Role of vector nonlinearity in soliton stability in a magnetized plasma

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The role played by effects of a vector nonlinearity in the three-dimensional stability of one-dimensional drift and ion acoustic solitons of a magnetic plasma is analyzed. These effects are shown to be important if the soliton amplitude is not too small. They shut off the instability of ion acoustic solitons propagating at a large angle from the magnetic field. They have a more complex effect on drift solitons, stabilizing transverse perturbations but destabilizing oblique perturbations.

1. INTRODUCTION

The existence of a fairly large variety of solitons in a magnetized plasma has now been demonstrated theoretically. These solitons propagate either exactly or nearly across the equilibrium magnetic field (see Ref. 1 and the bibliography there). Such solitons are pertinent to the problem of plasma confinement in a magnetic field. Their stability must accordingly be analyzed. A first step in this direction was taken by Petviashvili,[2] who continued the general approach which he and Kadomtsev had proposed.[3] One aspect of this approach is that the stability of one-dimensional solitons with respect to two- or three-dimensional perturbations is analyzed on the basis of equations whose nonlinear terms are the same as those in the corresponding one-dimensional equations.

In the present paper we wish to point out that in this magnetized plasma case the approach of Refs. 2 and 3 is, generally speaking, not sufficient; nonlinear effects of an essentially multidimensional nature are important in this case. These effects are of the same nature as those which figure in vortex theory.[4] They are characterized by terms $[\nabla a, \nabla b]_z \neq 0$, where a and b are functions determining the wave field, and z is the direction of the equilibrium magnetic field. We refer to these terms as a "vector nonlinearity," while the nonlinearity which is conventionally incorporated in the general theory of the stability of solitons is the "scalar nonlinearity." This terminology was introduced in Ref. 1. In the same paper, the problem of steady-state waves was analyzed with a simultaneous treatment of both types of nonlinearity (a similar problem was studied in Ref. 5).

According to Ref. 1, a problem of this sort has solutions of two types: solutions of the type $b = b(a)$, which correspond to a situation in which the vector nonlinearity vanishes identically (the case of scalar solitons or simply "solitons"), and solutions of the type $b \neq b(a)$, for which the role of the scalar nonlinearity is unimportant (the case of vector solitons or "vortices"). In this sense we may say that it is unproductive to incorporate the vector nonlinearity in the problem of steady-state solitons (because this nonlinearity vanishes identically). In the stability problem, on the other hand, the situation is radically different: Instead of the steady-state solutions of the type $b_0 = b_0(a)$ we are dealing in this case with a solution of the type $b = b_0(a) + \tilde{b}$, where $a = a_0 + \tilde{a}$, and $\tilde{a}$ and $\tilde{b}$ are perturbations of the functions a and b $[\tilde{b} \neq \tilde{b}(a)]$. In this case we have $[\nabla a, \nabla b]_z \neq 0$, so that vector-nonlinearity effects play a nontrivial role.

Vector-nonlinearity effects are crucial to the nonlinear mechanism for the generation of convection cells in a magnetized plasma[6,7] and to the theory of the turbulence of such plasmas.[8–10] The incorporation of this nonlinearity in soliton stability theory thus fills an obvious gap in the nonlinear theory of magnetized plasmas. The analogy in terms of wave properties between a magnetized plasma and a rotating liquid[11] suggests that vector-nonlinearity effects may also be important in the problem of the stability of scalar Rossby-wave solitons in such liquids (such solitons were studied in Refs. 5, 12, and 13).

It is clear that incorporating nonlinear effects of the type $[\nabla a, \nabla b]_z$ in the problem of the linear stability of one-dimensional solitons is meaningful only for solitons which are propagating at some angle from the magnetic field. In this connection there is a series of papers, going back to one by Zakharov and Kuznetsov,[14] on the stability of ion acoustic solitons which are propagating strictly along the magnetic field (see the bibliography in Ref. 15). Obviously, vector nonlinearity plays no role in this case.

To make our analysis more specific we consider the simple and quite familiar example of drift-ion-acoustic waves propagating nearly across the magnetic field.[2,16,17] The three-dimensional nonlinear equation for these waves is given in Section 2. It is simplified for the case of small perturbations of a one-dimensional soliton in Section 3. A dispersion relation for three-dimensional perturbations of a soliton in the case with a vector nonlinearity is derived in Section 4 and analyzed in Section 5. The results are discussed in Section 6.

2. THREE-DIMENSIONAL NONLINEAR EQUATION FOR DRIFT-ION-ACOUSTIC WAVES

We assume a plasma in a uniform magnetic field $\mathbf{B}_0 \| \mathbf{z}$. The plasma pressure is negligible in comparison with the magnetic pressure, and the ion temperature is negligible in comparison with the electron temperature T_e. The equilibrium plasma density n_0 and the electron temperature are nonuniform along x; i.e., we have $\nabla n_0, \nabla T_e \| \mathbf{x}$. We consider waves in this plasma with typical frequencies which are small in comparison with the ion cyclotron frequency, and

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we assume that the electric field of these waves is a potential field: $\mathbf{E} = -\nabla\varphi$, where φ is the electrostatic potential. We are assuming that the waves are quasineutral, thereby ignoring effects of the finite electron Debye length, which are taken into account in Refs. 14 and 15. These are standard assumptions for the elementary theory of drift-ion-acoustic waves.[18]

These waves are described in the linear approximation by the dispersion relation

$$u^2-uV_*-\alpha^2c_s^2=0. \tag{2.1}$$

It is assumed that the waves are propagating along the y axis at a velocity u and along the z axis at a velocity u/α. Here $c_s=(T_e/m_i)^{1/2}$ is the ion acoustic velocity, $V_*=cT_e\varkappa_n/e_eB_0$ is the electron drift velocity along the density gradient, $\varkappa_n=\partial\ln n_0/\partial x$ is the reciprocal of the scale size of the density inhomogeneity, e_e is the electron charge, m_i is the ion mass, and c is the velocity of light. The typical frequencies of waves of the type in (2.1) are low in comparison with the ion cyclotron frequency $\omega_{Bi}=e_iB_0/m_ic$, where $e_i=-e_e$ is the ion charge.

