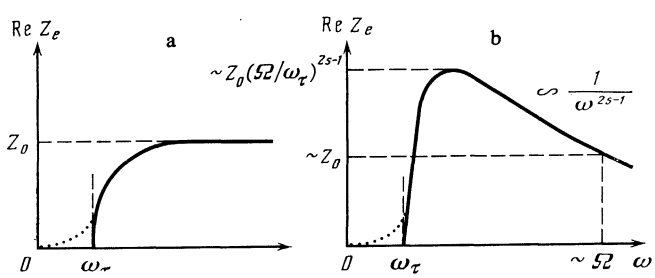


FIG. 1. The real part of the effective impedance of the lattice $LC$ model as a function of the frequency $\omega$. a—Two-dimensional case ($s = 1/2$); b—three-dimensional case ($s > 1/2$).

allows us to take a physical approach to the rather formal question of the singularities of the function $f(p, z)$.

**1.** The existence of a nonvanishing (at $\omega \neq 0$) real part of the impedance means that the system has eigenfrequencies $\omega_\nu \neq 0$ (the concept of the existence of local oscillations in an $LC$ model was introduced by Dykhne[11]). To determine the nature of these frequencies we consider the case of a low density of "defect" connections. We start from Kirchhoff 's equations for the lattice[15]:

$$\sum_{\Delta} \sigma_{\mathbf{r},\mathbf{r}+\Delta}(V_{\mathbf{r}}-V_{\mathbf{r}+\Delta})=0. \tag{33}$$

Here $V_{\mathbf{r}}$ is the potential at lattice node $\mathbf{r}$; the vectors $\Delta$ connect a node with the $n$ nearest-neighbor nodes; and $\sigma_{\mathbf{r},\mathbf{r}+\Delta}$ is the conductivity of the coupling between nodes $\mathbf{r}$ and $\mathbf{r}+\Delta$. We assume that all the couplings of the lattice have a conductivity $\sigma^{(e)}$ except the connection (of conductivity $\sigma^{(i)}$) between nodes $\mathbf{r}=0$ and $\mathbf{r}=\Delta_x$. Transferring the inhomogeneity in (33) to the right side, and solving the resulting equation by Fourier transforms, we find in the usual way[17] an equation for local (impurity) oscillations:

$$\frac{\sigma^{(e)}-\sigma^{(i)}}{\sigma^{(e)}}=\frac{n}{2}. \tag{34}$$

If $\sigma^{(e)}=Z_2^{-1}$ and $\sigma^{(i)}=Z_1^{-1}$ with $Z_1$ and $Z_2$ from (27) (a single $L$ connection in a lattice of $C$ connections), then we find from (34) that the system has impurity frequencies $\omega_\nu = \pm\omega_{01}$ where

$$\omega_{01}=\Omega(n/2-1)^{-1/2}. \tag{35}$$

Here $\Omega$ is the same as in (29). If, on the other hand, there is a single $C$ connection in a lattice $L$ connections, then the system has impurity levels $\pm\omega_{02}$ where

$$\omega_{02}=\Omega(n/2-1)^{1/2} \tag{36}$$

with the same $\Omega$. For a square lattice ($n=4$) we have $\omega_{01}=\omega_{02}=\Omega$. Results (35) and (36) have a simple meaning: The introduction of an $L$ connection into a $C$ matrix (or vice versa) forms an $LC$ circuit with a resonant frequency (35) or (36).

Analysis of a lattice with two (or three, etc.) defect connections shows that each of the impurity levels (35), (36) splits into two (or more) sublevels. It follows that a nonzero defect concentration causes a local level to "spread out" into an impurity band, as usual.[17] If the frequency of the external electric field falls in this band, the system absorbs energy. It can therefore be suggested that a cut along the negative real semiaxis in the complex $z$ plane corresponds to an impurity band in the $LC$ model. The gaps discussed in §3, which separate the branch point $z=-h_1$ from the origin in the case $p \neq p_c$, correspond in this picture to an energy gap.

