Singularities of the kinetic characteristics of metals with a Fermi-liquid interaction between conduction electrons

M. I. Kaganov and A. G. Plyavenek

Institute of Physical Problems, Academy of Sciences of the USSR; All-Union Scientific-Research Institute of Opticophysical Measurements

(Submitted 2 July 1984)
Zh. Eksp. Teor. Fiz. **88**, 249–259 (January 1985)

The singularities in the components of the dielectric tensor and in other kinetic characteristics of metals are analyzed as functions of the wave vector in the semiclassical approximation. No assumption is made regarding the shape of the Fermi surface. For certain directions of the wave vector, a quantum-mechanical treatment may be necessary. The role played by the Fermi-liquid interaction in the structure of the singularities is determined. When this interaction is taken into account, the divergence in the kinetic characteristics is eliminated.

1. INTRODUCTION

The singularities in the components of the dielectric tensor $\varepsilon_{\alpha\beta}$ and other kinetic characteristics of metals as functions of the wave vector $\mathbf{k}$ at vanishing temperature T and at a mean free path $l=\infty$ stem from the Fermi degeneracy, which gives rise to a nonuniform distribution of conduction electrons in quasimomentum space.

These singularities can be classified conveniently by using the gas approximation for $\varepsilon_{\alpha\beta}(\omega,\mathbf{k})$ (i.e., by ignoring the Fermi-liquid interaction); in this case, the analysis can be restricted to the longitudinal component $\varepsilon_{\parallel}(\omega,\mathbf{k})$. In this approximation we have

$$\varepsilon_{\parallel}^{(g)}(\omega,\mathbf{k})\equiv\langle\varkappa\hat{\varepsilon}^{(g)}\varkappa\rangle=1-\frac{4\pi e^2}{k^2}\int\frac{n(\varepsilon_{\mathbf{p}})-n(\varepsilon_{\mathbf{p}+\hbar\mathbf{k}})}{\varepsilon_{\mathbf{p}}+\hbar\omega-\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}+i0}\frac{2d^3p}{(2\pi\hbar)^3},\qquad \varkappa=\mathbf{k}/k, \tag{1}$$

where $n(\varepsilon)$ is the Fermi step function,

$$n(\varepsilon)=\begin{cases}1, & \varepsilon<\varepsilon_F,\\ 0, & \varepsilon>\varepsilon_F,\end{cases} \tag{2}$$

$\varepsilon_{\mathbf{p}}$ is the energy of an electron with quasimomentum $\mathbf{p}$, and ε_F is the Fermi energy. In the case of a quadratic and isotropic dispersion law, function (1) is called the "longitudinal Lindhard function." According to (2), the integration in (1) is carried out over that region in $\mathbf{p}$ space in which either (a) $\varepsilon_{\mathbf{p}}<\varepsilon_F$, but $\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}>\varepsilon_F$, or (b) $\varepsilon_{\mathbf{p}}>\varepsilon_F$, but $\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}<\varepsilon_F$ (Fig. 1). Since

$$\frac{1}{\varepsilon_{\mathbf{p}}+\hbar\omega-\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}+i0}=P\frac{1}{\varepsilon_{\mathbf{p}}+\hbar\omega-\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}}-i\pi\delta(\varepsilon_{\mathbf{p}}+\hbar\omega-\varepsilon_{\mathbf{p}+\hbar\mathbf{k}}), \tag{3}$$

where $\mathbf{P}$ means the principal value, the singularities of $\varepsilon_{\parallel}^{g}(\omega,\mathbf{k})$ are observed only at those values of the wave vector $\mathbf{k}=\mathbf{k}_c$ at which the surfaces are tangent:

$$\varepsilon_{\mathbf{p}}=\varepsilon_F,\qquad \varepsilon_{\mathbf{p}+\hbar\mathbf{k}_c}=\varepsilon_F+\hbar\omega, \tag{4}$$

or

$$\varepsilon_{\mathbf{p}}=\varepsilon_F,\qquad \varepsilon_{\mathbf{p}-\hbar\mathbf{k}_c}=\varepsilon_F-\hbar\omega. \tag{4'}$$

In the static case ($\omega=0$), expression (1) can have either Kohn singularities[2] or Taylor singularities.[3,4] The Taylor singularities occur only for metals whose Fermi surfaces have lines of parabolic points. At $\omega\neq0$ these singularities are accompanied by some which stem from the separating of the surfaces $\varepsilon_{\mathbf{p}}=\varepsilon_F$ and $\varepsilon_{\mathbf{p}\pm\hbar\mathbf{k}}=\varepsilon_F\pm\hbar\omega$ in energy [see (4) and (4')]. At $\hbar\omega\ll\varepsilon_F$, the critical wave vector corresponding to these singularities is $k_c\sim\omega/v_F$ (v_F is the Fermi velocity).

For an isotropic dispersion law we find from (1) with $k\approx k_c=\omega/v_F$

$$\varepsilon_{\parallel}^{(g)}(\omega,k)=1+\frac{3\Omega^2}{2\omega^2}\left\{1-\frac{2\varepsilon_F}{\hbar\omega}\left[\frac{\Delta k_+}{k_c}\ln\left(-\frac{\Delta k_+}{2k_c}\right)-\frac{\Delta k_-}{k_c}\ln\left(-\frac{\Delta k_-}{2k_c}\right)\right]\right\}. \tag{5}$$

Here $\Omega^2=4\pi Ne^2/m$ is the square of the plasma frequency (we are using the standard notation), $\Delta k_{\pm}=k-k_c^{\pm}$, $k_c^{\pm}=k_c(1\pm\hbar\omega/4\varepsilon_F)$, and $\ln x=\ln|x|+i\pi$ for $x<0$. Expression (5) shows that $\varepsilon_{\parallel}^{(g)}(\omega,k)$ has two closely positioned singularities on each side of $k_c=\omega/v_F$. The splitting is of quantum origin (in the limit $\hbar\omega/\varepsilon_F\to0$ the singularities merge and, as is easily shown, become stronger). In Section 2 (see also Refs. 5 and 6), we describe the singularities at $\varepsilon_c\sim\omega/v_F$ in the case of an electron gas with an arbitrary dispersion law.

