Solitons in a random force field

F. G. Bass, V. V. Konotop, and Yu. A. Sinitsyn

Institute of Radiophysics and Electronics, Academy of Sciences of the Ukrainian SSR
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We study the dynamics of a soliton of the sine-Gordon equation in a random force field in the adiabatic approximation. We obtain an Einstein-Fokker equation and find the distribution function for the soliton parameters which we use to evaluate its statistical characteristics. We derive an equation for the averaged functions of the soliton parameters. We determine the limits of applicability of the delta-correlated in time random field approximation.

One of the basic characteristics of a dynamical system is its response to an external action. This refers, in particular, to a system of solitons described by completely integrable evolution equations. It is well known that such a set of solitons can be treated as a system of noninteracting particles. If the external force is a random function of the coordinates and the time, we must essentially be dealing with the Brownian motion of a soliton. In its general formulation such a problem is very complex as the dynamical equation is nonlinear in partial derivatives while the equation for the distribution function, i.e., the Hopf equation, is an equation in functional derivatives. There are at present practically no methods to solve that kind of equations. However, if the random force in some sense can be assumed to be small the use of some variants of perturbation theory[1–3] allows us to reduce the dynamical equations to ordinary differential equations for the soliton parameters, which can be treated as Langevin equations after which we can, under well defined assumptions about the nature of the external force, use the well developed apparatus of Markov processes. Notwithstanding its seeming simplicity such a program encounters, when one tries to realize it, appreciable technical difficulties; nonetheless is can be completed in certain well defined cases.

In the present paper we have chosen as the evolution equation the sine-Gordon equation which describes the propagation of nonlinear waves in a semiconductor with a superlattice,[4–6] Josephson junctions,[7,8] the motion of dislocations[9,10] and so on. The random force in this model corresponds to dynamic and static deviations of the superlattice from periodicity, to the presence of random contact inhomogeneities, or to slowing down of the dislocations. The starting equations of the problem has the form

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}+\sin\varphi=\varepsilon f(x,t). \tag{1}$$

We study the evolution of a single-soliton solution of this equation—a kink, which in the unperturbed case has the form

$$\varphi=4\,\mathrm{arctg}\exp[(x-X)(1-u_0^2)^{-1/2}], \tag{2}$$

where $X = u_0t - x_0$ is the center of the unperturbed kink, u_0 its velocity, and x_0 its phase.

We obtain in the first section the distribution function of the kink and evaluate the statistical characteristics of its parameters. In the second section we obtain an equation for the average values of functions of the parameters of the nonlinear wave. In the third section we discuss the limits of applicability of the proposed method.

1. When a soliton moves under the action of external perturbations its parameters change and its shape is also distorted; moreover, emission may occur in the system. In the present paper we study the effect of random inhomogeneities on the parameters u and X of the kink. The evolution these quantities under the action of a perturbation is described by the equations[2]

$$\frac{du}{dt}=-\varepsilon\frac{1-u^2}{4}\int_{-\infty}^{\infty}dx\,f(x,t)\,\mathrm{sech}\,\theta, \tag{3a}$$

$$\frac{dX}{dt}=u-\varepsilon\frac{u}{4}(1-u^2)^{1/2}\int_{-\infty}^{\infty}dx\,f(x,t)\,\theta\,\mathrm{sech}\,\theta,$$
$$\theta=(x-X)(1-u^2)^{-1/2}, \tag{3b}$$

which in the field of random forces are stochastic equations. To describe the statistical properties of the parameters that characterize the kink we introduce a distribution function (probability density)

$$P(\psi_1,\psi_2,t)=\langle\delta(\psi_1-u(t))\,\delta(\psi_2-X(t))\rangle. \tag{4}$$

The angle brackets indicate averaging over the realizations of the random force $f(x,t)$. We shall in what follows consider the case when the field is a Gaussian (in time) delta-correlated process:

$$\langle f(x,t)\rangle=0,\quad \langle f(x_1,t_1)f(x_2,t_2)\rangle=B(x_1-x_2)\,\delta(t_1-t_2). \tag{5}$$