The existence of such a gap in the oscillation spectrum is plausible for systems with a periodic arrangement of inclusions. For a randomly inhomogeneous medium, on the other hand, we would naturally expect that the energy gap would contain the small "tails" which are characteristic of disordered system.[17] To evaluate these tails,[17] we need to find the composition fluctuation of nonzero probability with which the low-lying impurity levels are coupled.

In the lattice $LC$ model, low-frequency oscillations are exhibited by, for example, line defects ("dislocations")—$L$ connections arranged in an infinite line.

The spectrum of such oscillations as $k \to 0$ ($k$ is the wave vector) is

$$\omega^2(k)\sim\Omega^2 k \quad (D=2), \quad \omega^2(k)\sim\Omega^2 k^2 \ln(1/k) \quad (D=3), \tag{37}$$

where $D$ is the dimensionality of the space, and $\Omega$ is the same as in (29). The probability for the formation of a long, straight "dislocation" is extremely small. Nevertheless, defects of this type may contribute to the tails mentioned above.

At significant concentrations, the "normal" (most probable) defects in a lattice are finite-size clusters: bound formations consisting of linear chains. It is natural to suggest that these formations are also coupled with low-lying levels; the minimum oscillation frequency $\omega_{\min}$ of a cluster of size $r$ decreases with increasing $r$. The frequency $\omega_\tau$ in (32) is apparently related to a finite cluster of size equal to the correlation radius $r_c$ (Ref. 8). Clusters of size $r \gg r_c$—the probability for whose existence is exponentially small but nonzero—then contribute to $\mathrm{Re}Z_e$ at $\omega \ll \omega_\tau$. Since the $r$ dependence of $\omega_{\min}$ is not known, we cannot draw more definite conclusions regarding $\mathrm{Re}Z_e$ at $\omega \ll \omega_\tau$. Nevertheless, it may be assumed that there is no gap in the oscillation spectrum and that $\mathrm{Im}f$ is nonzero in the frequency interval $0<\omega<\omega_\tau$. A further study of the lattice $LC$ model should resolve these questions, which are of importance to percolation theory; in particular, further study should reveal the nature of the singularity of $f(p, z)$ at $z=0$.

**2.** The properties of the $LC$ model can also be studied in a continuous problem by assigning the first component a conductivity $\sigma_1=Z_1^{-1}$ and the second a conductivity $\sigma_2=Z_2^{-1}$, with $Z_i$ from (27). Below we restrict the discussion to the $2D$ case, for which the analysis is comparatively simple.

It is not difficult to see that the eigenfrequency $\omega_\nu=\Omega$ is associated with an inclusion of circular shape. Local oscillations with frequencies $\sim\Omega$ are apparently characteristic of other "volume" figures. Highly elongated inclusions have low-lying levels. An analysis of an infinite band of width $2d$, for example, shows that two oscillation branches are associated with it:

$$\omega_1^2(k)=\Omega^2 \operatorname{th} kd, \quad \omega_2^2(k)=\Omega^2 \operatorname{cth} kd. \tag{38}$$

The probability for the formation of a highly elongated defect is extremely small, however (as in a lattice, the most common defects are finite clusters; the question of the spectrum of the oscillations associated with these finite clusters remains open). On the other hand, it can be expected that a spectrum analogous to (38) will arise when two inclusions of smooth shape come close together, since a rather long and narrow channel forms in the "contact" region. The probability for such a close approach ($\sim c^2$, where $c$ is the concentration of inclusions) is not extremely small even as $c \ll 1$. We should therefore expect that even in the approximation quadratic in $c$ the quantity $\mathrm{Im}f$ will be nonzero at all $z<0$.