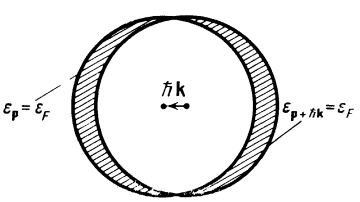

FIG. 1. Intersection of the surfaces $\varepsilon_{\mathbf{p}}=\varepsilon_F$ and $\varepsilon_{\mathbf{p}}+\hbar\mathbf{k}>\varepsilon_F$. The hatched region is that region in the plane of the intersection in which we have $\varepsilon_{\mathbf{p}}<\varepsilon_F$ but $\varepsilon_{\mathbf{p}}+\hbar\mathbf{k}>\varepsilon_F$ and $\varepsilon_{\mathbf{p}}>\varepsilon_F$, but $\varepsilon_{\mathbf{p}}+\hbar\mathbf{k}<\varepsilon_F$.

 0038-5646/85/010145-07$04.00

It is clear that the only singularities which can be manifested in the macroscopic properties of a metal are those for which the condition $\hbar k_c \ll p_F \sim \hbar/a$ holds (a is the interatomic distance). If $\hbar\omega \ll \varepsilon_F$, this condition holds for singularities with $k_c \sim \omega/v_F$. As for Kohn and Taylor singularities, we note that the condition $\hbar k_c \sim p_F$ usually holds for them. Exceptional cases are singularities which stem directly from the line of parabolic points on the Fermi surface and its vicinity.[6-8,4]

Briefly, here are the properties of a metal in which singularities of $\varepsilon_{\alpha\beta}$ are manifested: a) the static and dynamic screening of the field of a charge in the metal[9,10]; b) the anomalous penetration of electromagnetic and sound waves into the interior of a metal[11-13]; c) the angular anomalies of the sound absorption coefficient[6-8,4]; d) the frequency dependence of the effective cross section for inelastic scattering of light by conduction electrons.[14]

We must emphasize that in most of the studies mentioned above the Fermi-liquid interaction has been ignored. The Fermi-liquid interaction between electrons, which plays an important role in shaping the spectrum of conduction electrons, is manifested primarily in the quantitative characteristics of metals. Under the condition $\omega\tau \ll 1$ (τ is the electron relaxation time), the linearized kinetic equation for the electron distribution function can be rewritten in such a manner that the Landau matrix function $f(\mathbf{p},\mathbf{p}')$, which incorporates the Fermi-liquid interaction,[15] does not appear at all in the "answers."[16] Under the condition $\omega\tau \gg 1$ the function $f(\mathbf{p},\mathbf{p}')$ cannot be eliminated, but in general the equations which incorporate the Fermi-liquid interaction differ from the "gas" equations only by numerical factors. It has been found that there are very few phenomena in whose existence the Fermi-liquid interaction dominates. Among these few are spin waves in normal (nonferromagnetic) metals.[17] As we will see below (see also Ref. 5), the incorporation of the Fermi-liquid interaction leads to qualitative changes in the structure of the singularities in $\varepsilon_{\alpha\beta}$ and in other kinetic characteristics of a metal which have a spatial dispersion (which depend on $\mathbf{k}$). In this paper we will be concerned only with the singularities in the long-wave region: $k_c \sim \omega/v_F$ with $\hbar\omega \ll \varepsilon_F$.

2. DIELECTRIC TENSOR OF AN ELECTRON GAS WITH AN ARBITRARY DISPERSION LAW

In the semiclassical approximation, which is legitimate under the conditions $\hbar k \ll p_F$ and $\hbar\omega/\varepsilon_F$, the expression for $\varepsilon_{\alpha\beta}$ becomes

$$\varepsilon_{\alpha\beta}^{(g)}(\omega,\mathbf{k}) = \delta_{\alpha\beta} + \frac{8\pi e^2}{\omega(2\pi\hbar)^3}\oint_F v_\alpha R v_\beta \frac{dS}{v} \equiv \delta_{\alpha\beta} + \frac{4\pi e^2}{\omega}\langle v_\alpha R v_\beta\rangle,$$
$$\mathbf{v} = \partial\varepsilon_{\mathbf{p}}/\partial\mathbf{p}; \qquad R = (\mathbf{kv} - \omega - i0)^{-1}. \tag{6}$$

The superscript (g) means, as before, that the Fermi-liquid interaction is being ignored (in this section); the integration is over the Fermi surface; and dS is an element of area on this surface.

It is clear from (6) that the singularities of $\varepsilon_{\alpha\beta}^{(g)}(\omega,\mathbf{k})$ stem from the multiple zeros of the denominator of R. The equations

$$W(\xi,\eta) \equiv \mathbf{kv} - \omega = 0, \quad \partial W/\partial\xi = \partial W/\partial\eta = 0, \tag{7}$$

where ξ and η are mutually orthogonal coordinates on the Fermi surface ($dS = d\xi d\eta$), determine the "critical point" p_c on the Fermi surface ($\xi = \xi_c$, $\eta = \eta_c$), and also that value $k = k_c$ at which the components of $\varepsilon_{\alpha\beta}$ are singular (the frequency ω and the direction of the vector $\mathbf{k} = \varkappa k$ are assumed given). In the case of a Fermi surface of complex shape, a single value of k_c may correspond to several critical points $\mathbf{p}_{ci}$ ($i = 1,2,\ldots,n$) on the Fermi surface and to an entire line of critical points.[1)]

If the Fermi surface is a sphere, then we have $k_c = \omega/v_F$ for any $\varkappa$. In general, there can be a spectrum of singularities: several values of k_c for a given $\varkappa$. The values of k_c are determined by the positive (if $\omega > 0$) extrema of $\varkappa\mathbf{v}$ as a function of ξ and η:

$$k_c = \omega/(\varkappa\mathbf{v})_{extr}. \tag{8}$$

When a single point $\mathbf{p}_c$ on the Fermi surface is responsible for a singularity, then under the condition $v_\alpha^c v_\beta^c \neq 0$, where $\mathbf{v}^c \equiv \mathbf{v}(\mathbf{p}_c)$, the single part (SP) of the tensor $\langle v_\alpha R v_\beta\rangle$ can be written as follows, according to (6) and (7):

$$\mathrm{SP}\ \langle v_\alpha R v_\beta\rangle = v_\alpha^c v_\beta^c\ \mathrm{SP}\ \langle R\rangle. \tag{9}$$

When n points $\mathbf{p}_{ci}$ are responsible for a singularity, then under the condition $v_{\alpha i}^c v_{\beta i}^c \neq 0$ ($\mathbf{v}_i^c \equiv \mathbf{v}(\mathbf{p}_{ci})$, $i = 1,2,\ldots,n$) we have

$$\mathrm{SP}\ \langle v_\alpha R v_\beta\rangle = \sum_{i=1}^{n} v_{\alpha i}^c v_{\beta i}^c\ \ \mathrm{SP}\ \langle R\rangle_i. \tag{10}$$

Here the angle brackets mean an integration over the neighborhood of the critical point [$\mathbf{p}_c$ in (9) or $\mathbf{p}_{ci}$ in (10)].

