Asymptotic stability of manifold of self-similar solutions in self-focusing

G. M. Fraĭman

Institute of Applied Physics, USSR Academy of Sciences
(Submitted 4 July 1984)
Zh. Eksp. Teor. Fiz. **88**, 390–400 (February 1985)

The dynamics of self-focusing of quasioptic beams is analytically investigated within the framework of the nonlinear Schrödinger equation. It is proved that the manifold of the self-similar solutions is asymptotically stable. The manifestation of the stability is that the energy influx into the singularity is equal to the critical flux. A procedure is proposed for deriving a set of equations that is equivalent to the nonlinear Schrödinger equation and yields explicitly the focusable and nonfocusable components of the beam field. The character of the singularity in the case of self-focusing is analyzed within the framework of this set of equations.

§I. INTRODUCTION

A large class of physical phenomena is described, under certain assumptions, by a nonlinear Schrödinger equation. These include self-focusing of quasioptic light beams (see, e.g., the reviews by Askar'yan[1] and by Prokhorov and Lugovoi,[2]) electron-phonon interaction in solids,[3] self-focusing of various waves in a plasma,[4] and others. In terms of nondimensional variables, the equation for all the foregoing problems can be reduced to the form

$$2i\frac{\partial u}{\partial t}+\Delta u+|u|^2u=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \tag{1.1}$$

where $u(x,y,t)$ is the nondimensionalized field amplitude; x, y, and t are the nondimensionalized coordinates.

The solution of Eqs. (1.1) becomes infinite within a finite time t_0 under a large class of initial conditions. Numerical calculations (see, e.g., Refs. 5–7) show that the character of this singularity is such that the energy flux

$$P_0=\int |u|^2\,dx\,dy, \tag{1.2}$$

into the singularity is of the order of the critical $P_{\rm cr}\approx 11.86$. An exact interpretation of P_0 entails certain difficulties. It is therefore of interest to solve this problem analytically. It is shown in the present paper that the field distribution realized near the singularity ($t\to t_0$) is asymptotically close to the Townes distribution,[8] in which case $P_0=P_{\rm cr}$. The law governing the variation of the beam width $a(t)$ is derived *in passim*.

For the reader's convenience, we summarize briefly the content of the paper. Its main idea is the following. As shown in Ref. 9, Eq. (1.1) has a three-parameter manifold of self-similar solutions (2.1). We propose that the solution of our problem near the instant of singularity formation is close to a distribution of type (2.1), except that unlike in (2.1) the law governing the variation of the width $a(t)$ is determined from the requirement that the distribution be stable in the linear approximation in the perturbation. In lieu of the initial equation (1.1) we arrive thus at an equivalent system of nonlinear equations [(3.12), (3.14) and (3.15)] for the amplitude and width of the distribution (2.1) (of the "peak") and for the perturbations orthogonal to it (the "background"). This system is derived by using essentially the results of Ref. 9, where it is shown that a lens transformation changes the entire manifold of the self-similar solutions (2.1) into the homogeneous beam (2.3). With allowance for this fact, the linearized problem is analyzed in §2 using the results of Ref. 10, in which the unstable modes were found. The requirement that the amplitudes of these modes vanish along the entire focusing path do in fact lead to Eq. (3.14) for the peak width $a(t)$ and to Eq. (3.1) for its amplitude.

The solution of the system (3.12), (3.14), and (3.15) is further simplified by a property revealed by analysis of the spectrum of the linearized problem (§2), viz., the presence of a broad band ($\sim 1/2a^2$) between the lines of the discrete spectrum and the boundary of the continuous spectrum. This permits, first, the self-action of the perturbations to be neglected in the analysis of the behavior of the solution near the instant of singularity formation (following the lens transformation (3.1), this instant corresponds to $\tau\to\infty$). This means that Eq. (3.15) can be solved in the linear approximation, and this is readily achieved with the aid of relations (2.34)–(2.37). Second, the slow variation of the background:

$$\Delta\omega_b \ll 1/2a^2(t) \tag{1.3}$$

permits Eqs. (3.14) and (3.15) to be averaged over the high frequency $\omega_0=1/2a^2$ of the peak.

In §§4 and 5 is analyzed the behavior of the solution near the instant of singularity formation ($t\to t_0$, i.e., $\tau\to\infty$). Two characteristic focusing regimes are observed here. First, (§4), if the peak amplitude differs only little from unity from the very beginning (the energy flux in the peak is close to critical), so that notwithstanding the rapid spreading of the background the excess photons manage to be reradiated into the background via nonlinear parametric interaction directly in the region of the peak, then the law governing the singularity (the quantity λ in the relation $a(t)\propto(t_0-t)^\lambda$) is close to the linear (4.11). If, however, the initial background is small, the principal effect is the emission, by tunneling, of the peak field into the background through an opacity region whose size is proportional to d^2a/dt^2 (§5). The singularity exponent λ is then close to $\frac{1}{2}$ [Eq. (5.17)], in good agreement with the results of Refs. 5–7.

