# The Doppler effect for higher order harmonics at the boundary of a moving nonlinear medium

A. V. Samokhvalov and Yu. M. Sorokin

*Gorkiĭ State University*

The dependence of the Doppler spectrum of secondary waves on the number of the harmonic is investigated. The Doppler formula for a harmonic of arbitrary order is presented. One corollary is that in the presence of dispersion in the medium the Doppler spectrum of the harmonics ceases to be equidistant. It is shown that for higher order harmonics ($p \gg 1$) due to the movement of the boundary, amplitude effects may also be considerable in those cases when the velocity is small compared with that of the secondary waves ($|\beta| \ll 1$). For the amplitude of Doppler harmonics of moderate multiplicity ($|\beta|p \ll 1$) the parameter ($p + 1$)/2 plays the same role as the effective slowing-down coefficient in the linear case.

## 1. INTRODUCTION

The transformation of electromagnetic waves at a moving boundary separating two linear media has been extensively described in the literature (see, for example, the review[1]). Here the key points are: the question of the frequency of the secondary waves originating at the boundary, the laws of frequency conversion (the double Doppler effect), and the secondary-wave intensity variation that results from the relativistic transformation of the fields. For relativistic boundary velocities (and in the presence of a "background" medium having a large effective refractive index also in the nonrelativistic case), considerable restructuring of the waves' energy spectrum may occur.

Another well-known possibility of the drastic conversion of frequencies is due to the nonlinearity of the medium (i.e., the finiteness of the incident wave's amplitude), including also in the presence of dielectric boundaries[2–4] or in an inhomogeneous plasma.[5–7] The problem considered below, of electromagnetic wave transformation at the boundary of a moving nonlinear dielectric, is the first attempt at a simultaneous analysis of the indicated possibilities of spectrum conversion.[1] To clarify the general laws of frequency-spectrum and wave-energy transformation associated with the Doppler effect, the simplest model of a non inertial local nonlinearity was chosen. This model admits of generation of high-order ($p \gg 1$) harmonics of the incident wave at an abrupt boundary. The harmonics of order $m \sim 10^2$ recorded in a laser plasma[8] and due to formation of abrupt discontinuities of the electron density[8–10] provide also a practical basis for such a formulation of the problem.

The transformation of waves at a moving boundary in a nonlinear medium has a number of peculiarities which are not present in the corresponding problem, nor for an immobile nonlinear-medium boundary. As well be shown below, the Doppler effect for high order harmonics preserves equidistance of the frequency spectrum of the secondary waves only if dispersion is absent. If dispersion occurs, the frequency spectrum of the harmonics (reflected or transmitted) will cease to be equidistant. The conditions for strong conversion (in comparison with the case of a stationary boundary) of the wave amplitude and energy at a moving boundary of a nonlinear medium must also be refined. Whereas to observe a noticeable Doppler effect in a linear medium, it is necessary that the boundary velocity $V$ be approximately that of the phase velocity $v_r$ of the reflected wave,[1] this effect becomes prominent at substantially lower velocities $V \sim v_r/m$ for high order ($m \gg 1$) harmonics at the boundary of a moving nonlinear medium. In other words, the relative effectiveness of generation of high-order harmonics increases as the boundary velocity and harmonic frequency increase.

Up to the present time, the Doppler frequency shift of harmonics has been observed only at nonrelativistic boundary velocities $V \leqslant 3 \cdot 10^7$ cm/s[11] for $m \leqslant 3$,[7,11,12] where the effect remains small. However, for velocities on the order of $m \sim 10^2$ we may expect a pronounced amplification of the harmonics, in which case dispersive or other cutoff of the spectrum from above may lead to a noticeable transfer of energy to several high-order harmonics.