In that case the probability density satisfies an Einstein-Fokker equation[11]

$$\frac{\partial P}{\partial t}=-\psi_1\frac{\partial P}{\partial\psi_2}+\frac{\varepsilon^2}{2}\iint_{-\infty}^{\infty}dx_1\,dx_2\,B(x_1-x_2)\frac{\partial}{\partial\psi_j}D_j(x_1)\frac{\partial}{\partial\psi_i}D_i(x_2)P, \tag{6}$$

where $i,j = 1,2$, and

$$D_1=\frac{1-\psi_1^2}{4}\,\mathrm{sech}\frac{x-\psi_2}{(1-\psi_1^2)^{1/2}},$$

$$D_2=\frac{\psi_1(x-\psi_2)}{4}\,\mathrm{sech}\frac{x-\psi_2}{(1-\psi_1^2)^{1/2}}. \tag{7}$$

We consider the case when the kink parameters initially are fixed:

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$$P(t=0)=P_0=\delta(\psi_1-u)\delta(\psi_2), \quad (8)$$

where without loss of generality we have set the initial phase of the kink equal to zero.

It is clear from (6) and (7) that in the case where the random field is statistically uniform the coefficients in the Einstein–Fokker equation are independent of the phase variable ψ_2 and the t which makes it possible to Fourier transform with respect of ψ_2 and Laplace transform with respect of t, after which (6) reduces to the form

$$A_{11}\frac{\partial^2\tilde P}{\partial\psi_1^2}+A_1\frac{\partial\tilde P}{\partial\psi_1}+[A_0-\omega-ik(\psi_1-A_2)-k^2A_{22}]\tilde P=-\tilde P_0, \quad (9)$$

where

$$\tilde P=\tilde P(\psi_1,k,\omega)=\frac{1}{2\pi}\int_0^\infty dt\int_{-\infty}^\infty d\psi_2 P(\psi_1,\psi_2,t)e^{-\omega t}e^{-ik\psi_2},$$

$$A_{ii}=\frac{1}{2}\iint_{-\infty}^{\infty} dx\,dx_1 D_i(x)D_i(x_1)B(x-x_1),$$

$$A_i=\frac{1}{2}\iint_{-\infty}^{\infty} dx\,dx_1\,D_i(x)\frac{\partial}{\partial\psi_j}D_j(x_1)B(x-x_1),$$

$$\tilde P_0=\delta(\psi_1-u_0), \qquad A_{12}=0.$$

We look for the solution of (9) in the form

$$\tilde P=P_1\exp\left\{-\frac{1}{2}\int A_1(\psi_1)A_{11}^{-1}(\psi_1)d\psi_1\right\}, \quad (10)$$

where P_1 satisfies the equation

$$\frac{\partial^2P_1}{\partial\psi_1^2}+\left[-\frac{\omega+ik(\psi_1-A_2)+k^2A_{22}}{A_{11}}\right.$$
$$\left.-\frac{1}{4}\left(\frac{A_1}{A_{11}}\right)^2-\frac{1}{2}\frac{\partial}{\partial\psi_1}\left(\frac{A_1}{A_{11}}\right)+\frac{A_0}{A_{11}}\right]P_1 \quad (11)$$
$$=-\tilde P_0A_{11}^{-1}\exp\left\{\frac{1}{2}\int\frac{A_1}{A_{11}}d\psi_1\right\}.$$

We have obtained an equation which contains a small parameter for the leading derivative so that we shall consider its solution for times $t\ll\varepsilon^{-2}$, meaning $\omega\gg\varepsilon^2$, in the WKB approximation:

$$\tilde P=\frac{1+O(\varepsilon^2)}{2}\frac{A_{11}^{1/4}(\psi_1)}{A_{11}^{3/4}(u_0)}[\varkappa(\psi_1)\varkappa(u_0)]^{-1/2}$$
$$\times\exp\left\{\frac{1}{2}\int_{\psi_1}^{u_0}\frac{A_1(x)dx}{A_{11}(x)}\right\}$$
$$\times\exp\left\{-\left|\int_{u_0}^{\psi_1}\frac{\varkappa(x)dx}{A_{11}^{1/2}(x)}\right|\right\}, \quad (12)$$

where

$$\varkappa^2(x)=\omega+ikx+k^2A_{22}(x).$$

We note that in the factor preceding the exponential we must retain the term of order ε^2.