§2. INDIFFERENT STABILITY OF MANIFOLD OF SELF-SIMILAR-SOLUTIONS

Equation (1.1) has an important class of exact solutions[9]—the manifold of self-similar solutions

 0038-5646/85/020228-06$04.00

$$u_0=\frac{1}{a}\varphi_0\left(\frac{r}{a}\right)\exp\left\{\frac{1}{2}\left[\frac{\dot a}{a}r^2+\int\frac{dt}{a^2}+\psi_0\right]\right\},\quad (2.1)$$

where ψ_0, a, and $\dot a$ = const are parameters of the manifold; $\varphi_0(r)$ is the localized positive solution of the ordinary differential equation[8]

$$\Delta\varphi_0+(\varphi_0^2-1)\varphi_0=0,\qquad \Delta=\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}.\quad (2.2)$$

The manifold (3.1) plays the same role in self-focusing theory as the solitons in the one-dimensional problem, which are typical of the asymptotic form of the initial problem.

The most important properties of the solution (2.2)

$$u_T=\frac{1}{a}\varphi_0\left(\frac{r}{a}\right)\quad (2.3)$$

are that it is extremal in the following problem. Assume that we are interested in the ground state with energy E_0 in a two-dimensional problem described by the stationary Schrödinger equation:

$$[\Delta+U(r)]\psi=E\psi\quad (2.4)$$

under the condition that

$$\frac{1}{2}\int U^2d\mathbf{r}=N=\text{const},\qquad d\mathbf{r}=dx\,dy.\quad (2.5)$$

For which potential $U(r)$ is the maximum of E_0 realized at constant N? We multiply (2.4) by ψ^* and integrate with respect to $\mathbf{r}$. Obvious transformations yield

$$E_0=\max_{U,\psi}\left.\frac{\int U|\psi|^2d\mathbf{r}-\int|\nabla\psi|^2d\mathbf{r}}{\int|\psi|^2d\mathbf{r}}\right|_{N=\text{const}}.\quad (2.6)$$

Assuming that the distribution of interest to us is symmetric (this can be demonstrated), we have

$$\int U|\psi|^2d\mathbf{r}\leqslant\left(\int U^2d\mathbf{r}\right)^{1/2}\left(\int|\psi|^4d\mathbf{r}\right)^{1/2}.\quad (2.7)$$

We note that equality in (2.7) is realized at

$$U=\text{const}\cdot|\psi|^2.\quad (2.8)$$

On the other hand, it is easy to show that

$$\frac{1}{2}\int|\psi|^4d\mathbf{r}\leqslant\frac{1}{P_{\text{cr}}}\int|\psi|^2d\mathbf{r}\int|\nabla\psi|^2d\mathbf{r}.\quad (2.9)$$

Indeed, it is known (see, e.g., Ref. 12) that the functional $\int|\psi|^4d\mathbf{r}$ is bounded from above by an inequality such as (2.9). The exact value of the constant can be established by noting that $\psi=u_T$ [Eq. (2.3)] changes (2.9) into an equality and is a solution of the corresponding variational problem.

Substituting (2.7) and (2.8) in (2.6) we have

$$E_0\leqslant\frac{N}{P_{\text{cr}}}=\frac{1}{2P_{\text{cr}}}\int U^2d\mathbf{r}.\quad (2.10)$$

Taking (2.8) into account, we easily verify that equality in (2.10) is reached at

$$U=u_T^2,\quad (2.11)$$

when

$$\max E_0=1/a^2.\quad (2.12)$$

Thus, the set (2.11) and (2.12) solves the problem (2.4) and (2.5). The equation for the sought fundamental mode reduces,[1)] of course, to (2.2).

It follows from this reasoning, in particular, that if (1.1) is regarded as a Schrödinger equation with potential $U=|u(r,t)|^2$ the distribution (2.3) maximizes the natural frequency of the fundamental modes at a fixed value of

$$N[u]=\frac{1}{2}\int|u|^4d\mathbf{r}.\quad (2.13)$$

On this basis we can expect this manifold to be stable, i.e., that small perturbations do not grow.