Analysis of a pair of circular inclusions shows that it is

associated with a set of discrete frequencies

$$\omega_{1n}^2=\Omega^2 \operatorname{th} n\xi_0, \quad \omega_{2n}^2=\Omega^2 \operatorname{cth} n\xi_0 \quad (n=1, 2, \dots),$$

$$\xi_0=\ln \frac{\rho+(\rho^2-4R^2)^{1/2}}{2R}, \tag{39}$$

where $\rho$ is the distance between the centers of the circles, and $R$ is their radius. As $\rho \to 2R$, we have $\xi_0 \to 0$, and the frequencies in (39) form a quasicontinuum of the type in (38), which stretches from 0 to $\infty$. A calculation of the effective conductivity of such a system incorporating terms $\sim c^2$ shows that the region in which the function $f$ is nonanalytic is the entire negative real semiaxis in the $z$ plane. At small values of $z$ the function $f$ contains a nonanalytic term $z^3 \ln z$, so that $f$ cannot be expanded in a series around $z = 0$. On the other hand, the first two derivatives of $f$ with respect to $z$ do exist at $z = 0$, so that the singularity of the function $f(p, z)$ at $z = 0$ is weak. [Analogous conclusions hold for an infinitely remote point $z = \infty$, as can also be shown by using the reciprocity relation (3).]

The conclusion that there is no gap in the oscillation spectrum of the $LC$ model may also hold in the general case (including $3D$ systems), so that $z = 0$ is a singularity of the function $f(p, z)$ for all concentrations for randomly inhomogeneous media. As discussed above, however, the imaginary part of $f$ is small in the "energy gap" ($-h_1 \lesssim z < 0$). Furthermore, the derivative of $f$ with respect to $z$ at $z = 0$ exists at least for $p < p_c$, since it determines the conductivity of a system with ideally conducting inclusions [see (5c)]. If $f'(p,0)$ exists at $p > p_c$ also, the singularity of $f(p, z)$ at $z = 0$ will be weak at all concentrations other than the critical concentration. On the other hand, as we have already mentioned (§3), the point $z = -h_1$ is a special one in that as we go from $z \gtrsim -h_1$ to $z \lesssim -h_1$ there is a rapid increase in $\operatorname{Im} f$. Although this point is apparently no singular from the mathematical standpoint, it is natural to call it a point of a "strong singularity" of the function $f(p, z)$.

I wish to thank A. M. Dykhne for critical comments offered in a discussion of this study.

[1]T. G. Castner, N. K. Lee, G. S. Cieloszyk, and G. L. Salinger, Phys. Rev. Lett. **34**, 1627 (1975).
[2]A. Zilberstein, B. Pannetier, and P. Merenda, Phys. Lett. **54A**, 145 (1975).
[3]N. F. Mott and E. A. Davis, Philos. Mag. **17**, 1269 (1968).
[4]V. L. Bonch-Bruevich, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 198 (1974) [JETP Lett. **4**, 122 (1974)].
[5]A. L. Éfros and B. I. Shklovskii, Phys. Status Solidi **b76**, 475 (1976).
[6]V. E. Dubrov, M. E. Levinshteĭn, and M. S. Shur, Zh. Eksp. Teor. Fiz. **70**, 2014 (1976) [Sov. Phys. LETP **43**, 1050 (1976)].
[7]A. N. Vinogradov, A. M. Karimov, A. T. Kunavin, *et al.*, Dokl. Akad. Nauk SSSR **275**, 590 (1984) [Sov. Phys. Dokl. **29**, 214 (1984)].
[8]B. I. Shklovskiĭ and A. L. Éfros, Usp. Fiz. Nauk **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1975)].
[9]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **82**, 1333 (1982) [Sov. Phys. JETP **55**, 774 (1982)].
[10]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **85**, 568 (1983) [Sov. Phys. JETP **58**, 331 (1983)].
[11]A. M. Dykhne, Zh. Eksp. Teor. Fiz. **59**, 110 (1970) [Sov. Phys. JETP **32**, 63 (1971)].
[12]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **81**, 665 (1981) [Sov. Phys. JETP **54**, 355 (1981)].
[13]L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred, Nauka, Moscow, 1982 (Electrodynamics of Continuous Media, Pergamon Press, Oxford).
[14]L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika, Nauka, Moscow §129, 1974 (Quantum Mechanics, Pergamon Press, Oxford).
[15]S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[16]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **79**, 1561 (1980) [Sov. Phys. JETP **52**, 787 (1980)].
[17]I. M. Lifshitz, Usp. Fiz. Nauk **83**, 617 (1964) [Sov. Phys. Usp. **7**, 549 (1964)].

Translated by Dave Parsons