In working with (2.1) we are treating both the case of

$$u=V_*, \tag{2.2}$$

which corresponds to the approximation $\alpha c_s < V_*$, and the case of purely ion acoustic waves of a homogeneous plasma, $V_*\to 0$, which are propagating at some angle from the magnetic field ($\alpha\neq\infty$),

$$u=\alpha c_s. \tag{2.3}$$

The nonlinear equation for the waves of type (2.1) is derived in Appendix A. This equation is

$$\rho_0^2\hat{Q}\varphi=-\frac{1}{u}\left(\frac{\partial}{\partial t}+\frac{\alpha c_s^2}{u}\frac{\partial}{\partial z}\right)\varphi-\left(\alpha\frac{\partial}{\partial\eta}+\frac{\partial}{\partial z}+\frac{\alpha}{u}\frac{\partial}{\partial\eta}\hat{D}^{-1}\frac{\partial}{\partial t}\right)Z. \tag{2.4}$$

Here $\hat{Q}\varphi$ is that part of the nonlinear equation which figures in the theory of two-dimensional steady-state x, η waves[1]:

$$\hat{Q}\varphi\equiv\hat{D}\Delta_\perp\varphi-\frac{\partial}{\partial\eta}(\Lambda\varphi-S\varphi^2), \tag{2.5}$$

where the operator $\hat{D}$ is defined in the standard way,

$$\hat{D}=\frac{\partial}{\partial\eta}-\frac{c}{uB_0}[\nabla\varphi,\nabla]_z, \tag{2.6}$$

and the coefficients Λ and S are

$$\Lambda=\frac{1}{\rho_0^2}\left(1-\frac{V_*}{u}-\frac{\alpha^2c_s^2}{u^2}\right), \tag{2.7}$$

$$S=\frac{e_e}{2T_e\rho_0^2u}\left[V_{*T}-\frac{\alpha^2c_s^2}{u^2}\left(1+\frac{\alpha^2c_s^2}{u^2}\right)\right]; \tag{2.8}$$

the function Z is

$$Z\equiv\frac{c_s^2}{u^2}\hat{D}^{-1}\left(\frac{\partial}{\partial z}+\frac{\alpha}{u}\frac{\partial}{\partial t}\right)\varphi, \tag{2.9}$$

$\rho_0^2=T_e/m_i\omega_{Bi}^2$ is the square of the ion Larmor radius calculated from the electron temperature, $V_{*T}=V_*\varkappa_T/\varkappa_n$ is the electron drift velocity along the temperature gradient, and $\varkappa_T=\partial\ln T_e/\partial x$ is the reciprocal of the scale dimension of the electron temperature gradient. We assume $\varphi=\varphi(x,\eta,z,t)$ where $\eta\equiv y-ut+\alpha z$.

The second term on the right side of (2.6) describes the vector nonlinearity. Here a question posed in Section 1 arises: What role does this nonlinearity play in the problem of soliton phenomena in a magnetized plasma? It follows from (2.4), in accordance with the discussion in Section 1, that this role depends on the nature of the phenomena. If, for example, we are interested in steady-state two-dimensional (x,η) phenomena, with $(\partial/\partial t)_\eta=(\partial/\partial z)_\eta=0$, then we see, in view of the discussion in Appendix A of the effect of the operator $[\nabla\varphi,\nabla]_z$ on the functions φ, that (2.4) implies

$$\hat{D}(\Delta_\perp\varphi-\Lambda\varphi+S\varphi^2)=0. \tag{2.10}$$

In the case of analytic functions φ (functions which have no discontinuities in their higher-order derivatives[1]), this equation can be integrated; it reduces to the equation

$$\Delta_\perp\varphi-\Lambda\varphi+S\varphi^2=0, \tag{2.11}$$

which now has no vector nonlinearity. (This point was first demonstrated by Petviashvili.[5]) Equation (2.11) therefore holds for both a weak and a strong vector nonlinearity, and in this case there is no need to determine the relative importance of the various terms on the right side of (2.6). This question does, however, arise in the case of time-varying and/or three-dimensional problems, in which we would have $(\partial/\partial t)_\eta\neq 0$ or $(\partial/\partial z)_\eta\neq 0$. It may turn out that the second term on the right side of (2.6) is negligible in comparison with the first. In such a case we would find from (2.4)

$$\frac{\partial^2}{\partial\eta^2}(\rho_0^2\Delta_\perp\varphi-\varepsilon\varphi+q\varphi^2)=-\frac{1}{u}\frac{\partial}{\partial\eta}\left[\left(1+\frac{\alpha^2c_s^2}{u^2}\right)\frac{\partial}{\partial t}+\frac{2\alpha c_s^2}{u}\frac{\partial}{\partial z}\right]\varphi-\frac{c_s^2}{u^2}\left(\frac{\partial}{\partial z}+\frac{\alpha}{u}\frac{\partial}{\partial t}\right)^2\varphi. \tag{2.12}$$

Here $\varepsilon=\Lambda\rho_0^2$; i.e., according to (2.7) we have

$$\varepsilon=1-V_*/u-\alpha^2c_s^2/u^2 \tag{2.13}$$

and, analogously, $q=S\rho_0^2$. Equation (2.12) was derived by Petviashvili[2] for the case $\alpha c_s \ll V_*$.