To verify this, we discuss now the properties of the Hamiltonian $H[u]$ of Eq. (1.1):

$$H[u]=\int\left(|\nabla u|^2-\frac{1}{2}|u|^4\right)d\mathbf{r}\quad (2.14)$$

on near-Townes distributions. Let

$$u=u_T+u_\perp,\qquad \int u_Tu_\perp\,d\mathbf{r}=0.\quad (2.15)$$

We substitute (2.15) in (2.14). Recognizing that

$$\int|\nabla u_T|^2d\mathbf{r}=\int|u_T|^2d\mathbf{r}=\frac{1}{2}\int|u_T|^4d\mathbf{r}=P_{\text{cr}},\quad (2.16)$$

we have accurate to terms of second order in u

$$H[u]=H_2[u]$$
$$=\int\left[|\nabla u_\perp|^2-|u_\perp|^2|u_T|^2-\frac{1}{2}(u_Tu_\perp^*+\text{c.c.})\right]d\mathbf{r}.\quad (2.17)$$

It is easy to verify that the ground state [the minimum of (2.16)] at $P_\perp=\int|u_\perp|^2d\mathbf{r}$ = const is realized if

$$u_{\perp 0}=\partial u_T/\partial a,\quad (2.18)$$

wherein

$$H_2[u_\perp]_{min}=-\frac{1}{a^2}P_\perp.\quad (2.19)$$

Indeed, substituting

$$u_\perp=q+i\eta\quad (2.20)$$

in (2.16) and (2.18), we get

$$H_2=-\int(q\hat L_1q+\eta\hat L_0\eta)\,d\mathbf{r},\qquad P_\perp=\int(q^2+\eta^2)\,d\mathbf{r},$$
$$\hat L_0=\Delta+|u_T|^2,\qquad \hat L_1=\Delta+3u_T^2.\quad (2.21)$$

On the other hand, substituting (2.15) in (2.9) and recognizing that u_T makes (2.9) a rigorous equality, we have

$$G[u]=\frac{1}{2}\int|u|^4\,d\mathbf{r}-\frac{1}{P_{\text{cr}}}\int|u|^2\,d\mathbf{r}\int|\nabla u|^2\,d\mathbf{r}$$
$$=-H_2[u_\perp]-\frac{P_\perp}{a^2}\leqslant 0.$$

Putting

$$u_\perp=C_\perp u_{\perp 0}+u_{\perp\perp},\qquad \int u_{\perp 0}u_{\perp\perp}\,d\mathbf{r}=0,\quad (2.22)$$

and taking (2.19) into account, we have

$$H_2[u_{\perp\perp}]+\frac{1}{a^2}P[u_{\perp\perp}]\geqslant 0.\quad (2.23)$$

It follows hence, in particular, that the homogeneous beam (2.3) is stable (in the linearized approximation) to exponentially growing small perturbations (this results was obtained numerically in Ref. 13).

To conclude this section, we discuss the spectral properties of the linearized problem. Substituting in (1.1)

$u=(u_T+u_\perp)\exp(it/2a^2)$

and separating the terms linear in $u_\perp$, we have

$$2i\frac{\partial u_\perp}{\partial t}+\hat L_0 u_\perp+u_T^2(u_\perp+u_\perp^*)=j_\perp,\qquad j=\frac{u_T}{P_{cr}}\int u_T^3(u_\perp+u_\perp^*)\,d\mathbf{r}. \tag{2.24}$$

Here $j_\perp$ ensures orthogonality of u_T and $u_\perp$. The explicit form of $j_\perp$ is obtained after multiplying (2.24) by u_T and integrating with respect to **r**. Separating the real and imaginary parts in (2.24)

$$u=q+i\eta, \tag{2.25}$$

we have

$$\hat L_1 q=2\frac{\partial\eta}{\partial t}+j_q,\qquad \hat L_0\eta=-2\frac{\partial q}{\partial t},\quad j_q=\frac{2u_T}{P_{cr}}\int u_T^3 q\,d\mathbf{r}. \tag{2.26}$$

Since the solution of (2.26) is exponentially stable, it is expedient to seek the eigenvector of this problem in the form

$$\begin{pmatrix} q\\ \eta\end{pmatrix}_\omega=\mathrm{Re}\begin{pmatrix} q_\omega\\ -i\eta_\omega\end{pmatrix}e^{i\omega t}. \tag{2.27}$$

In this case we have for q_ω and η_ω the system of equations

$$\hat L_1 q_\omega=2\omega\eta_\omega+j_q[q_\omega],\qquad \hat L_0\eta_\omega=2\omega q_\omega. \tag{2.28}$$

Taking (2.23) into account, we have therefore

$$2\omega\left(\int \eta_\omega^* q_\omega\,d\mathbf{r}+\text{c.c.}\right)=0. \tag{2.29}$$

It can be seen from (2.29) that the eigenvalue ω are pure real. In addition, inasmuch as according to (2.28)

$$q_{-\omega}=q_\omega,\qquad \eta_{-\omega}=-\eta_\omega, \tag{2.30}$$

it suffices to discuss the linear-problem spectrum properties at $\omega>0$.