If $\Delta k = k - k_c = 0$, the expressions $\langle R\rangle$ and $\langle R\rangle_i$ become infinite. The nature of this divergence by the local structure of the Fermi surface near $\mathbf{p}_c$ (or near $\mathbf{p}_{ci}$). In general (for arbitrary $\varkappa$) we would have (a) an O-type singularity, with $\mathrm{Re}\langle R\rangle \propto \ln|\Delta k|$ and with a discontinuity in $\mathrm{Im}\langle R\rangle$, and (b) an X-type singularity, with $\mathrm{Re}\langle R\rangle$ discontinuous and $\mathrm{Im}\langle R\rangle \propto \ln|\Delta k|$. These singularities are named on the basis of the shape of the belt-shaped line along which the Fermi surface intersects the surface $\mathbf{kv} = \omega$ under the condition $\Delta k \ll k_c$ (cf. Refs. 7, 4, and 6). Logarithmic singularities are the weakest. In metals whose Fermi surfaces have lines of parabolic points (i.e., depressions and connecting necks) there are necessarily directions along which a singularity is strengthened and we have $\langle R\rangle \propto |k_c/\Delta k|^\nu$, $0 < \nu < 1$ (cf. Refs. 5 and 6).

The components $\varepsilon_{\alpha\beta}$ may remain finite at $k = k_c$ if we have $v_{\alpha i}^c v_{\beta i}^c = 0$ for all i ($i = 1,2,\ldots,n$). In this case

$d\varepsilon_{\alpha\beta}^{(g)}/dk$ [or a higher derivative, depending on the relation between the multiplicities of the zeroes of the numerator and the denominator in (6)] will become infinite. If the Fermi surface is spherical, only the longitudinal component will diverge: $\mathrm{Re}\varepsilon_{\parallel}^{(g)} \propto \ln|\Delta k|$. The imaginary part, $\mathrm{Im}\varepsilon_{\parallel}^{(g)}$, will be discontinuous. The strengthening of the singularity [in comparison with (5)] results from the neglect of the quantum splitting. The singularities of the transverse components have the behavior $\mathrm{Re}\varepsilon_{\perp}^{(g)} \propto \Delta k \ln|\Delta k|$, while $\partial \mathrm{Im}\varepsilon_{\perp}^{(g)}/\partial k$ is discontinuous. If the Fermi surface is anisotropic we have $\mathbf{v}_{\perp}^{c} = \mathbf{v}^{c} - (\varkappa \mathbf{v}^{c})\ \varkappa \neq 0$ (at least for an arbitrary value of $\varkappa$), and the transverse components $\varepsilon_{\alpha\beta}$ may become infinite.

If the Fermi surface has a complex shape, a quantum-mechanical treatment may become *necessary*, since at certain values $\varkappa = \varkappa_c$ the values of k_c given by the "classical" expression becomes infinite. For clarity we assume a dumbbell-shaped Fermi surface; we will show that $\varkappa_c$ coincides with the direction of the tangent to the parabolic points (points A and B in Fig. 2; $\varkappa_c \perp \mathbf{v}_A, \mathbf{v}_B$). On the Fermi surface we construct $\varkappa \mathbf{v} = 0$ belts at $\varkappa \approx \varkappa_c$ (Fig. 2, a and c) and at $\varkappa = \varkappa_c$ (Fig. 2b). The region with $\varkappa \mathbf{v} > 0$ is hatched. Also shown in these figures is a plot of $\varkappa \mathbf{v}$ versus the angle θ, which describes the position on the Fermi surface. We see that as $\varkappa \to \varkappa_c$ the values of $\varkappa \mathbf{v}$ at the extrema vanish: at the point $\theta = \theta_A$ when approached from one direction and at the point $\theta = \theta_B$ when approached from the other direction. A quantum-mechanical treatment is therefore necessary at $\varkappa \approx \varkappa_c$. We wish to emphasize that this result is independent of the particular shape of the Fermi surface. This result occurs exclusively because the belt $\varkappa_c \mathbf{v} = 0$ has a self-intersection point. An analogous situation prevails if at some $\varkappa = \varkappa_c$ a loop of the belt appears or disappears (cf. Ref. 7). Again in this case we find $k_c \to \infty$ as $\varkappa \to \varkappa_c$. To determine the finite value of k_c at $\varkappa = \varkappa_c$ (we denote this value by k_{cc}) we must use the quantum conditions (4), (4′), under the condition $\hbar\omega \ll \varepsilon_F$, of course. Retaining the quadratic terms in the expansion of $\varepsilon_{\mathbf{p} \pm \hbar\mathbf{k}}$ in powers of $\hbar\mathbf{k}$, we easily find (first)

$$k_{cc} \sim (\varepsilon_F/\hbar\omega)^{1/2}\omega/v_F \sim (\hbar\omega/\varepsilon_F)^{1/2} p_F/\hbar,$$

which means that k_{cc} lies in the intermediate region between the classical and quantum-mechanical values[2] ($\omega/v_F \ll k_{cc} \ll p_F/\hbar$), and (second)

$$|k_{cc}^{A} - k_{cc}^{B}| \sim \omega/v_F, \qquad |1 - (\varkappa_c^{A}\varkappa_c^{B})^2|^{1/2} \sim (\hbar\omega/\varepsilon_F)^{1/2},$$

which means that the parabolic antipodes (A and B in our example) generate singularities at several different vectors $\varkappa$. The values of k_{cc}^{A} and k_{cc}^{B} are approximately equal ($\omega/v_F \ll k_{cc}^{A,B}$).