We will write (2.12) in a simpler form, noting that the quantity ε, the dimensionless dielectric constant, is a small parameter in our problem. For this reason, we can use the approximation $u=u_0$ on the right side of (2.12), where u_0 satisfies the dispersion relation [cf. (2.1)]

$$\varepsilon(u_0,\alpha)=0. \tag{2.14}$$

We regard (2.14) as an equation for u_0, which determines the function $u_0=u_0(\alpha)$. We can then use the concept of the longitudinal wave group velocity $v_g=\partial u_0/\partial\alpha$, which in our case is

$$v_g=-\frac{\partial\varepsilon/\partial\alpha}{\partial\varepsilon/\partial u_0}=\frac{2\alpha c_s^2}{u_0(1+\alpha^2c_s^2/u_0^2)}. \tag{2.15}$$

Transforming to a coordinate system which is moving along the z direction at a velocity v_g, i.e., replacing z by the new variable $\zeta = z - v_g t$, we put (2.12) in the form

$$\frac{\partial}{\partial\eta}\left[\frac{\partial\varepsilon}{\partial u_0}\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial\eta}(\rho_0{}^2\Delta_\perp\varphi-\varepsilon\varphi+q\varphi^2)\right]=-s\frac{\partial^2\varphi}{\partial\zeta^2}, \quad (2.16)$$

$$s=\frac{1}{2}\frac{\partial\varepsilon}{\partial u_0}\frac{\partial^2 u_0}{\partial\alpha^2}\equiv\frac{c_s{}^2}{u_0{}^2}\left(\frac{1-\alpha^2c_s{}^2/u_0{}^2}{1+\alpha^2c_s{}^2/u_0{}^2}\right)^2. \quad (2.17)$$

As is usual in problems of this sort, we have assumed $(\partial/\partial t)_\zeta$ to be small, and we have ignored terms on the order of $(\partial^2/\partial t^2)_\zeta$ and $(\partial/\partial t)_\zeta\partial/\partial\zeta$. We now replace η by the new variable

$$\theta=\eta+\frac{\varepsilon}{\partial\varepsilon/\partial u_0}t. \quad (2.18)$$

From (2.16) we then find

$$\frac{\partial}{\partial\theta}\left[\frac{\partial\varepsilon}{\partial u_0}\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial\theta}(\rho_0{}^2\Delta_\perp\varphi+q\varphi^2)\right]=-s\frac{\partial^2\varphi}{\partial\zeta^2}. \quad (2.19)$$

The canonical form of this equation,

$$\frac{\partial}{\partial y}(u_t+6uu_y+u_{yyy})=3\varkappa_1\frac{\partial^2 u}{\partial z^2}+\varkappa_2\frac{\partial^4 u}{\partial x^2\,\partial y^2}, \quad (2.20)$$

where $\varkappa_1$ and $\varkappa_2$ are coefficients, differs from the three-dimensional Kadomtsev-Petviashvili equation.

3. INITIAL EQUATION FOR SMALL PERTURBATIONS OF A SOLITON

In (2.4) we set $\varphi=\varphi_0+\tilde\varphi$, where $\varphi_0=\varphi_0(\eta)$ and $\tilde\varphi=\tilde\varphi(\eta,t,x,z)$ are the steady-state and perturbed parts of the potential. For the steady-state part we find the equation [cf. (2.11)]

$$\partial^2\varphi_0/\partial\eta^2-\Lambda\varphi_0+S\varphi_0{}^2=0. \quad (3.1)$$

We assume

$$\tilde\varphi\propto\exp(-i\omega t+ik_\perp x+ik_\parallel z), \quad (3.2)$$

where ω, $k_\perp$, and $k_\parallel$ are the frequency and transverse and longitudinal wave numbers of the perturbation. We find the following equation for $\tilde\varphi$:

$$\begin{aligned}\rho_0{}^2\hat D&\left(\frac{\partial^2\tilde\varphi}{\partial\eta^2}-\Lambda\tilde\varphi+2S\tilde\varphi\varphi_0\right)\\&=\frac{i}{u_0}\left(\omega-\frac{\alpha c_s{}^2k_\parallel}{u}\right)\tilde\varphi-i\frac{\alpha c_s{}^2}{u^2}\left(k_\parallel-\frac{\alpha\omega}{u}\right)\frac{\partial}{\partial\eta}\hat D_0{}^{-1}\tilde\varphi\\&+\rho_0{}^2k_\perp{}^2\hat D_0\tilde\varphi+\frac{c_s{}^2k_\parallel}{u^2}\left(k_\parallel-\frac{\alpha\omega}{u}\right)\hat D_0{}^{-1}\tilde\varphi\\&-\frac{\alpha c_s{}^2\omega}{u^3}\left(k_\parallel-\frac{\alpha\omega}{u}\right)\frac{\partial}{\partial\eta}\hat D_0{}^{-2}\tilde\varphi, \end{aligned}\quad (3.3)$$

where the operator $\hat D_0$ is defined by (cf. (2.6)]

$$\hat D_0=\frac{\partial}{\partial\eta}+ik_\perp\frac{c\varphi_0{}'}{uB_0},\quad\varphi_0{}'=\frac{\partial\varphi_0}{\partial\eta}. \quad (3.4)$$