The spectrum point $\omega=0$ is the ground state and corresponds to the eigenvector (2.18)

$$q_\omega=\frac{\sqrt{2}}{\xi_{\text{eff}}}\frac{\partial u_T}{\partial a},\qquad \eta_\omega=0,\qquad \xi_{\text{eff}}^2=\int \xi^2\varphi_0^2\,d\xi. \tag{2.31}$$

Note that for Eqs. (2.26) this solution corresponds to a mode that increases with time:

$$q(r,t)=\frac{\partial u_T}{\partial a}t,\qquad \eta(r,t)=-\frac{1}{a}\left(\frac{r^2}{a^2}-\xi_{\text{eff}}^2\right)u_T.$$

It is easy to deduce from (2.27) the completeness relation for the eigenfunction system

$$\int \eta_\omega^* q_{\omega'}\,d\mathbf{r}+\text{c.c.}=\delta(\omega-\omega'). \tag{2.32}$$

Taking (2.32) into account, we can find the solution of the initial linearized problem (2.26). If

$$u_\perp(r,0)=q_0(r)+i\eta_0(r), \tag{2.33}$$

we seek the solution of (2.26) in the form

$$\begin{pmatrix} q\\ \eta\end{pmatrix}=\mathrm{Re}\sum_\omega C_\omega\begin{pmatrix} q_\omega\\ -i\eta_\omega\end{pmatrix}e^{i\omega t}. \tag{2.34}$$

At the initial instant of time we have

$$2\sum_\omega \dot C_\omega\eta_\omega e^{i\omega t}+q_0\delta(t)=0,\qquad 2\sum_\omega \dot C_\omega q_\omega e^{i\omega t}+\eta_0\delta(t)=0. \tag{2.35}$$

Taking the scalar product of (2.35) and the vector (q_ω,η_ω) and integrating with respect to **r**, we get with allowance for (2.32)

$$C_\omega=-\frac{1}{2}\int(q_0q_\omega+\eta_0\eta_\omega)\,dr,\qquad \omega<0. \tag{2.36}$$

Recognizing now that q_ω and η_ω are pure real, we obtain from (2.36) and (2.34) ultimately

$$q(r,t)=\sum_\omega C_\omega q_\omega\cos\omega t,\qquad \eta(r,t)=\sum_\omega C_\omega\eta_\omega\sin\omega t. \tag{2.37}$$

§3. DERIVATION OF ABBREVIATED EQUATIONS

We now obtain, for the width of the peak and for the background, abbreviated equations based on the requirement that the manifold of solutions of type (2.1) be stable. We note first that in the case of the manifold (2.1), when $\ddot a\equiv 0$ the lens transformation with arbitrary variation of $a(t)$

$$u(r,t)=\frac{1}{a}V(\xi,\tau)\exp\frac{i\dot a}{2a}r^2,\qquad \xi=\frac{r}{a},\qquad \tau=\int_0^t\frac{dt}{a^2}, \tag{3.1}$$

which leads to the equation

$$2i\frac{\partial V}{\partial\tau}+\Delta V+|V|^2V+\xi_r^{-2}\xi^2V=0,\qquad \xi_r^{-2}=-\frac{d^2a}{dt^2}a^3, \tag{3.2}$$

transforms the manifold and all the focused beams into a homogeneous beam[8]:

$$V_0(\xi,\tau)=\varphi_0(\xi)\exp(i\tau/2). \tag{3.3}$$

Within the framework of (3.2), the stability condition means that if the solution is sought in the form

$$V=C_0\varphi_0+V_\perp\exp(i\tau/2), \tag{3.4}$$

the "lens" in (3.2), specified by the coefficient ξ^2/ξ_r^2, should be chosen such that the initially small perturbations $V_\perp(\xi,0)$ and $|C_0|^2-1$ not grow any more.

Moreover, if we stipulate that a pure Townes mode be separated as $t\to t_0$ (the instant at which the singularity corresponding to $\tau\to\infty$), we must have $|C_0|^2\to 1$ and $V_\perp\to 0$. This means that $\xi_r^{-2}\to 0$ as $\tau\to\infty$, with ξ_r^{-2} decreasing quite rapidly. Indeed, if

$$a(t)=\alpha(t_0-t)^\mu, \tag{3.5}$$

we have at $\lambda\neq\frac{1}{2}$

$$\xi_r^{-2}=\alpha^4\mu(1-\mu)(t_0-t)^{4\mu-2},\qquad (\tau+\tau_0)^{-1}=\alpha^2(2\mu-1)(t_0-t)^{2\mu-1}, \tag{3.6}$$