The solution of Eq. (11), which is found using a well known method,[12] looks very unwieldy in the general case and we shall therefore give in what follows the form of the distribution function in two limiting cases: small and large spatial correlation ranges of the random force as compared to the kink dimensions.

We use below the model of a random field with a Gaussian correlation function

$$B(x-x_1)=\frac{1}{l\sqrt{2\pi}}\exp\left[-\frac{(x-x_1)^2}{2l^2}\right]. \quad (13)$$

We consider the motion of a kink in the field of inhomogeneities which are large-scale as compared with the kink width:

$$l\gg l_0=(1-u_0^2)^{1/2}, \quad (14)$$

where l_0 is the width of the unperturbed kink. In that case a soliton which is in a field which fluctuates with time manages to adjust itself to the spatial inhomogeneities of the system. Evaluating the coefficients A_i and A_{ii} and using (12) we find

$$\tilde P=\frac{\nu^{1/2}l^{1/2}}{2^{1/4}\pi^{5/4}}(1-\psi_1^2)^{-3/4}[\varkappa(\psi_1)\varkappa(u_0)]^{-1}$$
$$\times\exp\left\{-\frac{l^{1/2}\nu^{1/2}2^{1/4}}{\pi^{3/4}}\left|\int_{u_0}^{\psi_1}\frac{\varkappa(x)dx}{(1-x^2)^{3/4}}\right|\right\}$$
$$\times\left\{1-\frac{\pi^{3/4}}{2^{3/4}\nu^{1/2}l^{1/2}}\left|\int_{u_0}^{\psi_1}dx\left[-\frac{15}{32}\frac{k^2}{\varkappa(x)}(1-x^2)^{1/2}\right.\right.\right.$$
$$\left.\left.\left.+\frac{3ikx(1-x^2)^{1/4}}{8\varkappa^3(x)}\right]\right|\right\}, \quad (15)$$

where $\varkappa^2(x)=\omega+ikx$, $\nu=16\varepsilon^{-2}$.

We have dropped in (15) terms of higher order in l^{-1} and ε. It is convenient for the evaluation of the statistical characteristics of the kink parameters to have in what follows an expression for the distribution function $P^{(1)}(\psi_1,t)$ which is obtained from the probability density $P(\psi_1,\psi_2,t)$ by integrating over ψ_2 and has the form

$$P^{(1)}(\psi_1,t)=\frac{2^{3/4}\nu^{1/2}l^{1/2}}{t^{1/2}\pi^{5/4}}(1-\psi_1^2)^{-3/2}$$
$$\times\exp\left\{-\frac{\nu l}{2^{1/2}\pi^{3/4}t}\left[\frac{\psi_1}{(1-\psi_1^2)^{1/2}}-\frac{u_0}{(1-u_0^2)^{1/2}}\right]^2\right\}. \quad (16)$$

The distribution function is obtained in the time interval

$$0\leqslant t\ll\nu; \quad (17)$$

hence, it is a rapidly decreasing function. This makes it possible to calculate asymptotically the characteristics of the parameters u and X.

For the mean value and the variance of $u(t)$, and also for the width of the adiabatic part of the kink $\Delta=(1-u^2)^{1/2}$, we get, using (16),

$$\langle u\rangle=u_0-3\pi^{3/4}2^{-1/2}u_0(1-u_0^2)^2t(\nu l)^{-1}, \quad (18)$$

$$\sigma_u=\pi^{3/4}2^{1/2}(1-u_0^2)^3t(\nu l)^{-1}, \quad (19)$$

$$\langle\Delta\rangle=l_0-\pi^{3/4}2^{-1/2}(1-u_0^2)^{3/2}(1-3u_0^2)t(\nu l)^{-1}, \quad (20)$$

$$\sigma_\Delta=\pi^{3/4}2^{1/2}u_0^2(1-u_0^2)^2t(\nu l)^{-1}. \quad (21)$$

To determine the width of the perturbed kink it is necessary to take into account the first correction to the adiabatic approximation. Similar calculations lead to the following statistical characteristics of $X(t)$:

$$\langle X\rangle = u_0 t - 3\pi^{3/2}2^{-3/2}u_0(1-u_0^2)^2 t^2(\nu l)^{-1}, \tag{22}$$

$$\sigma_X = \frac{1}{3\sqrt{2}}\pi^{3/2}(1-u_0^2)^3 t^3(\nu l)^{-1}. \tag{23}$$

It is clear from (18) and (22) that the average velocity of the kink is equal to $\langle u(t)\rangle$. It also follows immediately from Eqs. (3) that $\langle \dot{X}^n(t)\rangle = \langle u^n(t)\rangle$. This expresses the fact that when the kink moves in a field with large-scale spatial inhomogeneities $P^{(1)}(\psi_1,t)$ is, apart from terms of order l^{-2}, the velocity distribution function (in the general case a similar statement is not true—see below and also Refs. 11, 13). The statistics of the phase variables is thus in the case (14) completely determined by the fluctuations in the velocity u.

It follows from (18) that a wave which propagates in a medium with random inhomogeneities is slowed down and the most effective deceleration is found for waves which are incident with a velocity $u_0 = \sqrt{0.2}$ upon a randomly inhomogeneous region.

It is also clear from (18), (19) that if the kink is initially at rest the switching on of the external field sets it to oscillate about its original position and the speed of the oscillations increases with time. A similar behavior of a kink is also observed in the case of small-scale spatial inhomogeneities (see (27) to (29) below). In general the qualitative nature of this behavior is independent of the scale of the inhomogeneities, as a kink at rest does not feel the spatial changes in the medium. The scale of the inhomogeneities only affects the magnitude of the accceleration of the wave.

The other limiting case is the evolution of a kink in the field of small-scale spatial inhomogeneities:

$$l \ll l_0. \tag{24}$$

In that case the distribution function has the form

$$\tilde{P} = \frac{\nu^{1/2}}{2}\frac{(1-u_0^2)^{1/8}}{(1-\psi_1^2)^{11/8}}[\varkappa(\psi_1)\varkappa(u_0)]^{-1/2}\exp\left\{-\nu^{1/2}\left|\int_{u_0}^{\psi_1}\frac{\varkappa(x)\,dx}{(1-x^2)^{5/4}}\right|\right\}$$
$$\times\left\{1-\frac{1}{\nu^{1/2}}\left|\int_{u_0}^{\psi_1}d\psi_1\left[-\frac{3}{32}\frac{k^2(1-\psi_1^2)^{5/4}}{\varkappa(\psi_1)^5}+ik\cdot\frac{5}{8}\frac{\psi_1(1-\psi_1^2)^{1/4}}{\varkappa(\psi_1)^3}\right.\right.\right.$$
$$\left.\left.\left.+\frac{1-{}^5/_4\psi_1^2}{4\varkappa(\psi_1)(1-\psi_1^2)^{3/4}}\right]\right|\right\}. \tag{25}$$

Integrating over ψ_2 and taking the inverse Laplace transform we find

$$P^{(1)}(\psi_1,t) = \frac{1}{2}\left(\frac{\nu}{\pi t}\right)^{1/2}\frac{(1-u_0^2)^{1/8}}{(1-\psi_1^2)^{11/8}}$$
$$\times\exp\left\{-\frac{\nu}{4t}\left[\int_{u_0}^{\psi_1}\frac{dx}{(1-x^2)^{5/4}}\right]^2\right\}$$
$$-\frac{t}{4\nu}\left(1-\frac{5}{4}u_0^2\right)(1-u_0^2)^{1/2}\delta(\psi_1-u_0). \tag{26}$$

Using (26) we calculate

$$\langle u\rangle = u_0 - 2u_0(1-u_0^2)^{3/2}t\nu^{-1}, \tag{27}$$

$$\langle\Delta\rangle = l_0 - (1-u_0^2)(1-2u_0^2)t\nu^{-1}, \tag{28}$$

$$\sigma_\Delta = u_0^2(1-u_0^2)^{3/2}t\nu^{-1}. \tag{29}$$

The phase variable X has the following statistical characteristics:

$$\langle X\rangle = u_0 t - u_0(1-u_0^2)^{3/2}t^2\nu^{-1}, \tag{30}$$

$$\sigma_X = (1-u_0^2)^{3/2}t\nu^{-1}\left[\pi^2u_0^2+\frac{2}{3}(1-u_0^2)t^2\right]. \tag{31}$$