In concluding this section of the paper, we find the locus of the singularities: that surface in $\mathbf{k}$ space which is formed by the vectors $\mathbf{k} = k_c(\varkappa)\varkappa$. For definiteness we assume that the Fermi surface is a solid of revolution of the dumbbell type; this example reveals the characteristic features of the locus of singularities. Figure 3 shows the intersection of the locus of singularities with the $k_x = 0$ plane. The k_z axis runs parallel to the axis of revolution. On the solid lines we ob-

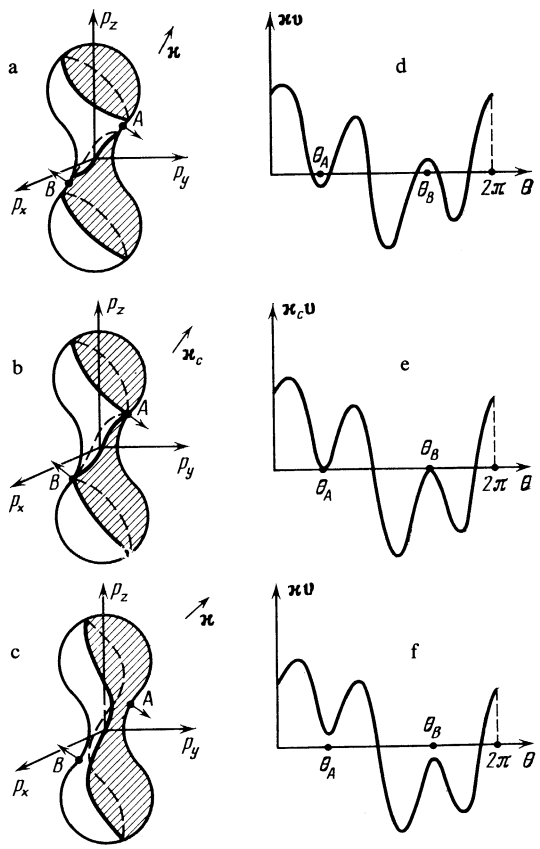

FIG. 2. a, b, c—The belts $\varkappa \mathbf{v} = 0$ (the heavy lines) on a Fermi surface which is a dumbbell-shaped solid of revolution; d, e, f— $\varkappa \mathbf{v}$ versus the angle θ. a, d—$\varkappa \mathbf{v}_A < 0$, $\varkappa \mathbf{v}_B > 0$; b, d—$\varkappa_c \mathbf{v}_{A,B} = 0$; c, f—$\varkappa \mathbf{v}_A > 0; \varkappa \mathbf{v}_B < 0$. The points A and B are parabolic points; θ is the angle reckoned from the p_z axis in the $p_x = 0$ plane ($0 \leqslant \theta \leqslant 2\pi$).

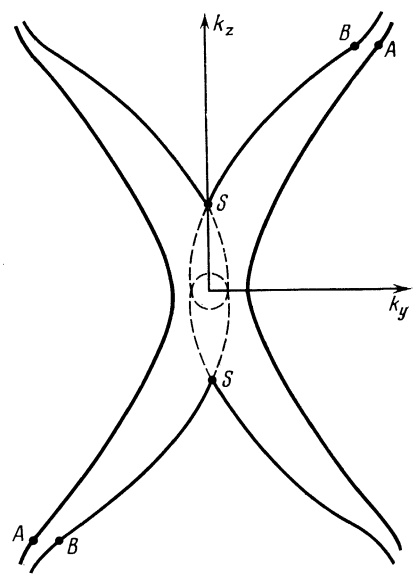

FIG. 3. Intersection of the locus of singularities with the $k_x = 0$ plane for a Fermi surface which is a dumbbell-shaped solid of revolution.

serve singularities of the X type, while on the dashed lines we observe singularities of the O type. At points of transition from the X-type to the O-type singularities (at the points S) the components $\varepsilon_{\alpha\beta}$ have enhanced singularities of the type $(k-k_c)^{-1/2}$. Points A and B in Fig. 3 correspond to the values k_{cc}^{A} and k_{cc}^{B}. We stress that the lines of the locus of singularities are terminated because of our use of the semiclassical expansion. In a quantum treatment, beyond the points A and B the lines would move off to values $\hbar k_c \sim p_F$ (according to Ref. 4, the locus of points has no discontinuities).

3. DIELECTRIC TENSOR OF THE ELECTRON FERMI LIQUID

In this section we show that the renormalization of the components $\varepsilon_{\alpha\beta}$ made necessary by the Fermi-liquid interaction eliminates the divergence of these components (this topic was covered briefly in Ref. 5).

The linearized kinetic equation incorporating the Fermi-liquid interaction is written in the ω, $\mathbf{k}$ representation in the limit $kl\to\infty$ as follows[16,18–20]:

$$-i\omega\delta n+i\mathbf{kv}\delta\tilde{n}+e\mathbf{v}\mathbf{E}\partial n_0/\partial\varepsilon=0, \tag{11}$$

where $\delta n = n - n_0(\varepsilon_0)$ is the derivation of the electron distribution function from equilibrium, and

$$\delta\tilde{n}=n-n_0(\varepsilon)=\delta n-(\partial n_0/\partial\varepsilon)\,\delta\varepsilon$$

is the deviation from a "locally equilibrium" distribution function. Its argument ε is a functional of the distribution function:

$$\delta\varepsilon=\int f(\mathbf{p},\mathbf{p}')\,\delta n(\mathbf{p}')\,\frac{2d^3p'}{(2\pi\hbar)^3}.$$

We write δn and $\delta\tilde{n}$ in the following form:

$$\delta n=-\frac{\partial n_0}{\partial\varepsilon}e\psi\mathbf{E},\quad \delta\tilde{n}=-\frac{\partial n_0}{\partial\varepsilon}e\chi\mathbf{E}. \tag{12}$$

By virtue of the definition of $\delta\varepsilon$ we have

$$\chi=\psi+\langle f(\mathbf{p},\mathbf{p}')\psi(\mathbf{p}')\rangle'\equiv\psi-i\mathbf{J}. \tag{13}$$

Using (11)–(13) we easily find

$$\varepsilon_{\alpha\beta}^{(l)}(\omega,\mathbf{k})=\delta_{\alpha\beta}+\frac{4\pi e^2}{\omega}\langle v_\alpha R(v_\beta-\omega J_\beta)\rangle, \tag{14}$$

which differs from the "gas expression" (6) only in the presence of a term $-\omega J_\beta$. The vector function $\mathbf{J}=\mathbf{J}(\mathbf{p})$ is the solution of the integral equation

$$\mathbf{J}(\mathbf{p})+\langle\mathbf{kv}'f(\mathbf{p},\mathbf{p}')R'\mathbf{J}'\rangle'=\langle\mathbf{v}'f(\mathbf{p},\mathbf{p}')R'\rangle',\quad \mathbf{v}'\equiv\mathbf{v}(\mathbf{p}'),\dots \tag{15}$$

where $\langle\dots\rangle'$ means an average over the Fermi surface in $\mathbf{p}'$ space.