$$\tau_0^{-1}=\alpha^2(2\mu-1)t_0^{2\mu-1}$$

or

$$\xi_r^{-2}=\frac{\mu(1-\mu)}{(2\mu-1)^2}(\tau+\tau_0)^{-2},\qquad \mu\neq{}^1/_2, \tag{3.7}$$

$$\xi_r^{-2}=\alpha^4/4,\qquad \mu={}^1/_2. \tag{3.8}$$

It can be seen from (3.6) that we are dealing with a defocusing lens at $0<\lambda\leqslant 1$ and with a focusing one at $\lambda>1$. Since $P>P_{cr}$ for focusing, dispersal of the background calls for a defocusing lens, so that as $t\to t_0$ we should have

$$\tfrac{1}{2}<\mu\leqslant 1. \tag{3.9}$$

Nor is the strict equality $\lambda=\frac{1}{2}$ possible, since at a distant $0<\tau<\infty$, at a constant lens strength, the tunneling through the barrier ξ^2/ξ_r^2 causes the "peak" also to disperse. Under the conditions (3.8), however, the tunneling effect is exponentially small. In principle, therefore, the law obtained in Refs. 5 and 7 is not prohibited by this effect. We shall show later that as $t\to t_0$, depending on the initial conditions, λ tends either to 1 or to $\frac{1}{2}$.

Having made these remarks, we continue the derivation of the abbreviated equations. Substituting (3.3) and (3.4) in (3.2) we obtain

$$\begin{gathered}[2i\partial C_0/\partial\tau+\xi_p^{-2}\xi^2C_0+2(|C_0|^2-1)\varphi_0^2C_0\\ +2|V_\perp|^2C_0]V_0+V_\perp^2C_0^*V_0^*+2i\partial V_\perp/\partial\tau\\ +(\Delta+2|C_0|^2\varphi_0^2-1)V_\perp+(C_0^2V_0^2+|V_\perp|^2)V_\perp=-\xi^2V_\perp/\xi_r^2.\end{gathered} \tag{3.10}$$

The equations of interest to us for the amplitude C_0 of the peak, its width a, and the background field $V_\perp$ are obtained from (3.10) when account is taken of the properties, known to us from §2, of the linearized-problem solutions. Since $V_\perp$ is orthogonal to φ_0 by assumption:

$$\int V_\perp\varphi_0\,d\xi=0, \tag{3.11}$$

after multiplying (3.10) by $\varphi_0(\xi)$ and integrating with respect to ξ we arrive at an equation for C_0:

$$\begin{gathered}(2i d/d\tau+\xi_{\text{eff}}^2/\xi_r^2)C_0P_{cr}+\int(\delta U+\xi_r^{-2}\xi^2V_\perp)\varphi_0\,d\xi=0,\\ \delta U=(|C_0\varphi_0+V_\perp|^2-\varphi_0^2)(C_0\varphi_0+V_\perp).\end{gathered} \tag{3.12}$$

The equation for the width of the peak is obtained from the condition that the amplitude of the unstable mode (2.31) be identically zero. To satisfy this condition we must, according to (2.31) and (2.36), multiply (3.10) by

$$C_0^*\left(\frac{\partial u_\tau}{\partial a}\right)_{a=1}=-C_0^*\frac{\partial}{\partial\xi}\xi\varphi_0, \tag{3.13}$$

integrate with respect to ξ, and take the real part of the result. We get then

$$\operatorname{Re}\int[\delta U+\xi_r^{-2}\xi^2(C_0\varphi_0+V_\perp)]C_0^*\frac{d}{d\xi}(\xi\varphi_0)\,d\xi=0. \tag{3.14}$$

The multiplication by C_0^* in (3.13) takes into account the fact that the phases of the eigenmodes of the linearized problem in §2 were reckoned from the phase of the field at the peak.

Finally, a third equation is obtained from (3.10) for the background $V_\perp$ by taking the orthogonality condition (3.11) into account:

$$\begin{gathered}\frac{\partial V_\perp}{\partial\tau}+\hat L_0V_\perp+\xi_r^{-2}\xi^2(C_0\varphi_0+V_\perp)+\delta U=j_\perp,\\ j_\perp=\frac{\varphi_0}{P_{cr}}\int[\xi_r^{-2}\xi^2(C_0\varphi_0+V_\perp)+\delta U]\varphi_0\,d\xi.\end{gathered} \tag{3.15}$$

The current $j_\perp$ is determined here by the condition (3.11) that the background and the peak field be orthogonal. The operator $\hat L_0=\Delta+(\varphi_0^2-1)$ is the same as in the linearized problem (2.21).