We see from (3.4) that the vector nonlinearity is clearly important under the condition

$$k_\perp c\bar\varphi_0/uB_0\gtrsim1, \quad (3.5)$$

where $\bar\varphi_0$ is a typical value of the soliton potential. (In Section 5 we will show that in some cases the vector nonlinearity is important at even smaller values of $k_\perp$.) From (3.1) we have

$$\varphi_0(\eta)=\frac{\delta}{\rho_0{}^2S}\Phi_0(\xi)\equiv\frac{2T_eu\delta\Phi_0(\xi)/e_e}{V_{*T}-\alpha^2c_s{}^2(1+\alpha^2c_s{}^2/u^2)/u}, \quad (3.6)$$

where $\Phi_0(\xi)$ is the solution of the dimensionless equation

$$\partial^2\Phi_0/\partial\xi^2-\Phi_0+\Phi_0{}^2=0, \quad (3.7)$$

with the variable $\xi\equiv\delta^{1/2}\eta/\rho_0$, and the dimensionless small parameter δ is defined by

$$\delta=\Lambda\rho_0{}^2\equiv1-V_*/u-\alpha^2c_s{}^2/u^2. \quad (3.8)$$

From (3.6) we find the estimate

$$\bar\varphi_0\sim T_e\delta/e_e, \quad (3.9)$$

and from the expression for ξ we find an estimate of the scale size of the soliton,

$$\lambda\sim\rho_0/\delta^{1/2}. \quad (3.10)$$

In our analysis we are assuming $k_\perp\lambda<1$. Using (3.5), (3.9), and (3.10), we then find that the vector nonlinearity clearly plays an important role in our problem if $k_\perp$ lies in the interval

$$u/c_s\delta<k_\perp\rho_0<\delta^{1/2}. \quad (3.11)$$

This double inequality is noncontradictory if

$$\delta^{3/2}>u/c_s. \quad (3.12)$$

In the case of drift waves, with $u\sim V_*$, condition (3.12) means

$$\delta>(\rho_0/L_n)^{2/3}, \quad (3.13)$$

where $L_n\sim1/\varkappa_n$ is the scale size of the plasma gradient. For tokamaks we would have $\rho_0\sim0.1$ cm and $L_n\sim10$ cm, and condition (3.13) would hold even at $\delta>1\%$. The level of the potential fluctuations observed experimentally is typically on the order of 3% (Ref. 19, for example). It follows, in particular, that the treatment of the stability of drift solitons in a tokamak should incorporate the vector nonlinearity. Our analysis is also pertinent to ion acoustic solitons which are propagating through a homogeneous plasma at a large angle from a magnetostatic field, $\alpha\ll1$. In this case we have $u\sim\alpha c_s$, so that condition (3.12) means

$$\delta^{3/2}>\alpha. \quad (3.14)$$

It is also clear that at sufficiently large soliton amplitudes, $\delta\sim1$, the interval of the parameter α in which the vector nonlinearity is important extends to values on the order of unity. Quasineutral ion acoustic solitons with $\delta\sim1$ and $\alpha\sim1$ were studied in Refs. 20 and 21, but their stability was apparently not studied.

4. DERIVATION OF A DISPERSION RELATION FOR PERTURBATIONS OF A SOLITON

We now seek solutions of Eq. (3.3), and, correspondingly, we derive the relation among ω, $k_\perp$, and $k_\parallel$: the "dispersion relation" for perturbations of the soliton. To simplify the calculations we put Eq. (3.3) in dimensionless form. We replace η by ξ, introduced above, while $\tilde{\varphi}$ is replaced by the function $\tilde{\Phi}$, defined by analogy with (3.6):

$$\tilde{\Phi}=\tilde{\varphi}\rho_0{}^2S/\delta. \tag{4.1}$$

We also introduce

$$K_\perp=k_\perp\rho_0/\delta^{1/2},\quad K_\parallel=k_\parallel c_s\rho_0/u\delta,\quad \Omega=\omega\rho_0/u\delta^{3/2},$$
$$\sigma=\alpha c_s/u,\quad \mu=c(\delta/\rho_0{}^2)^{3/2}/uB_0S, \tag{4.2}$$
$$\hat{d}_0=\partial/\partial\xi+i\mu K_\perp\Phi_0',$$

where the prime means the derivative with respect to ξ. Equation (3.3) then becomes

$$\hat{d}_0(\tilde{\Phi}''-\tilde{\Phi}+2\tilde{\Phi}\Phi_0)=i\Big(\Omega-\frac{\sigma}{\delta^{1/2}}K_\parallel\Big)\tilde{\Phi}+K_\perp{}^2\hat{d}_0\tilde{\Phi}$$
$$-i\sigma\Big(\frac{K_\parallel}{\delta^{1/2}}-\sigma\Omega\Big)\frac{\partial}{\partial\xi}\hat{d}_0{}^{-1}\Phi+K_\parallel(K_\parallel-\sigma\delta^{1/2}\Omega)\hat{d}_0{}^{-1}\tilde{\Phi}$$
$$-\sigma\delta^{1/2}\Omega(K_\parallel-\sigma\delta^{1/2}\Omega)\frac{\partial}{\partial\xi}\hat{d}_0{}^{-2}\tilde{\Phi}. \tag{4.3}$$

