Kinetic theory of the absorption of low-frequency sound and second sound in insulators

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The attenuation factors for low-frequency sound and second sound in pure insulators are calculated. The finite lifetime and dispersion of the longitudinal phonons are taken into account. The results are compared with results derived previously.

1. INTRODUCTION

The absorption of low-frequency sound, at frequencies below the reciprocal of the relaxation time (the collision rate) of thermal phonons, was first studied in Refs. 1–3. In this case the sound may be regarded as an external factor which alters the distribution function and energy of the phonons. These changes give rise to an irreversible dissipation of the energy of the sound waves. This dissipation is determined by the temperature and by the rate of change of the entropy of the thermal phonons.

In addition to the external agent—the sound—the probabilities of various microprocesses in the phonon system go into determining the phonon entropy. In pure crystals the simplest of these are those three-phonon processes which are allowed by the rigorous energy and momentum conservation laws. When these interaction processes alone are taken into account, one is dealing with the so-called problem of long-wave longitudinal phonons, which arises in the calculation of the rate of change of the entropy of the phonons and the thermal conductivity of insulators.

This problem has been solved in the model of an isotropic, unbounded crystal by taking the four-phonon interactions into account through anharmonicities of fourth order[2] and of third order in the second Born approximation.[4] For certain real, unbounded crystals, this problem has been solved by taking into account the effect of degeneracy points on the probabilities for the three-phonon interactions.[5]

The finite lifetime of the longitudinal phonons and their dispersion have not been taken into account.

As is shown below (Section 3), the governing processes in this problem are not the four-phonon interactions or interactions of higher order but the interactions of three longitudinal phonons; in addition, the finite lifetime and dispersion of these phonons are important. We take this interaction into account in calculations of the attenuation factors for low-frequency sound and for second sound. We compare the results with results derived previously. We find that the attenuation factor for low-frequency sound has a different temperature dependence and is smaller in magnitude than in Refs. 2 and 3.

The effect of these processes on the attenuation factor of high-frequency sound and on the thermal conductivity was studied in Ref. 14.

2. DISSIPATION FUNCTION

The energy of a quasiparticle of species j, with wave vector $\mathbf{k}$, can be written in the following form for the case in which the spatial and temporal changes in the parameters on which this energy depends caused by a modulated external field are small:

$$\omega^j(\mathbf{k},\ \mathbf{r},\ t)=\omega_0{}^j(\mathbf{k})\,(1+b^j(\mathbf{r},\ t)), \tag{2.1}$$

where $\omega_0^j(\mathbf{k})$ is the energy ($\hbar=1$) of the quasiparticle in the absence of the modulated field, and $|b^j(\mathbf{r},t)|\ll 1$ is the modulation depth.

The modulation of the energy causes the distribution function of the quasiparticles to deviate from an equilibrium Planck function:

$$\overline{N}^j=\left(\exp\frac{\omega_0{}^j}{T}-1\right)^{-1}. \tag{2.2}$$

Under the adiabatic conditions

$$\left|\tau^j\frac{d}{dt}\ln|b^j(\mathbf{r},t)|\right|\ll 1,\qquad |l^j\nabla\ln|b^j(\mathbf{r},t)||\ll 1, \tag{2.3}$$

where τ^j and $l^j=\tau^j g^j$ are the average relaxation time and the mean free path of the quasiparticles, and $\mathbf{g}^j=d\omega_0^u/d\,\mathbf{k}$ is the group velocity of the quasiparticles, the interaction between quasiparticles causes this deviation of the distribution function to relax to a new quasiequilibrium distribution

$$N_0{}^j=\left[\exp\left(\frac{\omega_0{}^j-\mathbf{ku}}{T}\right)-1\right]^{-1}. \tag{2.4}$$

Here $T=T_0(1+\vartheta(\mathbf{r},t))$ and $\mathbf{u}\equiv\mathbf{u}(\mathbf{r},t)$ are the temperature field and the field of the drift velocity of the quasiparticles, and T_0 is the equilibrium temperature.

If the interactions of the quasiparticles obey energy and momentum conservation, then ϑ and $\mathbf{u}$ are determined from the equations for stimulated secondary waves (waves of the "second-sound" type).[8] If momentum is not conserved in an interaction, the drift velocity will be zero ($\mathbf{u}=0$), there will be no secondary waves, and energy conservation is used to determine ϑ.