The system (3.12), (3.14), and (3.15) is fully equivalent to Eq. (1.1). The initial conditions for $V(\xi,\tau)$ must be chosen here in a strictly prescribed manner, so that $C_0(0)$, $a(0)$ and $\dot a(0)$ are defined. For simplicity we present the required relations only for the particular case of initially axisymmetric beams with plane phase fronts:

$$\begin{gathered}\int u(r,0)\frac{\partial}{\partial a}\frac{1}{a}\varphi_0\left(\frac{r}{a}\right)d\mathbf{r}=0,\qquad \dot a(0)=0,\\ C_0(0)=\frac{1}{P_{cr}}\int u(r,0)\frac{1}{a}\varphi_0\left(\frac{r}{a}\right)d\mathbf{r}.\end{gathered} \tag{3.16}$$

The first equation in (3.16) is in fact the equation for a_0. It follows formally from the condition that the unstable mode (2.31) of the linearized problem not be contained in $V_\perp$ at the initial instant.

In contrast to (1.1), the system (3.12), (3.14–3.16) lends itself readily to analysis because of the explicit resolution of the total field into focusing and dispersing components. In particular, in the case of weak supercriticality, $||C_0|^2-1|\ll 1$ the dynamics of the entire focusing process can be tracked from the beginning up to the instant when the singularity is formed. In this important case[2)] the system (3.12), (3.14), (3.15) can be greatly simplified since background self-action effects (the term $|V_\perp|^2V_\perp$ in (3.11)) can be neglected. The point is that if the background self-action is small at the initial instant $t=0$, it can subsequently only weaken because of the rapid "dispersal" of the background out the region of interaction with the peaks.[3)] In the analysis of the background-field dynamics near the instant of singularity formation we can therefore confine ourselves to the approximation linear in the background. On the other hand, this enables us to simplify also Eq. (3.14) for the width of the peak. When account is taken of the background dispersal, this equation takes the form

$$\begin{gathered}\frac{d^2a}{dt^2}=-\frac{\mathcal{U}}{a^3},\\ \mathcal{U}=\left[(|C_0|^2-1)P_{cr}-\int\left(\frac{\partial}{\partial\xi}\overline{|V_\perp|^2}\right)\xi\varphi_0^2\,d\xi\right]\Big/\xi_{\text{eff}}^2.\end{gathered} \tag{3.17}$$

We have averaged here over the period τ of the fast peak-field oscillations, during which the background changes little. It can be seen that the peak focusing is due both to the degree of its supercriticality ($|C_0|^2-1\neq 0$) and to the background lens.[4)]

Before we proceed with the analysis, notice must be taken of the possibility of emission of peak-field photons. This is due, on the one hand, to the direct parametric interaction of the background with the peak (to the term in (3.10) which is quadratic in the background), and on the other hand to the possibility of tunnel emission through the potential barrier ξ^2/ξ_r^2 at $\tau\to\infty$ $(t\to t_0)$, i.e., to asymptotic stability of the manifold of self-similar solutions (2.1). However, depending on which of these channels is decisive in the dynamics of the emission, two different regimes of variation of $a(t)$ near the instant of singularity formation are possible.

§4. PEAK FOCUSING IN THE BACKGROUND FIELD

We proceed now to an analysis of (3.17). We consider first the case $|C_0|^2 = 1$. Returning to $\mathscr{U}$ from (3.17), we have

$$\mathscr{U}=-\xi_{\text{eff}}^{-2}\int\left(\xi\frac{\partial}{\partial\xi}|V_{\perp\perp}|^2\right)\varphi_0^2\,d\xi. \tag{4.1}$$

According to (2.34) we have for $V_{\perp\perp} = q + i\eta$ in the linear approximation

$$\begin{pmatrix} q \\ \eta \end{pmatrix}=e^{i\tau/2}\,\mathrm{Re}\sum C_k(0)\begin{pmatrix} q_k \\ -i\eta_k \end{pmatrix}e^{-ik^2\tau}, \tag{4.2}$$

where q_k and η_k are eigenfunctions of the linearized problem (see §2 for details).

Since the main contribution to $V_{\perp\perp}(\xi,\tau)$ as $\tau \to \infty$ is made by the low-frequency modes of the continuous spectrum ($k^2 \lesssim 1/\tau$), we shall examine the structure of these modes in greater detail. The asymptotic form of these modes at $\xi \gg 1$ is obvious:

$$q_k\approx J_0(k\xi)\alpha_k+N_0(k\xi)\beta_k, \tag{4.3}$$

where J_0 and N_0 are Bessel functions. The condition for normalization of the eigenfunctions of the continuous spectrum yields one constraint on α_k and β_k:

$$|\alpha_k|^2+|\beta_k|^2=1. \tag{4.4}$$

To assess the structure of the eigenmodes at $k^2 \ll 1$ we can reason as follows. Assume that we have found at $k = 0$ a solution of q_0 of the system (2.3) with initial conditions $q_0(0) = 1$ and $q'(0) = 0$. Clearly, at $\xi \gg 1$ we have

$$q_0(\xi)=\gamma_0+\delta_0\ln\xi. \tag{4.5}$$

On the other hand, if $k^2 \ll 1$ we have

$$\gamma_k=\gamma_0+o(k^2),\qquad \delta_k=\delta_0+o(k^2). \tag{4.6}$$

This means that at $k^2 \ll 1$ the functions q_k (4.3) are close to q_0 at $\xi \gtrsim 1$. The normalized functions take therefore in this region the form

$$q_k=C_k(\gamma_0+\delta_0\ln\xi),\quad \xi\sim1. \tag{4.7}$$

With account taken of (4.4) we have then

$$|C_k|^2=\left[\left(\frac{\pi\delta_0}{2}\right)^2+\left(\gamma_0+\delta_0\ln\frac{2}{\gamma k}\right)^2\right]^{-1},\qquad \gamma\approx0{,}5772. \tag{4.8}$$

If we now asume that $\delta_0 \neq 0$, i.e., that the mode does not belong to the discrete spectrum, we get at $k^2 \ll 1$

$$C_k^2\sim[\delta_0^2\ln^2(\gamma k/2)]^{-1}. \tag{4.9}$$

Estimating now (4.1) under the assumption, say, that

$$C_k(0)=(P_\perp/\pi b^2)\exp(-k^2b^2/4),$$

where b is the characteristic initial width of the background, it is easy to verify that $\mathscr{U} \propto /1\tau^2\ln^2\tau$ and

$$a(t)=\dot a(t_0-t)+o\left(\frac{(t_0-t)^2}{\ln^2 t-t}\right),\qquad \dot a=\text{const}. \tag{4.10}$$

Thus, if the fundamental mode is not overexcited at the initial instant of time ($|C_0|^2 = 1$), the singularity exponent $\lambda \to 1$ as $\tau \to \infty$.

§5. SELF-FOCUSING IN THE TUNNELING EMISSION OF THE PEAK

We proceed now to analyze the case

$$|C_0|^2-1\neq0. \tag{5.1}$$

The analysis of the preceding section did not touch upon at all on questions connected with photon spillover from the modes of the discrete spectrum into the continuum (nonadiabatic effects). This is formally possible if the total number of photons (the sum of the occupation numbers) of the discrete modes is exactly equal to P_{cr}.

Indeed, the problem involves two discrete-spectrum modes, the Townes mode and the fundamental mode of the linearized problem. The frequencies of these modes are separated in the spectrum from the continuum frequencies (in terms of which $V_{\perp\perp}$ is expanded) by the large quantity $\Delta\omega \gtrsim \frac{1}{2}$. The photons can spill over into the continuum in two ways. First, via the parametric effect [the terms $\propto V_\perp^2$ in (3.10)]. This effect is relatively weak. Simple quantum-mechanical estimates show that this effect changes little the number of photons in the discrete modes:

$$\frac{d}{d\tau}\Delta P_0\leqslant-|V_{\perp\perp}|^4_{max}P_0,\qquad \Delta P_0=P_0-P_{\text{cr}}. \tag{5.2}$$

It can be seen from (5.1) that at $P_0 \sim P_{\text{cr}}$ and $|V_{\perp\perp}|^2 < \varepsilon^2/\tau$, where ε is a small parameter,

$$\Delta P_{max}\sim o(\varepsilon^4P_{\text{cr}}). \tag{5.3}$$

This means that, in the scope of this effect, a larger number of photons than P_{cr} remains in the discrete spectrum if $\Delta P_0 > \Delta P_{\max}$. For the estimates of §4 to be valid it is thus necessary, accurate to (5.3), to have $P_0 = P_{\text{cr}}$.

If, however, the supercriticality of the mode is low but finite, for example if

$$u(r,0)=\frac{C_0}{a}\varphi_0\left(\frac{r}{a}\right),\qquad |C_0|^2-1\ll1, \tag{5.4}$$

i.e.,

$$V_{\perp\perp}(\xi,\,0)=0, \tag{5.5}$$

another photon spillover mechanism becomes substantial. The point is that in this case it is necessary to take into account in (3.10) and thereafter a perturbation of the form