We solve (4.3) in the spirit of Refs. 2 and 3, assuming that Ω, $K_\parallel$, $K_\perp$ are small, but, going beyond Refs. 2 and 3, we assume that the product $\mu K_\perp$ is of order unity. We write $\tilde{\Phi}$ as a series in these small parameters:

$$\tilde{\Phi}=\Phi_1+\Phi_2+\Phi_3+\dots. \tag{4.4}$$

The function Φ_1 is derived in the same way as in Ref. 2:

$$\Phi_1=A\Phi_0', \tag{4.5}$$

where A is an arbitrary constant [we are taking into account the Fourier representation of (3.2)]. The function Φ_2 turns out to be

$$\Phi_2=iA\Big[\Big(\Omega-\frac{\sigma}{\delta^{1/2}}K_\parallel\Big)\Phi_2^{(1)}-\sigma\Big(\frac{K_\parallel}{\delta^{1/2}}-\sigma\Omega\Big)\Phi_2^{(2)}\Big], \tag{4.6}$$

where

$$\Phi_2^{(1)}=\frac{\Phi_0'}{(K_\perp\mu)^2}\int^{\xi}\frac{1-e^{-i\psi}-i\psi}{\Phi_0'^2}d\zeta,$$
$$\Phi_2^{(2)}=-\frac{\Phi_0'}{(K_\perp\mu)^2}\int^{\xi}\frac{1-e^{-i\psi}-i\psi e^{-i\psi}}{\Phi_0'^2}d\zeta, \tag{4.7}$$

and $\psi\equiv K_\perp\mu\Phi_0$. The functions in the integrands are evaluated at $\xi=\zeta$, where ζ is the integration variable. The integrals are understood as functions of their upper limits. An orthogonality condition on Φ_3 (a dispersion relation) is found by integrating the corresponding third-order equation with a weight $(e^{i\psi}-1)/iK_\perp\mu$. This equation is

$$a_{11}\Big(\Omega-\frac{\sigma K_\parallel}{\delta^{1/2}}\Big)^2-2a_{12}\frac{\sigma}{\delta^{1/2}}\Big(\Omega-\frac{\sigma}{\delta^{1/2}}K_\parallel\Big)(K_\parallel-\sigma\delta^{1/2}\Omega)$$
$$+a_{22}\frac{\sigma^2}{\delta}(K_\parallel-\sigma\delta^{1/2}\Omega)^2+a_\perp K_\perp{}^2$$
$$-(K_\parallel-\sigma\delta^{1/2}\Omega)(a_\parallel^{(1)}K_\parallel-a_\parallel^{(2)}\sigma\delta^{1/2}\Omega)=0, \tag{4.8}$$

where

$$a_{11}=\int d\xi\Phi_0\Phi_2^{(1)}\frac{e^{i\psi}-1}{i\psi},$$
$$a_{12}=\frac{1}{2}\int d\xi\Phi_0\Big(\Phi_2^{(1)}e^{i\psi}+\Phi_2^{(2)}\frac{e^{i\psi}-1}{i\psi}\Big), \tag{4.9}$$
$$a_{22}=\int d\xi\Phi_0\Phi_2^{(2)}e^{i\psi},$$

$$a_\perp=\int d\xi\Phi_0'^2,\quad a_\parallel^{(1)}=\frac{2}{(K_\perp\mu)^2}\int(1-\cos\psi)d\xi,$$
$$a_\parallel^{(2)}=\frac{i}{(K_\perp\mu)^2}\int\psi(1-e^{i\psi})d\xi. \tag{4.10}$$

The quantities a_{ik} $(i, k=1, 2)$ are calculated in Appendix B, where the explicit expression for the function $\Phi_0(\xi)$ which follows from (3.7) is taken into account:

$$\Phi_0=3/2\,\mathrm{ch}^2(\xi/2). \tag{4.11}$$

As a result we find

$$a_{11}=a_{22}=\frac{1}{(K_\perp\mu)^2}\int d\xi(1-\cos\psi)G(\xi),$$
$$a_{12}=\frac{1}{2(K_\perp\mu)^2}\int d\xi\psi\sin\psi G(\xi), \tag{4.12}$$

$$G(\xi)=\frac{15}{2\,\mathrm{ch}^4(\xi/2)}-1-\frac{5}{2\,\mathrm{ch}^2(\xi/2)}-\frac{15}{4}\xi\frac{\mathrm{sh}(\xi/2)}{\mathrm{ch}^5(\xi/2)}. \tag{4.13}$$

It is interesting to note that the coefficient $a_\parallel^{(2)}$ is complex [see (4.10)], so that dispersion relation (4.8) is also complex. According to (4.10), the imaginary part of $a_\parallel^{(2)}$ stems from the finite value of $K_\perp\mu$, i.e., the finite value of the transverse wave number. It is interesting here to note the analogy with a result derived by Zakharov[22]: The dispersion relation in a problem of the Kadomtsev-Petviashvili type[3] also becomes complex at finite wave numbers.