The amount of energy (Q) absorbed by the gas of quasiparticles from the external modulating field per unit time (the dissipation function) is equal to the product of the temperature and the rate of change of the entropy of the quasiparticles. To determine the latter we need the solution of the

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kinetic equation for $N_{\mathbf{k}}^j$. However, we can find a good estimate of Q without solving the kinetic equation[9]:

$$Q \sim cT\langle |\delta^j|^2/\nu^j\rangle, \tag{2.5}$$

where c is the specific heat of the gas of quasiparticles,

$$\dot{\delta}^j = \dot{b}^j - \left(\frac{\partial}{\partial t} + \mathbf{g}^j\nabla\right)\left(\vartheta + \frac{\mathbf{ku}}{\omega_0{}^j}\right), \tag{2.6}$$

$\nu^j \equiv \nu^j(\mathbf{k}) = 1/\tau^j(\mathbf{k})$ is the quasiparticle collision rate, defined as a functional derivative of the collision integral,[10]

$$|\vartheta| = |T - T_0|/T \ll 1; \qquad |\mathbf{ku}/\omega_0{}^j| \ll 1.$$

Here $\langle \ldots \rangle$ means an average

$$\langle f\rangle = \frac{1}{cT^2}\sum_{\mathbf{k},j} (\omega_0{}^j)^2 \bar{N}^j(\bar{N}^j+1) f^j. \tag{2.7}$$

As can be seen from (2.5), the dependence of the collision rate ν^j on the wave vector $\mathbf{k}$ is important in the calculation of Q.

3. ABSORPTION OF LOW-FREQUENCY SOUND

Let us use expression (2.5) to estimate the attenuation factor for low-frequency sound with a wave vector $\mathbf{q}$ and a frequency $\Omega_s = qV_s$ (V_s is the sound velocity) which satisfy conditions (2.3): $\Omega_s\tau^j \ll 1$, $ql^j \ll 1$.

The sound wave changes the elastic properties of the medium and modulates the energy of the thermal phonons. The modulation depth b^j can be written in the following form, as in Ref. 1:

$$b^j = \lambda_{ik}{}^j u_{ik}^{(0)}, \tag{3.1}$$

where $u_{ik}^{(0)}$ is the strain tensor caused by the sound, while the tensor λ_{ik}^j is determined by the crystalline symmetry (the Grüneisen tensor). In this case, the quantity δ^j in (2.6) is

$$\delta^j = \left[\lambda_{ik}{}^j - \left(1 - \frac{\mathbf{g}^j\mathbf{q}}{\Omega_s}\right)\left(1 - \frac{\mathbf{kg}_{II}}{\omega_0{}^j}\right)\frac{V_s{}^2}{V_s{}^2 - V_{II}{}^2}\langle\lambda_{ik}{}^j\rangle\right] u_{ik}^{(0)}, \tag{3.2}$$

where V_{II} and $\mathbf{g}_{II}$ are the phase and group velocities of the second sound. If there is no second sound, we have $V_{II} = 0$ and $\mathbf{g}_{II} = 0$.

We can determine the characteristic features of the absorption of the sound from the model of an isotropic crystal. In this model there are longitudinal (l) and transverse (t) phonons, whose frequencies (corrected for dispersion) are

$$\omega_{l,t} = kV_{l,t}[1 + \xi_{l,t}(\mathbf{k})], \tag{3.3}$$

where $V_{l,t}$ is the phase velocity of the phonons, and the quantity $\xi_{l,t}(\mathbf{k})$ is a measure of the deviation of the phonon frequency from a linear dispersion law. In an isotropic crystal, the tensor λ is $\lambda_{ik}^j = \lambda^j\delta_{ik}$, where λ^j is on the order of unity. The phase and group velocities of the second sound, uncorrected for dispersion, are[11]

$$V_{II} = \frac{V_t}{3^{1/2}}\left[\frac{1 + {}^1/_2(V_t/V_l)^2}{1 + {}^1/_2(V_t/V_l)^5}\right]^{1/2}; \qquad \mathbf{g}_{II} = V_{II}\frac{\mathbf{q}}{q}. \tag{3.4}$$

Since the velocities V_t, V_l, V_s, and V_{II} are comparable in magnitude, we find the following estimate of Q, averaged over a period of the sound wave:

$$\bar{Q} = \bar{\varepsilon}_s\frac{cT}{\rho V_s{}^2}\left\langle\frac{1}{\nu}\right\rangle\Omega_s{}^2, \qquad \bar{\varepsilon}_s = {}^1/_2\rho V_s{}^2\overline{(u_{ik}^{(0)})^2}, \tag{3.5}$$

where $\bar{\varepsilon}_s$ is the average energy density of the sound wave, and ρ is the density of the crystal.