It can be shown that expression (14) is not symmetric under a permutation of the indices α and β. If, however, we write $\mathbf{J}$ in the form

$$\mathbf{J}(\mathbf{p})=\langle\mathbf{v}'G(\mathbf{p},\mathbf{p}')R'\rangle'$$

with a matrix $G(\mathbf{p},\mathbf{p}')$ which satisfies the matrix integral equation

$$G(\mathbf{p},\mathbf{p}')+\langle\mathbf{kv}''f(\mathbf{p},\mathbf{p}'')R''G(\mathbf{p}'',\mathbf{p}')\rangle''=f(\mathbf{p},\mathbf{p}'),$$

then it can be shown that the symmetry of the matrix $f(\mathbf{p},\mathbf{p}')$ leads to a symmetry

$$\varepsilon_{\alpha\beta}^{(l)}(\omega,\mathbf{k})=\varepsilon_{\beta\alpha}^{(l)}(\omega,\mathbf{k}). \tag{14'}$$

In view of the proportionality of all the divergent integrals $\langle R\rangle$ [see (9)], when a single point $\mathbf{p}_c$ on the Fermi surface is responsible for a singularity we have

$$\mathbf{J}_c=\mathbf{v}^c/(\mathbf{k}_c\mathbf{v}^c)=\mathbf{v}^c/\omega, \tag{16}$$

since $\mathbf{k}_c\mathbf{v}^c=\omega$ according to (7). The tensor $\varepsilon_{\alpha\beta}^{(l)}$ thus has no divergent components.[3)]

This assertion remains true when n points $\mathbf{p}_{ci}$ on the Fermi surface are responsible for a singularity. From (15) we find in this case

$$\sum_{i=1}^{n}f(\mathbf{p},\mathbf{p}_{ci})\{\mathbf{v}_i^c-(\mathbf{k}_c\mathbf{v}_i^c)\mathbf{J}_{ci}\}\ \text{SP}\ \langle R\rangle_i=0. \tag{17}$$

The sum must vanish for any value of the quasimomentum $\mathbf{p}$ on the Fermi surface. This condition can be satisfied only if

$$\mathbf{J}_{ci}=\mathbf{v}_i^c/(\mathbf{k}_c\mathbf{v}_i^c)=\mathbf{v}_i^c/\omega, \tag{16'}$$

since the velocities $\mathbf{v}_i^c$ at all the critical points satisfy Eq. (7): $\mathbf{k}_c\mathbf{v}_i^c=\omega$ (this assertion does not mean that all the $\mathbf{v}_i^c$ are equal to each other). We wish to stress that we have made use of the circumstance that $f(\mathbf{p},\mathbf{p}')$ cannot be represented as a product of functions of $\mathbf{p}$ and $\mathbf{p}'$—this is not a limitation, since there is no basis of any sort for such a degeneracy.

Finally, it is clear that the conclusion that the tensor $\varepsilon_{\alpha\beta}^{(l)}$ has no divergent components can be generalized in a natural way to the case in which a singularity is generated by a line on the Fermi surface [this conclusion is based on the replacement of the sum in (17) by an integral].

The fact that the integrals over the Fermi surface do not have divergences does not imply that the components $\varepsilon_{\alpha\beta}^{(l)}$ are regular functions of the wave vector $\mathbf{k}$: The Fermi-liquid interaction weakens but does not eliminate the singularities. Restricting the discussion to the case of a single critical point $\mathbf{p}_c$ (to avoid unnecessary complications), we can determine the general structure of the components $\varepsilon_{\alpha\beta}^{(l)}$.

Singling out the divergent part of each of the integrals in (15), we can write the solution of the integral equation at

$k \approx k_c$ with the required accuracy as follows:

$$\mathbf{J}=(f_c/f_{cc})\mathbf{v}^c/\omega+\mathbf{I}+(F/\langle R\rangle)\Phi+\Psi/\langle R\rangle^2, \tag{18}$$

where

$$f_c=f(\mathbf{p},\mathbf{p}_c)=f(\mathbf{p}_c,\mathbf{p}), \quad f_{cc}=f(\mathbf{p}_c,\mathbf{p}_c),$$
$$\Phi=\omega^{-1}f_{cc}^{-2}\{\langle f_cR_c(\mathbf{V}-(\mathbf{k}_c\mathbf{v})\mathbf{I})\rangle-(\mathbf{v}^c/\omega)(1+f_{cc}^{-1}\langle f_c^2\rangle)\},$$
$$\Psi=-\omega^{-1}\{1+f_{cc}^{-1}\langle R_c(\mathbf{k}_c\mathbf{v}f_cF-\omega f_{cc}^2)\rangle\}\Phi,$$
$$\mathbf{V}=\mathbf{v}-(f_c/f_{cc})\mathbf{v}^c, \quad R_c=(\mathbf{k}_c\mathbf{v}-\omega-i0)^{-1},$$

and the functions $\mathbf{I}=\mathbf{I}(\mathbf{p})$ and $F=F(\mathbf{p})$ are determined by integral equations with renormalized kernels,

$$\mathbf{I}(\mathbf{p})+\langle\mathbf{k}_c\mathbf{v}'\xi(\mathbf{p},\mathbf{p}')R_c'\mathbf{I}'\rangle'$$
$$=\langle[\mathbf{V}'R_c'-(\mathbf{v}^c/\omega)(f_c'/f_{cc})]\xi(\mathbf{p},\mathbf{p}')\rangle',$$
$$F(\mathbf{p})+\langle\mathbf{k}_c\mathbf{v}'\xi(\mathbf{p},\mathbf{p}')R_c'F'\rangle'=f_c,$$
$$\xi(\mathbf{p},\mathbf{p}')=f(\mathbf{p},\mathbf{p}')-f_cf_c'/f_{cc}. \tag{19}$$

According to (19), we have $F_c=f_{cc}$ and $\mathbf{I}_c=0$; these results, along with the relation $V_c=0$, guarantee that there are no divergences in the expressions for Φ and Ψ.