## 2. THE DOPPLER SPECTRUM OF HARMONICS

Consider the case where linear as well as nonlinear wave transformation occurs at a boundary separating two media and traveling at a velocity **V**. The nonlinear transformation is accompanied by the generation of harmonics $p\omega'$ up to an order $m$ in the reference frame $K'$ moving with the. The following relation between the frequencies $\omega_j^{(p)}$ and the wave vectors $k_j^{(p)}$ follows them from the relativistic invariance of the phase in the laboratory reference frame $K$:

$$\omega_j^{(p)} - \mathbf{k}_j^{(p)}\mathbf{V} = \gamma^{-1}p\omega', \quad p=1,\ldots,m, \quad j=(0,i), \quad i=1,\ldots,s.$$

(1)

Here the index $j = 0$ corresponds to the incident wave and $s$ is the number of secondary waves produced at the boundary. Obviously, the well-known Doppler formular for linear waves at a moving boundary is obtained from (1) at $p = 1$. Generalizing the formula to include the cases $p = 2, \ldots, m$ allows us to analyze the Doppler spectrum for a harmonic of any order. It follows in particular, from (1) that in the presence of dispersion in the medium, the Doppler spectrum $\omega_j^{(p)}$

is not equidistant—in contrast to the case of an immobile boundary.

The condition of equidistance of the secondary-wave spectrum in any spectral interval $p\omega_i^{(q)} = q\omega_i^{(p)}$ presumes the absence of dispersion in this interval in the laboratory frame $K$, where the boundary moves with velocity $\mathbf{V}$, i.e.

$$n(\omega_i^{(q)}) = n(\omega_i^{(p)}).$$

## 3. AMPLITUDE RELATIONS

To clarify the wave-amplitude and wave-energy conversion laws associated with the motion, we consider the simple scalar case of normal incidence of a plane electromagnetic wave from a vacuum on a half-space of a moving nonlinear medium described in the comoving frame $K'$ by the material equation

$$\varepsilon'(p\omega', E') = n'^2(p\omega') + \delta_p'(E')^{p-1}. \qquad (2)$$

Confining ourselves to the case of weak nonlinearity

$$\delta_p'(E')^{p-1}/n'^2(p\omega') \ll 1 \qquad (2A)$$

in the absence of synchronism: $n'(\omega') \neq n'(p\omega')$, we use a parametric approximation in which the field of frequency $p\omega'$ in the nonlinear medium satisfies the wave equation[2]

$$-\Delta' \mathbf{E}'^{(p)} + \frac{n'^2(p\omega')}{c^2}\frac{\partial^2}{\partial t'^2}\mathbf{E}'^{(p)} = -\frac{4\pi}{c^2}\frac{\partial^2}{\partial t'^2}\mathbf{P}'^{(p)}, \qquad (3)$$

where $\mathbf{P}'^{(p)}$ is the effective nonlinear polarization at the frequency $p\omega'$ and is determined by the field $\mathbf{E}'$ at the fundamental frequency:

$$\mathbf{P}'^{(p)} = \frac{\delta_p'}{4\pi}(E')^{p-1}\mathbf{E}'. \qquad (4)$$

Because of the inhomogeneity of equation (3), the field of frequency $p\omega'$ in the nonlinear medium can be represented as the sum of two waves having waves numbers

$$k_1'^{(p)} = \frac{p\omega'}{c}n'(p\omega'), \quad k_2'^{(p)} = \frac{p\omega'}{c}n'(\omega'). \qquad (5)$$

By its nature, the second wave is nonlinearly polarized and its amplitude $E_2'^{(p)}$ is determined in the parametric approximation by the field $E'$ at the fundamental frequency:

$$E_2'^{(p)} = \frac{\delta_p'(E')^p}{n'^2(p\omega') - n'^2(\omega')}. \qquad (6)$$

From Eq. (6), the relation between the harmonic amplitudes $E_i'^{(p)}$ and the amplitude $E_0'$ in the comoving reference frame $K'$ will be a power-law function:

$$E_r'^{(p)} = r'^{(p)}(E_0')^p, \quad E_{t1,2}'^{(p)} = t_{1,2}'^{(p)}(E_0')^p. \qquad (7)$$

The "structural" coefficients $r'^{(p)}$ and $E_{1,2}'^{(p)}$ do not depend in our case at all[2] on the velocity $V$. They pertain to the reference frame $K'$ comoving with the nonlinear medium and are expressed in terms of the Fresnel transmission coefficient $t'$ of the fundamental-frequency wave:

$$r'^{(p)} = \frac{n'(\omega') - n'(p\omega')}{n'(p\omega') + 1} \frac{\delta_p'(t')^p}{n'^2(p\omega') - n'^2(\omega')}$$

$$t_1'^{(p)} = \frac{n'(\omega') + 1}{n'(p\omega') + 1} \frac{\delta_p'(t')^p}{n'^2(p\omega') - n'^2(\omega')}$$

$$t_2'^{(p)} = -\frac{\delta_p'(t')^p}{n'^2(p\omega') - n'^2(\omega')}. \qquad (8)$$