The sound attenuation factor is given in order of magnitude by

$$\gamma_s \equiv \frac{\bar{Q}_s}{\bar{\varepsilon}_s} \sim \frac{cT}{\rho V_s{}^2}\Omega_s{}^2\left\langle\frac{1}{\nu}\right\rangle. \tag{3.6}$$

We write the collision rate ν^i in the form

$$\nu^j(\mathbf{k}) = \nu_0\sum_p \alpha_p{}^j f_p{}^j(x), \tag{3.7}$$

where the sum over p means a sum over the processes in which the phonons of species j may participate; ν_0 is the collision rate of the thermal phonons, which is given at high temperatures ($T \gg \Theta$) by $\nu_0 = (T/mV_s^2)\omega_D$ and at low temperatures ($T \ll \Theta$) by $\nu_0 = (T/mV_s^2)(T/\Theta)^4\omega_D$; $\alpha_p^j \sim 1$; m is the mass of a unit cell of the crystal; Θ and ω_D are the Debye temperature and Debye frequency; and the function $f_p^j(x)$ depends on the type of process (p) and the argument x, which is ω^j/Θ at $T \gg \Theta$ and $x = \omega^j/T$ at $T \ll \Theta$.

When dispersion is ignored, the rigorous energy and momentum conservation laws allow the following three-phonon interactions: $l \leftrightarrow l + t$ and $l \leftrightarrow t + t$. The collision rates due to these processes were calculated in Refs. 2–4 and 12. In the region $\omega^j \ll T$ we have $f_p^t \propto x$ for transverse phonons and $f_p^l \propto x^4$ for longitudinal phonons.

If we consider these processes alone, we find that most of the attenuation in (3.6) is caused by long-wave longitudinal phonons, and γ_s becomes infinite. In Refs. 2 and 3, this singularity was eliminated by taking into account four-phonon interactions of longitudinal phonons through fourth-order anharmonicity and third-order anharmonicity in the second Born approximation. The contribution of these processes to the collision rate ν^l at high temperatures is

$$f_4{}^l \sim \frac{T}{mV_s{}^2}\left[1 + \beta'\left(\frac{\omega}{\Theta}\right)^2\right] \quad (\beta' \sim 1), \tag{3.8}$$

and the contribution at low temperatures is

$$f_4{}^l \sim \frac{T}{mV_s{}^2}\left[1 + \beta''\left(\frac{\omega}{T}\right)^2\right]\left(\frac{T}{\Theta}\right)^3 \qquad (\beta'' \sim 1). \tag{3.9}$$

On this basis, the attenuation factor of the sound was found to have a temperature dependence $T^{-1/4}$ at high temperatures and T^{-2} at low temperatures. The finite lifetime of the longitudinal phonons and their dispersion were not considered.

Here we take into account the interaction of three longitudinal phonons having a finite lifetime $\tau_0^l = 1/\nu_0$ and dispersion—a case not considered in Refs. 1–5. In this case, the δ-function in the energy in the expression for the probability for the process $l + l \leftrightarrow l$ must be replaced:

$$\delta(\omega_1{}^l + \omega_2{}^l - \omega_3{}^l) \to \frac{\nu_0}{\pi}[\nu_0{}^2 + (\omega_1{}^l + \omega_2{}^l - \omega_3{}^l)^2]^{-1}. \tag{3.10}$$

The rate of collisions due to these processes is[6,7]

$$\nu_{l+l\leftrightarrow l}^l \sim \nu_0 x\left\{\operatorname{arctg}\frac{2\omega^l}{\nu_0} + \operatorname{arctg}\left[\frac{\omega^l}{\nu_0}\left(\frac{g^l(k_m) - V_l}{V_l}\right)\right]\right\}, \tag{3.11}$$

where the quantity in parentheses, a measure of the phonon

dispersion, is taken for Debye phonons ($k_m \sim \omega_D/V_l$) at high temperatures, and is independent of the temperature; at low temperatures, it is taken for thermal phonons ($k_m \sim T/V_l$).