Substituting (18) and (19) into (14), we find an expression for $\varepsilon_{\alpha\beta}$ which incorporates, along with the term $\varepsilon_{\alpha\beta}^{(l)}(\omega,k_c)\equiv\varepsilon_{\alpha\beta}^c$, the main term (insofar as the nature of the singularity is concerned), which contains the singularity, $\varepsilon_{\alpha\beta}^{(s)}$:

$$\varepsilon_{\alpha\beta}^{(l)}=\varepsilon_{\alpha\beta}{}^c+\varepsilon_{\alpha\beta}^{(s)},$$
$$\varepsilon_{\alpha\beta}{}^c=\delta_{\alpha\beta}+(4\pi e^2/\omega)\{\langle V_\alpha R_c(V_\beta-\omega I_\beta)\rangle+(v_\alpha{}^c/f_{cc})[\langle f_cI_\beta\rangle$$
$$+(v_\beta{}^c/\omega)(1+\langle f_c^2\rangle/f_{cc})]\}, \quad \varepsilon_{\alpha\beta}^{(s)}=A_{\alpha\beta}/\langle R\rangle.$$
$$A_{\alpha\beta}=(4\pi e^2/\omega)\{v_\alpha{}^c-\omega\langle V_\alpha R_cF\rangle+(v_\alpha{}^c/f_{cc})\langle f_cF\rangle\}\Phi_\beta. \tag{20}$$

Expressions (20) hold for $\langle R\rangle\propto\ln|\Delta k|$ or $|\Delta k|^{-1/6}$. If $\langle R\rangle\propto|\Delta k|^{-1/4}$ [the case of a belt of the type $(\xi-\xi_c)^2-(\eta-\eta_c)^4=\Delta k$; cf. Refs. 4 and 6], or if $\langle R\rangle$ diverges more rapidly, then the conclusion that the divergences cancel out remains in force, but the expressions for $\varepsilon_{\alpha\beta}^c$ and $A_{\alpha\beta}$ become more complicated.

The condition for the applicability of expressions (20) contains f_{cc}:

$$\omega|f_{cc}\langle R\rangle|\gg 1. \tag{21}$$

If $|k_c/\Delta k|$ is replaced by k_cl, and if the Fermi surface is spherical, this applicability condition can be put in a more specific form:

$$|F_{cc}|\ln k_cl\gg 1, \tag{22}$$

where

$$F_{cc}=F(\mathbf{p}_c,\mathbf{p}_c), \quad F(\mathbf{p},\mathbf{p}')=(m^*p_F/\pi^2\hbar^3)f(\mathbf{p},\mathbf{p}') \tag{23}$$

is the dimensionless Landau function. [The values of this function are known only poorly for real metals, and there is no basis for assuming that they are anomalously small. For liquid He^3 we have[21] $F(\mathbf{p},\mathbf{p}')\sim 10$. For Na and K, the amplitudes of the spherical harmonics are given in Ref. 22; they fall off rapidly with the index of the harmonic. The zeroth harmonic is approximately -0.6.]

If the condition $A_{\alpha\beta}=0$ holds for any of the components, then the component $\varepsilon_{\alpha\beta}^{(s)}$ has a singularity weaker than the others, and for it we can write

$$\varepsilon_{\alpha\beta}^{(s)}=(4\pi e^2/\omega)\{\langle v_\alpha(R-R_c)(v_\beta-\omega I_\beta)\rangle-\omega\langle v_\alpha R_c\Delta I_\beta\rangle\}, \tag{24}$$

and $\Delta\mathbf{I}=\Delta\mathbf{I}(\mathbf{p})$ is determined by the integral equation

$$\Delta\mathbf{I}(\mathbf{p})+\langle\mathbf{k}_c\mathbf{v}'\xi(\mathbf{p},\mathbf{p}')R_c'\Delta\mathbf{I}'\rangle'$$
$$=\langle\xi(\mathbf{p},\mathbf{p}')(R'-R_c')(\mathbf{v}'-\omega\mathbf{I}')\rangle'. \tag{25}$$

Again in this case, the component $\varepsilon_{\alpha\beta}^{(s)}$ vanishes at $k=k_c$.

When the Fermi surface is a sphere, the latter case applies to the transverse components $\varepsilon_{\alpha\beta}$, and we must use Eqs. (24) and (25). As a result we find $\varepsilon_{\alpha\beta}^{\perp(s)}\propto\Delta k\ln|\Delta k|$.

It should be noted that in the case of a Fermi gas the singular part of the components $\varepsilon_{\alpha\beta}$ is determined by the electrons in the immediate vicinity of $\mathbf{p}_c$ on the Fermi surface. The Fermi-liquid interaction has the consequence that $\varepsilon_{\alpha\beta}^{(s)}$ is determined by all the Fermi electrons.

Table I shows the characteristic singularities of the real parts of the components $\varepsilon_{\alpha\beta}$. We see that the Fermi-liquid interaction may cause not only a weakening but also a strengthening of a singularity (see the second row in Table I).

Strictly speaking, Eq. (14) and the results which follow from it are valid under the conditions $\hbar kv_F\ll T$ and $\hbar\omega\ll\varepsilon_F$. As $T\to 0$, we cannot use the Boltzmann equation to determine the nature of the singularities; a quantum-mechanical treatment is necessary. What changes will be caused by allowance for the finite values of the momentum $\hbar\mathbf{k}$ and of the energy $\hbar\omega$? Unfortunately, we must restrict the discussion to the gas approximation—to the best of our knowledge, there is no equation analogous to (1) which incorporates Fermi-liquid interaction (but see Ref. 23). According to (5), the singularity at $k=\omega/v_F$ splits into two weaker singularities of the type $\Delta k^{\pm}\ln|\Delta k^{\pm}|$, where $k_c^{\pm}=k_c(1\pm\hbar\omega/4\varepsilon_F)$. This result means that the semiclassical expression for $\varepsilon_{\parallel}$ is valid unless we are too close to $k=k_c$, and it is limited by the condition $|\Delta k|/k_c\gg\hbar\omega/\varepsilon_F$. The maximum value which $\langle R\rangle$ can have is proportional to $\ln(\varepsilon_F/\hbar\omega)$. This conclusion can apparently be generalized by assuming that the finite value of $\hbar\omega$ [in the numerator; see (1)] splits each singularity,[4)] and all the equations of this section are meaningful under the condition $|\Delta k|/k_c\gg\hbar\omega/\varepsilon_F$.