It is easy to show that by virtue of Eq. (7) the dependence of the amplitudes $E_i^{(p)}$ of the harmonics on the amplitude $E_0$ of the incident wave in the laboratory frame $K$ will also obey a power law, and the coefficients $r^{(p)}$ and $t_{1,2}^{(p)}$ of transformation of the incident wave into a harmonic wave, determined in analogy with Eq. (7)

$$r^{(p)} = r'^{(p)}f_r^{(p)}, \quad t_{1,2}^{(p)} = t_{1,2}'^{(p)}f_{t1,2}^{(p)} \qquad (9)$$

are connected with $r'^{(p)}$ and $t_{1,2}'^{(p)}$ by the Doppler factors

$$f_r^{(p)} = \frac{1-\beta}{\gamma^{p-1}(1+\beta)^p}, \quad f_{t1}^{(p)} = \frac{1+\beta n'(p\omega')}{\gamma^{p-1}(1+\beta)^p},$$

$$f_{t2}^{(p)} = \frac{1+\beta n'(\omega')}{\gamma^{p-1}(1+\beta)^p}. \qquad (10)$$

## 4. DISCUSSION OF RESULTS

From a general physics viewpoint, the most interesting circumstance is that the Doppler factors (10) for harmonics $p \neq 1$ depend somewhat differently on the velocity $V$ than do the corresponding factors for waves at the fundamental frequency $p = 1$. Clearly, this difference will be more pronounced for a high-order harmonic ($p \gg 1$) in a nonlinear medium moving at relativistic velocities (or in the presence of a retarding system, when the phase velocity of the reflected wave is close to $V$). When this occurs the relative effectiveness of higher-harmonics generation may, because of the Doppler effect, increase by several orders and the harmonics energy may become comparable to the energy of the reflected wave of fundamental frequency. Naturally, in the latter case a more accurate, self-consistent description of the problem, beyond the scope of the parametric approximation, is necessary.

It is important, however, to emphasize that when the boundary velocity $V$ is much less than the phase velocity of the reflected wave and the fundamental-frequency waves are not subject to any noticeable effects, the intensity of high order harmonics may be increased by the Doppler effect to several times more than in an immobile medium. Actually, assume that $p \gg 1$, so that $|\beta| p \gg 1$. Then discarding in Eq. (10) the inessential terms of order $\beta$ and $\beta^2$ we obtain for opposite incidence ($\beta < 0$) obtain

$$f_r' \approx f_{t1,2} \gg (1 + 1/p)^p \to e.$$

Thus, at definitely nonrelativistic velocities the intensity of this harmonic increases by roughly an order of magnitude because of the Doppler effect. If, however, the multiplicity of the generated harmonics is relatively small ($|\beta| p \ll 1$) we find from (10) that $f_r^{(p)} \approx 1 + (p + 1)\beta$. In other words, for conversion of the amplitudes of the harmonics the factor $(p + 1)/2$ plays the same role as the effective refractive index

in a retarding system for linear transformation on a moving boundary.

Needless to say, within the scope of the parametric approximation examined above, the coefficients $r'^p$ and $t'^{(p)}_{1,2}$ (8) remain small. It is clear, however, that a qualitative conclusion that the higher-harmonic amplitudes depend more strongly on the velocity of the medium is more general. Investigations of this question for the self-consistent problem would shed light on the optimal conditions for generating high-order harmonics at a moving boundary.

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[1)] Introduction of a Doppler shift into the formulas for the second-harmonic generation in a laser plasma, as was done in Ref. 7, yields an adequate approximation only for the special case of nonrelativistic velocities for harmonics of low order $m \sim 1$.

[2)] If the nonlinear medium and the boundary move at different velocities (for example, a moving discontinuity of a parameter in an immobile medium), such a dependence is preserved also in the coefficients $r'^{(p)}$ and $t'^{(p)}_{1,2}$.

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