If the dispersion of the longitudinal thermal phonons is positive, $g^l_{km} > V_l$, or

$$(k_m/\nu_0)\,|g^l(k_m)-V_l| \ll 1, \tag{3.12}$$

the sound attenuation factor is

$$\gamma_s \sim \begin{cases} \Omega_s^2\Theta^{-1}, & T \gg \Theta, \\ \Omega_s^2 T^{-1}, & T \ll \Theta, \end{cases} \tag{3.13}$$

in agreement with the results derived in Ref. 1.

If the dispersion of the longitudinal phonons is negative, and we have

$$(k_m/\nu_0)\,|g^l(k_m)-V_l| \gg 1, \tag{3.14}$$

then the attenuation factor is

$$\gamma_s \sim \begin{cases} \dfrac{\Omega_s^2}{\Theta}\left[\dfrac{|g^l(k_m)-V_l|}{V_l}\dfrac{mV_s^2}{T}\right]^{1/4}, & T \gg \Theta \\ \dfrac{\Omega_s^2}{T}\dfrac{\Theta}{T}\left[\dfrac{|g^l(k_m)-V_l|}{V_l}\dfrac{mV_s^2}{\Theta}\right]^{1/4}, & T \ll \Theta \end{cases}. \tag{3.15}$$

Comparing (3.15) with (3.13), we see that in this case a large factor

$$\left[\frac{|g^l(k_m)-V_l|}{V_l}\frac{mV_s^2}{T}y\right]^{1/4}, \quad y = \begin{cases} 1, & T \gg \Theta, \\ (\Theta/T)^3, & T \ll \Theta, \end{cases} \tag{3.16}$$

appears in the attenuation. At high temperatures, $T \gg \Theta$, the sound attenuation factor has the same temperature dependence as in Refs. 2 and 3, but its magnitude is smaller by a factor of $[|g^l(k_m) - V_l|/V_l]^{1/4}$.

At low temperatures, $T \ll \Theta$, the dispersion has a strong effect on the temperature dependence of the attenuation factor. This can be seen in the case of a linear chain, for which the dispersion of longitudinal phonons is negative, $\xi_l \sim -a^2k^2$ (a is the lattice constant), and $|g^l(k_m) - V_l|/V_l \sim (T/\Theta)^2$, while the sound attenuation factor is

$$\gamma_s \sim \Omega_s^2 (mV_s^2\Theta)^{1/4}/T^{3/2}. \tag{3.17}$$

In contrast with the isotropic model, the presence of degeneracy points in a real crystal gives rise, for the longitudinal phonons, to a dependence of the function f^l_p in (3.7) on the frequency of these phonons which differs from the x^4 dependence in Ref. 5. For crystals of higher symmetry we have $f^l_p \sim x^2$, and the sound attenuation factor is determined by (3.13). For crystals of low symmetry, f^l_p may be proportional to x^3 or x^4. If $f^l_p \sim x^4$, the attenuation factor will be given by (3.13) or (3.15), depending on the dispersion. If $f^l_p \sim x^3$, and inequality (3.14) holds, the attenuation factor will acquire a logarithmic factor

$$\ln\left|\frac{g^l(k_m)-V_l}{V_l}\frac{mV_s^2}{T}y\right|$$

instead of (3.16).

Under conditions such that second-sound waves can propagate through a crystal, the hydrodynamic attenuation factor for the second sound, γ_{II}, can be determined by an approach similar to that taken for low-frequency sound[8,13]; the result is, in order of magnitude,

$$\gamma_{II} \approx \Omega_{II}^2 \langle \nu^{-1} \rangle, \tag{3.18}$$

where $\Omega_{II} = qV_{II}$ is the frequency of the second sound. Comparing γ_s in (3.6) with γ_{II} in (3.20), we find

$$\gamma_{II} \sim \gamma_s (\rho V_{II}^2/cT), \tag{3.19}$$

so that the attenuation factor for the second sound has the same characteristic features as the attenuation factor for low-frequency sound at low temperatures.

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