$$(|C_0|^2-1)C_0\varphi_0^3. \tag{5.6}$$

This introduces in (3.25) an additional term

$$\mathscr{U}=(|C_0|^2-1)P_{\text{cr}}/\xi^2_{\text{eff}}. \tag{5.7}$$

In the $\mathbf{r}$, t representation this term takes the form

$$\ddot a=(1-|C_0|^2P_{\text{cr}})/a^3\xi^2_{\text{eff}}. \tag{5.8}$$

It is precisely this part of the "potential energy" that determines the focusing process in this case. According to (5.8), if we neglect initially the change of $|C_0|^2$, it is easy to verify that as $t \to t_0$ we have $a(t) \propto (t_0 - t)^{\frac{1}{2}}$. As already noted above (§3), however, this law leads automatically to tuneling emission of the fundamental mode. Indeed, when the lens is taken into account in (3.2), the effective dielectric constant for the fundamental mode is

$$\varepsilon_{\text{eff}}=\varphi_0^2+\xi_r^{-2}\xi^2-1. \quad (5.9)$$

In our case of a defocusing lens we have $\xi_r^2>0$, and consequently at $\xi_r^2=$ const there exist no strictly trapped modes. We have seen that $\xi_r \gg 1$ and therefore the tunneling is exponentially weak. The "translucence point" ξ_r is obtained from the resonance condition

$$\varepsilon_{\text{eff}}=0, \quad \xi=\xi_r. \quad (5.10)$$

It is now easy to estimate the change of the number of photons produced in the fundamental modes by the tunnel effect:

$$\frac{d}{d\tau}|C_0|^2P_{\text{cr}}=-\Phi(\xi_r)\xi_r\varphi_0^2(\xi_r). \quad (5.11)$$

Here $\xi_r\Phi(\xi_r)$ is an inessential factor that precedes the exponential and is determined by the swelling effect and by the group velocity.

The field at the point $\xi=\xi_r$ can be obtained from the asymptotic form of the fundamental mode at $\xi_r \gg 1$:

$$\varphi_0^2(\xi_r)\approx A\exp(-\pi\xi_r/2), \quad A\sim 1. \quad (5.12)$$

To make the problem self-contained, we express ξ_r^2 of (3.10) in terms of (5.8):

$$\xi_r^2=\xi_{\text{eff}}^2/(|C_0|^2-1)P_{\text{cr}}. \quad (5.13)$$

Substituting (5.13) in (5.11) we get

$$\frac{d|C_0|^2}{d\tau}\infty\exp\left\{-\frac{\pi}{2}\left(\frac{P_{\text{cr}}(|C_0|^2-1)}{\xi_{\text{eff}}^2}\right)^{-1/2}\right\}. \quad (5.14)$$

This shows that as $\tau\to\infty$ we have with logarithmic accuracy

$$|C_0|^2-1\infty 1/\ln^2\tau. \quad (5.15)$$

Recognizing that $\tau\propto\ln(t_0-t)^{-1}$, we obtain ultimately from (5.8)

$$a(t)\infty\left(\frac{t_0-t}{\ln(t_0-t)^{-1}}\right)^{1/2}, \quad t\to t_0. \quad (5.16)$$

§6. CONCLUSION

The power influx into the singularity in the case of self focusing is exactly equal to P_{cr}. The exponent λ of the singularity in the relation $a(t)\propto(t_0-t)^\lambda$ can be close both to 1 and to $\frac{1}{2}$. We have found that the relation $\lambda\approx\frac{1}{2}$ is more approximate from the viewpoint of the permissible initial conditions. The qualitative reason is that we were unable to find for the parametric spillover of the photons from the peak into the background in the immediate vicinity of the peak a mechanism more effective than that described at the start of §5. The feasibility of such a mechanism is still moot. It is desirable at any rate to analyze the perturbation of the spectral characteristics of the linearized problem with account taken of weak supercriticality of the peak. Do lines corresponding to growing perturbations appear at $C_0^2-1\neq 0$ near the boundary of the continuous spectrum?

I thank A. G. Litvak, M. A. Miller, V. I. Talanov, and S. N. Vlasov for helpful discussions.

[1)]Similar properties appear also in the one- and three- dimensional problems.

[2)]As follows from the numerical experiments, $|C_0|^2-1\ll 1$ near the instant of singularity formation even for beams with $P\gg P_{\text{cr}}$.

[3)]The presence of a defocusing lens in Eq. (3.19) only accelerates the dispersal of the background.

[4)]We note that in the problem of focusing of a beam in an inhomogeneous medium $2i(\partial u/\partial t)+\Delta u+|u|^2u+\Phi(r,t)u=0$ similar operations lead to (3.17) in which U is supplemented by the term $[\int(\partial\Phi/\partial\xi)\xi\varphi_0^2d\xi]/2\xi_{\text{eff}}^2$. In the particular case $\Phi=\alpha(t)r^2$ this leads to an exact class of self-similar solutions (2.1) with width $a(t)$ satisfying the equation $\ddot{a}=\alpha a$.

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Translated by J. G. Adashko