The quantity σ in (4.8) is generally not an adjustable parameter. According to (3.8) and our assumption $\delta\ll1$, this quantity is related to u by the approximate relation

$$2(a_{11}+a_{12})\Omega'^2+a_\perp K_\perp{}^2=0. \tag{4.14}$$

In the case of ion acoustic waves in a homogeneous plasma ($V_* \to 0$), Eq. (4.14) would mean $\sigma^2=1$. Making use of the small value of δ, we find the following dispersion relation from (4.8):

$$a_{11}=a_{22}=a_{12}=\frac{1}{2}\int\Phi_0{}^2G(\xi)d\xi\equiv a^{(0)}, \tag{4.15}$$

where $\Omega'=\Omega-K_\parallel/\delta^{1/2}$.

5. ANALYSIS OF THE DISPERSION RELATION

5.1. The case $K_\perp\mu\to0$

We begin the analysis of dispersion relation (4.8) with the limiting case $K_\perp\mu\to0$, in which the vector nonlinearity is not important. In this approximation we find from (4.10) and (4.12)

$$a_\parallel^{(1)}=a_\parallel^{(2)}=\int\Phi_0{}^2d\xi\equiv a_\parallel^{(0)}. \tag{5.1}$$

The quantity $a_\perp$, on the other hand, does not depend on $K_\perp\mu$, as can be seen from (4.10), so for all $K_\perp\mu$ we have $a_\perp = \int d\xi \Phi'^2$. Using (5.1) and (4.14) we can put Eq. (4.8) in the form

$$(1+\sigma^2)^2 a^{(0)}\tilde{\Omega}^2 = -a_\perp K_\perp{}^2 + a_\parallel^{(0)} K_\parallel{}^2 \frac{(V_*/u)^2}{(1+\sigma^2)^2}, \quad (5.2)$$

$$\tilde{\Omega} = \Omega - \frac{2\sigma}{1+\sigma^2}\frac{K_\parallel}{\delta^{1/2}}. \quad (5.3)$$

For drift waves we would have $u \sim V_*$ and $\sigma \ll 1$, and Eq. (5.2) would mean

$$a^{(0)}\tilde{\Omega}^2 = -a_\perp K_\perp{}^2 + a_\parallel^{(0)} K_\parallel{}^2. \quad (5.4)$$

This result was derived in a different form by Petviashvili.[2] At $K_\perp > K_\parallel$, Eq. (5.4) describes an instability which, according to the interpretation of Ref. 2, should give rise to two-dimensional circular solitons. In the case $K_\perp < K_\parallel$ it follows from (5.4) that the perturbations are stable. This result becomes more understandable when we note that in the limit $K_\perp / K_\parallel \to 0$ the initial nonlinear equation reduces to the Kadomtsev-Petviashvili equation for a medium with negative dispersion and when we recall that one-dimensional solitons are stable in such media.[3]

For ion acoustic waves in a homogeneous plasma we have $V_*/u = 0$ and $\sigma^2 = 1$. In this case we find the following dispersion relation from (5.2) [or from (4.15)] in place of (5.4):

$$4a^{(0)}\tilde{\Omega}^2 = -a_\perp K_\perp{}^2. \quad (5.5)$$

This relation describes a transverse instability of the soliton. This instability is analogous to that studied in Ref. 23.

5.2. Case of small but nonvanishing $K_\perp\mu$

Assuming $K_\perp\mu \ll 1$ and taking into account the small terms on the order of $K_\perp\mu$ and $(K_\perp\mu)^2$, we find from (4.10) and (4.11) the following in place of (5.1):

$$a_\parallel^{(1)} \approx a_\parallel^{(0)}, \quad a_\parallel^{(2)} = a_\parallel^{(0)} + iK_\perp\mu c_\parallel,$$
$$a_{11} = a_{22} = a^{(0)} - (K_\perp\mu)^2 c^{(0)}, \quad a_{12} = a^{(0)} - 2(K_\perp\mu)^2 c^{(0)}. \quad (5.6)$$

Here

$$c_\parallel = \frac{1}{2}\int \Phi_0{}^3 d\xi = \frac{18}{5}, \quad c^{(0)} = \frac{1}{24}\int \Phi_0{}^4 G(\xi)\, d\xi = \frac{9}{20}. \quad (5.7)$$

In this case, dispersion relation (4.8) reduces to

$$(1+\sigma^2)^2 a^{(0)}\Omega_*{}^2 = -a_\perp K_\perp{}^2 + \frac{K_\parallel{}^2}{(1+\sigma^2)^2}\left(\frac{V_*}{u}\right)^2 \left[a_\parallel^{(0)} - 2c^{(0)}\sigma^2 \frac{(K_\perp\mu)^2}{\delta} \right] - i\frac{2K_\parallel{}^2\sigma^2 c_\parallel}{(1+\sigma^2)^2}\frac{V_*}{u} K_\perp\mu, \quad (5.8)$$

where Ω_* differs from $\tilde{\Omega}$ by terms of order $(K_\perp\mu)^2$, whose paticular form is not important to the discussion below. We see that, in contrast with the case $K_\perp\mu \to 0$ the leading terms with $K_\parallel^2$ in (5.8) change from stabilizing to destabilizing for waves with a finite $\sigma \neq 1$ even at rather small values of $(K_\perp\mu)^2$, specifically, at

$$(K_\perp\mu)^2 \gtrsim \delta. \quad (5.9)$$

These terms are larger than the usual terms with $K_\perp^2$ under the condition $K_\perp^2\mu^2 > \delta$. In dimensional form, inequality (5.9) gives us

$$k_\perp\rho_0 \gtrsim u/c_s\delta^{1/2}, \quad (5.10)$$

which is a weaker condition than the first inequality in (3.11).