The maximum value of $\langle R\rangle$ for a Fermi surface which has lines of parabolic points depends on the direction of $\varkappa$: For a random direction we would have $\max|\langle R\rangle|\propto\ln(\varepsilon_F/\hbar\omega)$, while for certain selected directions we would have $\max|\langle R\rangle|\propto(\varepsilon_F/\hbar\omega)^\nu, 0<\nu<1$. The quantum splitting does not eliminate the singularities but it does weaken them ($\langle d\langle R\rangle/dk$ diverges as $k\to k_c^{\pm}$]. The finite

TABLE I.

		Electron gas	Fermi liquid
Fermi surf. of arb. shape (not sph-al), $\varkappa \neq \varkappa_c$	$\varepsilon_{\parallel}$, $\varepsilon_{\perp}$ for $v_{\perp}^{c} \neq 0$	$\ln \lvert\Delta k\rvert$	$\ln^{-1} \lvert\Delta k\rvert$
	$\varepsilon_{\perp}$ for $v_{\perp}^{c} = 0$	$\Delta k \ln \lvert\Delta k\rvert$	$\ln^{-1} \lvert\Delta k\rvert$
Sph-al Fermi surf.	$\varepsilon_{\parallel}$	$\ln \lvert\Delta k\rvert$	$\ln^{-1} \lvert\Delta k\rvert$
	$\varepsilon_{\perp}$	$\Delta k \ln \lvert\Delta k\rvert$	$\Delta k \ln \lvert\Delta k\rvert$

temperature T and the mean free path eliminate the singularities (they spread them out). If T, $\hbar/\tau \gg \hbar\omega$, then we have $\max|\langle R \rangle| \propto \ln(\varepsilon_F/T^*)$ or $(\varepsilon_F/T^*)^{\nu}$, where $T^* = \max\{T, \hbar/\tau\}$; if instead T, $\hbar/\tau \ll \hbar\omega$, then the latter estimates are valid for $\max|d\langle R \rangle/dk|$.

Attention has recently been called[6,14,24] to the renormalization of the kinetic coefficients (in particular, the components $\varepsilon_{\alpha\beta}$) which "eliminates" the infinities. The arguments in those papers were based on a "field" renormalization involving the elimination of the longitudinal electric field for electrodynamic problems[14] and all components of the electric field for electroacoustic problems.[6,24] In our case eliminating the longitudinal field leads to the replacement of $\varepsilon_{\alpha\beta}$ by

$$\varepsilon_{\alpha\beta}^{\perp} = \varepsilon_{\alpha\beta} - \varepsilon_{\alpha\varkappa}\varepsilon_{\varkappa\beta}/\varepsilon_{\varkappa\varkappa}, \tag{26}$$

where the subscript $\varkappa$ means the projection on to the direction of $\varkappa$. Substituting $\varepsilon_{\alpha\beta}^{(l)}$ from (14) into (26), we find

$$\varepsilon_{\alpha\beta}^{(l)\perp} = \delta_{\alpha\beta} + \frac{4\pi e^2}{\omega}\left\{\langle v_\alpha R(v_\beta - \omega J_\beta)\rangle - \frac{\langle v_\alpha R(\varkappa\mathbf{v} - \omega\varkappa\mathbf{J})\rangle\langle \varkappa\mathbf{v} R(v_\beta - \omega J_\beta)\rangle}{\omega/4\pi e^2 + \langle \varkappa\mathbf{v} R(\varkappa\mathbf{v} - \omega\varkappa\mathbf{J})\rangle}\right\}. \tag{26'}$$

Here α and β are the coordinates in the plane perpendicular to $\varkappa$. Since all the components $\varepsilon_{\alpha\beta}^{(l)}$ are finite, renormalization (26′) does not qualitatively change the k dependence of the components $\varepsilon_{\alpha\beta}$.

In connection with the existence of various renormalizations (the field and Fermi-liquid renormalizations), we should make the following comment: In a sense, the Fermi-liquid renormalization is more fundamental. It redefines $\varepsilon_{\alpha\beta}$—the tensor of coefficients of the proportionality between the vectors **D** and **E**—in a natural way for an arbitrary value of the electric field **E** (**D** is the displacement vector; $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \dot{\mathbf{P}} = \mathbf{j}$). The field renormalization, on the other hand, is a result of a solution of Maxwell's equations which imposes a condition on the electric field which can actually exist in a metal. Furthermore, the field renormalization eliminates an infinity only if it results from a single critical point or from a group of completely equivalent points. If there are several critical points, then we find from (26′) discarding the Fermi-liquid term,

$$\varepsilon_{\alpha\beta}^{(g)\perp} \propto \sum_{i=1}^{n} v_{\alpha i}^{c}\ \mathrm{SP}\ \langle R\rangle_i \times\left(v_{\beta i}^{c} - \sum_{j=1}^{n} v_{\beta j}^{c}\ \mathrm{SP}\ \langle R\rangle_j \Big/ \sum_{r=1}^{n}\ \mathrm{SP}\ \langle R\rangle_r\right).$$

Assuming that the divergence is of the same type (e.g., $\mathrm{SP}\langle R\rangle_i = a_i \ln|\Delta k|$ or $a_i|\Delta k|^{-\nu}$, where $\nu > 0$) at all critical points, we find

$$\varepsilon_{\alpha\beta}^{(g)\perp} \propto \sum_{i=1}^{n} v_{\alpha i}^{c}\ \mathrm{SP}\ \langle R\rangle_i (v_{\beta i}^{c} - \tilde{v}_{\beta}^{c}),$$

where

$$\tilde{v}_{\beta}^{c} = \sum_{j=1}^{n} v_{\beta j}^{c} a_j \Big/ \sum_{r=1}^{n} a_r.$$

We see that in this case the infinities cancel out only if $\mathbf{v}_{cj}$ is independent of j.