We recall that (27) to (31) were obtained in the time interval (17). In all formulae describing the statistical characteristics of the wave the term following the zeroth order one is proportional to ε^2. This reflects the fact that $\langle f(x,t)\rangle = 0$; hence the small parameter in the problem is not ε but ε^2. It is, however, well known that there exist corrections to the adiabatic approximation which are connected with second order perturbation theory[11] and which lead to effects such as, for instance, the deceleration of a kink due to emission. It is clear that by virtue of the cause above indicated these terms will make a contribution of order ε^2 to the adiabatic-approximation equations only when they are quadratic functionals of the random external field, otherwise the averaging increases the order of smallness. Moreover, the presence of corrections in the equation for $u(t)$ is important for the kink dynamics, as up to terms of order ε^3 they can be replaced by a constant and the appearance of a constant in the equation for $X(t)$ is not reflected in the dynamics of the wave. One verifies by a direct calculation that the averaged second-order corrections to the equation for $u(t)$ are proportional to ε^3. This enables us to state that the effects of an interaction of the soliton with the radiation induced by random inhomogeneities are of higher order and are switched on at times for which the formulae obtained by us cease to hold. Problems referring directly to the radiation by a kink require a separate consideration. For this it is necessary to go beyond the framework of the approximation of a delta-correlated process, as noise contains all frequencies, including resonant ones.[11]

In contrast to the case of large-scale inhomogeneities, in the approximation (24) $P^{(1)}(\psi_1,t)$, notwithstanding the fact that $\langle \dot{X}(t)\rangle = \langle u(t)\rangle$, is not a velocity distribution function, since $\langle \dot{X}^n(t)\rangle \neq \langle u^n(t)\rangle$. To evaluate the moments $\langle \dot{X}^n(t)\rangle$ it is, in general, necessary to known the complete distribution function $P(\psi_1,\psi_2,t)$. We note that the equality for $n=1$ is a consequence of the statistical uniformity of the random field in the spatial variable.

The indicated difference between the motions in a field of large- and of small-scale inhomogeneities is connected with the fact that in the first case the distribution function of the phases of $X(t)$ retains, as time evolves, the form of a delta-function moving with a velocity u, whereas in the second case the distribution function spreads out symmetrically relative to a center which also moves with the kink velocity.

In concluding this section we note that the distribution function for the parameter $u(t)$ in the case (24) was considered in Ref. 15. However, a stationary distribution was observed there when there was friction present in the dynamic equation. In the present paper we study a nonstationary problem and find a two-parameter distribution function.

2. We obtained in Sec. 1 the probability density for the kink parameters which we used to evaluate any function of $u(t)$ and $X(t)$. However, in a number of cases it may turn out to be more convenient to analyze the equations directly for the average values, in particular, for the average kink and the dispersion variance σ_φ.

Using (5), such an equation has in the adiabatic approximation the form

$$\left\langle \frac{\partial \chi}{\partial t} \right\rangle = \left\langle u(t) \frac{\partial \chi}{\partial X} \right\rangle + \frac{1}{2} \left\langle \iint_{-\infty}^{\infty} dx_1\, dx_2\, B(x_1-x_2) \frac{\delta}{\delta f(x_1,t)} \frac{\delta}{\delta f(x_2,t)} \chi \right\rangle . \tag{32}$$

Here χ is a function of u and X. Equation (32) is not closed in $\langle \chi \rangle$. One can close it by decoupling the correlators, using the small parameter ε. However, there arises then a restriction on the time during which one can retain in the decoupling only the lowest terms of the expansion in ε.