Let us assume, however, that $K_\perp$ is small enough that condition (5.9) does not hold, and the leading terms with $K_\parallel$ in (5.8) are stabilizing, outweighing the destabilizing effect of the terms with $K_\perp^2$. In this case we find from (5.8)

$$(1+\sigma^2)^2 a^{(0)}\Omega_*{}^2 = \frac{K_\parallel{}^2}{(1+\sigma^2)^2}\frac{V_*}{u}\left(a_\parallel^{(0)}\frac{V_*}{u} - 2i\sigma^2 c_\parallel K_\perp\mu \right). \quad (5.11)$$

We see that an instability occurs for an arbitrarily small value of $K_\perp\mu$, although the growth rate for this instability is small in comparison with the real part of the wave frequency.

5.3. The case $K_\perp\mu \gg 1$

We now consider the limiting case of a strong vector nonlinearity, $K_\perp\mu \gg 1$. From (4.10) and (4.12) we find the approximate expressions

$$a_{11} = a_{22} \approx -\frac{2}{(K_\perp\mu)^2}\ln(6K_\perp\mu),$$
$$a_\parallel^{(1)} \approx \frac{4\ln(6K_\perp\mu)}{(K_\perp\mu)^2}, \quad a_\parallel^{(2)} \approx \frac{6i}{K_\perp\mu}. \quad (5.12)$$

For a_{12} we find the estimate $a_{12} \sim (K_\perp\mu)^{-2}$ from (4.12), so that this element is small in comparison with a_{11}, by a factor of $1/\ln(K_\perp\mu)$, so we can use the approximation $a_{12} \approx 0$. The dispersion relation then reduces to

$$(1+\sigma^4)\tilde{\Omega}^2 = a_\perp \frac{(K_\perp\mu)^2 K_\perp{}^2}{2\ln(6K_\perp\mu)} - K_\parallel{}^2 f(\sigma), \quad (5.13)$$

where

$$\tilde{\Omega} = \Omega - \frac{\sigma(1+\sigma^2)}{1+\sigma^4}\frac{K_\parallel}{\delta^{1/2}},$$
$$f(\sigma) = \frac{1-\sigma^2}{1+\sigma^4}\left\{ \frac{\sigma^2(1-\sigma^2)}{\delta} + 2\left[1 - \frac{\sigma^2(1+\sigma^2)}{1+\sigma^4}\frac{a_\parallel^{(2)}}{a_\parallel^{(1)}} \right] \right\}. \quad (5.14)$$

The most important new point seen in this short-wave approximation, $K_\perp\mu \gg 1$, is that the signs of a_{11} and a_{22} differ from those in the case of long waves [cf. (5.12) and (5.1)]. This result is important for both drift waves and ion acoustic waves. Accordingly, the purely transverse instability ($K_\parallel = 0$) which occurs at $K_\perp\mu < 1$ does not occur in this case [see (5.13) with $K_\parallel = 0$]. In this case, ion acoustic solitons are stable even if $K_\parallel \neq 0$, since we have $\sigma = 1$ and thus $f(\sigma) = 0$ for these solitons, according to the discussion above. In contrast with these solitons, drift solitons may be unstable at finite values of $K_\parallel$. With $\sigma = 0$ and sufficient small values of $K_\parallel$, the instability results from the complex nature of $f(\sigma)$, because of the imaginary nature of $a^{(2)}$ [see (5.12)]. If, on the other hand, we have $\sigma \neq 0$ and

$$K_{\parallel}/K_{\perp} \gtrsim K_{\perp}\mu\delta^{1/2}, \tag{5.15}$$

then there should be a more important aperiodic instability (in the sense that $\bar{\Omega}^2 < 0$). An aperiodic instability can also occur in the case of drift solitons with $\alpha \approx 0$, but here the condition $K_{\parallel}/K_{\perp} \gtrsim K_{\perp}\mu$ would have to hold. It should be kept in mind, however, that our analysis does not apply at large values of $K_{\parallel}$, since we have used the assumption that $\partial/\partial z$ quite small (Section 2).

6. DISCUSSION OF RESULTS

We have derived a dispersion relation for three-dimensional perturbations of one-dimensional drift-ion-acoustic solitons in a magnetized plasma [Eq. (4.8)], and we have analyzed this relation (Section 5). It follows from our analysis that vector-nonlinearity effects are important in problems of this sort if the soliton amplitudes are not too small. These effects may qualitatively change the stability picture. In particular, a vector nonlinearity can suppress the transverse instability of drift solitons ($K_{\perp} \neq 0$, $K_{\parallel} = 0$) which was discussed in Ref. 2, and it can also completely stabilize ion acoustic solitons. In this connection it is interesting to note the experimental fact[24] that plane ion acoustic solitons of sufficiently large amplitude are stable in a magnetized plasma (see Ref. 15 for a different interpretation of this fact). In the case of drift solitons, perturbations with $K_{\parallel}, K_{\perp} \neq 0$ may grow by virtue of the vector nonlinearity. The instability may be aperiodic, while in the absence of this instability an oscillatory instability may occur.

We believe that the approach formulated in this paper will also be useful for analyzing the stability of solitons of other types in a magnetized plasma, and it may be extended to problems involving solitons of Rossby waves in a rotating liquid.

APPENDIX A.

Derivation of Eq. (2.4)

In deriving (2.4) we use as the initial equations the continuity equation, the longitudinal equation of motion for the ions, and Boltzmann's law for electrons:

$$\frac{d_0 n}{dt} + n_0 \operatorname{div} \mathbf{V}_I + \frac{\partial}{\partial z}(nV_z) = 0, \tag{A1}$$

$$\frac{d_0 V_z}{dt} + V_z \frac{\partial V_z}{\partial z} = -\frac{e_i}{m_i}\frac{\partial \varphi}{\partial z}, \tag{A2}$$

$$n = n_0 \exp(-e_e\varphi/T_e). \tag{A3}$$

Here n is the density of each particle species (the sum of the equilibrium and wave components), V_z is the ion velocity along the magnetic field, and $\mathbf{V}_I$ is the velocity of the inertial motion of ions across the magnetic field. By definition we have

$$\operatorname{div} \mathbf{V}_I = -\frac{c}{B_0\omega_{Bi}}\frac{d_0}{dt}\Delta_{\perp}\varphi, \tag{A4}$$

where

$$\Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \frac{d_0}{dt} = \frac{\partial}{\partial t} + \frac{c}{B_0}[\nabla\varphi, \nabla]_z. \tag{A5}$$

The physical reason for the second term on the right side of (A5) is the electric drift of the particles, which occurs at a velocity $\mathbf{V}_E = c[\mathbf{e}_z, \nabla\varphi]/B_0$, where $\mathbf{e}_z$ is a unit vector along z. Accordingly, this term can also be written in the more customary form $\mathbf{V}_E\nabla$. We are assuming that the condition $\mathbf{V}_E\nabla \gg V_z\partial/\partial z$ holds in general. Using this condition, we have ignored the term on the order of $V_z\partial/\partial z$ on the right side of (A4). To avoid any misunderstanding, we will explain the specific although completely obvious nature of the operator $\mathbf{V}_E\nabla$. The result of the application of this operator to φ (or to any function of φ) is identically zero: $\mathbf{V}_E\nabla\varphi = 0$ (cf. the discussion in Section 1 of terms of the type $[\nabla a, \nabla b]_z$). Consequently, the relations of the type $(V_z\partial/\partial z + \mathbf{V}_E\nabla)\varphi = V_z\partial\varphi/\partial z$ used in Section 2 are identities, not the result of neglecting certain terms and thereby violating the original assumption $\mathbf{V}_E\nabla \gg V_z\partial/\partial z$.

As in Refs. 1 and 4, we replace y by the self-similar variable η, defined in Section 2. In contrast with Refs. 1 and 4, on the other hand, we assume that φ depends on both η and x, on the one hand, and z and t, on the other; the dependence φ on η is assumed to be stronger than that on the other arguments. We write V_z in the form (cf. Section 1)

$$V_z = \tilde{V}_z + V_{z0}(\varphi), \tag{A6}$$

where $V_{z0}(\varphi)$ satisfies Eq. (A2) in the approximation $(\partial/\partial t)_\eta, (\partial/\partial z)_\eta \to 0$, i.e.,

$$V_{z0} = \frac{e_i\alpha\varphi}{m_i u}\left(1 + \frac{\alpha^2}{2u^2}\frac{e_i\varphi}{m_i}\right). \tag{A7}$$

From (A2) we find an equation for $\tilde{V}_z$:

$$\hat{D}\tilde{V}_z = \frac{1}{u}\left[\frac{\partial V_z}{\partial t} + \frac{e_i}{m_i}\left(\frac{\partial\varphi}{\partial z} + \frac{\alpha}{u}\frac{\partial\varphi}{\partial t}\right)\right], \tag{A8}$$

where the operator $\hat{D}$ is defined by (2.6). Since $\partial/\partial t$ is small, as assumed above, we find from (A8) by the method of successive approximations

$$\tilde{V}_z = \frac{e_i}{m_i u}\left(\hat{D}^{-1} + \frac{1}{u}\hat{D}^{-1}\frac{\partial}{\partial t}\hat{D}^{-1}\right)\left(\frac{\partial\varphi}{\partial z} + \frac{\alpha}{u}\frac{\partial\varphi}{\partial t}\right). \tag{A9}$$

Substituting (A3), (A6), (A7), (A9) into (A1), we find (2.4).

APPENDIX B. CALCULATION OF a_{ik}

With Φ_0 as in (4.11) we have

$$\frac{1}{\Phi_0'^2} = \frac{8}{9}\frac{d}{d\xi}\left(\frac{15}{16}\xi + \frac{7}{8}\operatorname{sh}\frac{\xi}{2}\operatorname{ch}\frac{\xi}{2} + \frac{1}{4}\operatorname{sh}\frac{\xi}{2}\operatorname{ch}^3\frac{\xi}{2} - \operatorname{cth}\frac{\xi}{2}\right). \tag{B1}$$

Using (B1), and integrating by parts in (4.7), we find

$$\Phi_2^{(1)} = \frac{1}{(K_\perp\mu)^2}\left[(1 - e^{i\psi} - i\psi)G + i\psi'\int^{\xi}(1 - e^{i\psi})G\,d\zeta\right],$$

$$\Phi_2^{(2)} = -\frac{1}{(K_\perp\mu)^2}\left[(1 - e^{-i\psi} - i\psi e^{-i\psi})G + i\psi'\int^{\xi}\psi e^{-i\psi}G\,d\zeta\right], \tag{B2}$$

where G is defined by (4.13). Substituting (B2) into (4.9), and integrating by parts, we find (4.12).

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