To obtain a closed equation we consider the correlator $\langle \Phi(t), \chi(t) \rangle$, where $\Phi = \Phi(u, X, t)$ and χ are functionals of the random field $f(x,t)$. We expand Φ in a functional series (it is sufficient to retain the first three terms):

$$\begin{aligned}\langle \Phi(t), \chi(t) \rangle = {} & \Phi_0 \langle \chi \rangle + \int_{-\infty}^{\infty} dx \int_0^t d\tau \frac{\delta \Phi_0}{\delta f(x,\tau)} \langle f(x,\tau) \chi \rangle \\ & + \iint_{-\infty}^{\infty} dx_1\, dx_2 \iint_0^t dt_1\, dt_2 \frac{\delta^2 \Phi_0}{\delta f(x_1,t_1)\, \delta f(x_2,t_2)} \\ & \times \langle f(x_1,t_1) f(x_2,t_2) \chi \rangle ,\end{aligned} \tag{33}$$

where

$$\frac{\delta^n \Phi_0}{\delta f(x_1,t_1) \ldots \delta f(x_n,t_n)} \equiv \left. \frac{\delta^n \Phi}{\delta f(x_1,t_1) \ldots \delta f(x_n,t_n)} \right|_{f\equiv 0}$$

and the limits of integration are established taking into account that

$$\frac{\delta \Phi(t)}{\delta f(x_1,t_1)} = 0 \quad \text{at} \quad t_1 < 0 \text{ and } t_1 > t .$$

We consider the last term in (33). Using (5), we have, accurate to ε^2

$$\langle f(x_1, t_1) f(x_2, t_2) \chi \rangle = B(x_1 - x_2)\, \delta(t_1 - t_2)\, \chi_0 . \tag{34}$$

In the case when $\chi = \chi(\theta)$ (such a function is, e.g., the kink $\chi = \varphi_S$) we can write the last term in (33) in the form $t / \tilde{t}_v(u_0, \theta)$, where

$$\tilde{t}_v^{-1}(u_0, \theta) = \iint_{-\infty}^{\infty} dx_1\, dx_2 \frac{\delta^2 \Phi_0}{\delta f(x_1,t)\, \delta f(x_2,t)} B(x_1 - x_2) .$$

The second term also reduces to a similar form $t / t_v(u_0, \theta)$; here

$$t_v^{-1}(u_0, \theta) = \iint_{-\infty}^{\infty} dx_1\, dx_2 \frac{\delta \Phi_0}{\delta f(x_1,t)} \frac{\delta \chi_0}{\delta f(x_2,t)} B(x_1 - x_2) .$$

We thus have

$$\langle \Phi(t), \chi(t) \rangle = \Phi_0 \langle \chi \rangle + t \tau_v^{-1}(u_0, \theta) , \tag{35}$$

where $\tau_v^{-1} = t_v^{-1} + \tilde{t}_v^{-1}$. It is immediately clear from the expression for τ_v that for large-scale spatial inhomogeneities $\tau_v \sim vl$. Hence, for the decoupling of correlators of the form (33) we can limit ourselves to the first term in (33) for times $t \ll \tau_v$.

In the case (14) we have

$$\begin{aligned}&\frac{1}{2} \left\langle \iint_{-\infty}^{\infty} dx_1\, dx \frac{\delta^2 \chi}{\delta f(x,t)\, \delta f(x_1,t)} \right\rangle \\ &= \frac{\pi^{3/2} (1 - u_0^2)^{3/2}}{2^{3/2} vl} \frac{\partial}{\partial u_0} (1 - u_0^2)^{3/2} \frac{\partial}{\partial u_0} \langle \chi \rangle .\end{aligned} \tag{36}$$

Changing to a reference frame that moves with the soliton, we get an equation for the average value $\langle \chi \rangle$ in the form

$$\frac{\partial \langle \chi \rangle}{\partial t} = \frac{\pi^{3/2}}{2^{3/2} vl} (1 - u_0^2)^{3/2} \frac{\partial}{\partial u_0} (1 - u_0^2)^{3/2} \frac{\partial}{\partial u_0} \langle \chi \rangle . \tag{37}$$

This equation has the solution

$$\langle \chi \rangle = \frac{v^{1/2} l^{1/2}}{t^{1/2} 2^{1/4} \pi^{5/4}} \int_{-\infty}^{\infty} d\xi \exp\left[-\frac{vl (\xi_0 - \xi)^2}{2^{1/2} \pi^2 t} \right] \chi_0 (\xi (1 + \xi^2)^{1/2}) . \tag{38}$$

We introduced here the notion $\xi_0 = u_0 (1 - u_0^2)^{-1/2}$, $\zeta = x - u_0 t$, and χ_0 is the unperturbed value of the function χ. For the dispersion of the adiabatic term we get

$$\sigma_\varphi = \frac{2^{5/2} \pi^{3/2} t}{vl} \zeta^2 u_0^2 \operatorname{sech}^2 [\zeta (1 - u_0^2)^{-1/2}] . \tag{39}$$

3. In the preceding sections we studied the motion of a kink in the approximation of a Gaussian random process which is delta-correlated in time. We now discuss the limits of applicability of this approximation.

The Einstein-Fokker equation for a process with an arbitrary range of correlations has the form[13]

$$\begin{aligned}\frac{\partial P}{\partial t} = -\psi_1 \frac{\partial P}{\partial \psi_2} + \Bigg\langle \int_0^t d\tau \iint_{-\infty}^{\infty} dx\, dx_1\, \Psi(x, t | x_1, \tau) \frac{\delta}{\delta f(x,t)} \\ \times \frac{\delta}{\delta f(x_1,\tau)} \delta(\psi_1 - u)\, \delta(\psi_2 - X) \Bigg\rangle ,\end{aligned} \tag{40}$$

where $\Psi(x,t | x_1, \tau)$ is the correlator of the field $f(x,t)$. The expression under the averaging sign can be rewritten in the form

$$\begin{aligned}J = \int_0^t d\tau \iint_{-\infty}^{\infty} dx\, dx_1\, \Psi(x, t | x_1, \tau) \frac{\partial}{\partial \psi_i} D_i(t | x, t) \frac{\partial}{\partial \psi_j} D_j(t | x_1, \tau) \\ \times \delta(\psi_1 - u)\, \delta(\psi_2 - X) .\end{aligned} \tag{41}$$

We have here introduced the notation

$$D_i(t | x, \tau) = \delta \psi_i / \delta f(x, \tau) .$$

From the set (3) we find

$$D_1(t | x, t) = -2^{-2} \varepsilon (1 - u^2) \operatorname{sech} \theta , \tag{42}$$

$$D_2(t | x, t) = -2^{-2} \varepsilon u (1 - u^2)^{1/2} \theta \operatorname{sech} \theta . \tag{43}$$

Expanding $D_i(t | x, \tau)$ in a functional series in $f(x,t)$ we write $\langle J \rangle$ in the form

$$\langle J\rangle=\langle J_0\rangle+\langle J_1\rangle+\langle J_2\rangle+\ldots, \tag{44}$$

where $\langle J_{n-1}\rangle\langle J_n\rangle^{-1}\sim\varepsilon^{-2}$. The approximation of a delta-correlated process corresponds to the fact that, up to the accuracy in a small parameter η_0, $\langle J_0\rangle$ is the same as the corresponding expression for the random field $f(x,t)$ which is delta-correlated in time, and $\langle J_n\rangle\ll\langle J_0\rangle$ for $n\geqslant 1$. A direct calculation of $\langle J_n\rangle$ and a comparison of it with $\langle J_0\rangle$ shows that for $\eta_\varepsilon=\varepsilon^2 T\ll 1$, where T is the radius of the time correlations of the random field $f(x,t)$, we have $\langle J_n\rangle\ll\langle J_0\rangle$, and for $\eta_0=Tu_0 l_0^{-1}\ll 1$ the first requirement is met.

The conditions which we obtained here for the applicability of a process delta-correlated in time can be explained. This approximation means the smoothness of the change with time of functions of the kink parameters as compared to the correlation function of the field. The rate of change of the kink parameters is determined by the action of the external field $\varepsilon f(x,t)$. The condition $\eta_\varepsilon\ll 1$ means that the intensity of the action of the external field on the soliton is sufficiently small while the condition $\eta_0\ll 1$ is the requirement that the kink in its motion in space feel a smooth variation of the field. The region of applicability of the delta-correlated process approximation when a kink moves in a field with large-scale inhomogeneities is thus a broad one.

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