4. THE FERMI-LIQUID INTERACTION DURING THE PROPAGATION OF SOUND IN A METAL. CONCLUSION

Incorporating the Fermi-liquid interaction in the kinetic equation for the distribution function of the conduction electrons naturally leads to renormalization of the expressions for both the current density and the force density.[6,25,26] For the current density, for example, we have

$$\mathbf{j} = iek\omega\langle \mathbf{v} R\Lambda_\Upsilon\rangle u_\Upsilon - ie^2\langle \mathbf{v} R v_\Upsilon\rangle E_\Upsilon - ie\omega\langle \mathbf{v} R\tilde{J}\rangle, \tag{27}$$

and the function $\tilde{J} = \tilde{J}(\mathbf{p})$ satisfies an integral equation which generalizes Eq. (15):

$$\tilde{J}(\mathbf{p}) + \langle \mathbf{k}\mathbf{v}' f(\mathbf{p}, \mathbf{p}') R'\tilde{J}'\rangle' = k\omega\langle f(\mathbf{p}, \mathbf{p}') R'\Lambda_\Upsilon'\rangle' u_\Upsilon + e\langle f(\mathbf{p}, \mathbf{p}') R' v_\Upsilon'\rangle' E_\Upsilon, \tag{28}$$

where **u** is the displacement vector and **Λ** is a vector with the coordinates $\Lambda_{\gamma\delta}\varkappa_\delta$, where $\Lambda_{\gamma\delta} = \lambda_{\gamma\delta} - \langle\lambda_{\gamma\delta}\rangle/\langle 1\rangle$ is the renormalized strain energy.[27] Arguments analogous to those in the preceding section show that the Fermi-liquid interac-

tion causes the divergent terms to cancel out. It must be kept in mind, however, that the condition for the applicability of the resulting finite (nondivergent) expressions is considerably more stringent than that in the preceding case. Since for a sound wave we have $\omega \approx sk$, where s is the velocity of sound, we have the following in this case instead of (21) and (22):

$$|F_{cc}| \ln kl \quad \text{or} \quad |F_{cc}|(kl)^{\nu} \gg v_F/s \sim 10^2-10^3, \tag{29}$$

where $\nu = 1/6$ if the critical point is a flattening point.[6] These conditions of course do not hold. Furthermore, the terms containing the Fermi-liquid interaction can simply be omitted, since they are negligible even near a resonance.

We thus see that the singularity-cancellation mechanisms studied previously also operate during the propagation of sound: a) the field renormalization (elimination of the transverse field) and b) the resonant interaction of the sound wave with a quasi-electron wave with a dispersion law $\omega = \mathbf{kv}_c$, which means that we are going beyond perturbation theory. The elimination of the transverse field (like the elimination of the longitudinal field for $\varepsilon_{\alpha\beta}$) does away with a divergence only if it is generated by a single point on the Fermi surface. In more complex cases and also when no transverse field is excited at all (e.g., when longitudinal sound propagates along a high-symmetry crystallographic direction) the only mechanism which keeps the observable quantities (in particular, the sound absorption coefficient) finite is the resonant interaction.

The renormalization of $\varepsilon_{\alpha\beta}$ due to the Fermi-liquid interaction should be manifested in all the effects listed in the Introduction. To calculate the corresponding characteristics in the case with a Fermi-liquid interaction is a separate problem.

We have not considered here effects due to a magnetic field. We simply note that a magnetic field $\mathbf{H}$ (even in the classical limit, but under the condition $\omega_c \tau \gg 1$, where ω_c is the cyclotron frequency) effectively changes the "dimensionality of the resonance," since the denominator of R is a function of the single variable $p_H = \mathbf{pH}/H$, rather than two variables, if $\mathbf{H} \neq 0$; i.e.,

$$R \to R_H = (\mathbf{k}\bar{\mathbf{v}} - \omega - i0)^{-1}, \quad \bar{\mathbf{v}} = \frac{1}{T_H}\int_0^{T_H} \mathbf{v}(t_H)\, dt_H, \tag{30}$$

where T_H is the period at which the path is traced out in the magnetic field, and t_H is the time required for motion along this path [cf. Eq. (6)].

[1)]Model Fermi surfaces have cylindrical and planar regions comparatively often. Lines of critical points arise in a natural way because of cylindrical regions. If the vector $\varkappa$ is perpendicular to a plane region, then all its points will be critical. We will not discuss this case since, strictly speaking, a metal whose Fermi surface has planar regions is unstable.

[2)]Since $p_F \sim \hbar/a$, the condition $k^{cc}a \ll 1$ holds, and a wave with wave vector $\mathbf{k} = \varkappa_c k_{cc}$ can be treated macroscopically.

[3)]Analysis of the enhanced singularities[6] shows that in all cases studied the vanishing of the numerator at the critical point (at $\mathbf{p} = \mathbf{p}_c$) eliminates the divergence.

[4)]This comment of course also applies to the singularities at $k = k^A_{cc}$ and k^B_{cc}. We might note that the splitting of each singularity, $\Delta k_{cc} \sim (\omega/v_F)(\hbar\omega/\varepsilon_F)^{1/2}$, is considerably smaller than the difference $|k^A_{cc} - k^B_{cc}| \sim \omega/v_F$.

[5)]See Ref. 6, which has a bibliography on this question.

[1]J. Lindhard, Kgl. danske Videnskab. Selskab. Mat.-fys. medd. **28**, 8 (1954).
[2]W. Kohn, Phys. Rev. Lett. **2**, 393 (1959).
[3]P. L. Taylor, Phys. Rev. **131**, 1995 (1963).
[4]M.I. Kaganov, A. G. Plyavenek, and M. Khitshol'd, Zh. Eksp. Teor. Fiz. **82**, 2030 (1982) [Sov. Phys. JETP **55**, 1167 (1982)].
[5]M. I. Kaganov and A. G. Plyavenek, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 94 (1983) [JETP Lett. **38**, 109 (1983)].
[6]M. I. Kaganov, V. M. Kontorovich, T. Yu. Lisovskaya, and N. A. Stepanova, Preprint, 82-54, Donetsk Physicotechnical Institute, 1982; Zh. Eksp. Teor. Fiz. **85**, 1675 (1983) [Sov. Phys. JETP **58**, 975 (1983)].
[7]G. T. Avanesyan, M. I. Kaganov, and T. Yu. Lisovskaya, Zh. Eksp. Teor. Fiz. **75**, 1786 (1978) [Sov. Phys. JETP **48**, 900 (1978)].
[8]V. M. Kontorovich and N. A. Stepanova, Zh. Eksp. Teor. Fiz. **76**, 642 (1979) [Sov. Phys. JETP **49**, 321 (1979)].
[9]B. N. J. Persson, J. Phys. C **13**, 435 (1980).
[10]P. Apell, Phys. Scr. **23**, 284 (1981).
[11]G. E. H. Reuter and E. H. Sondheimer, Proc. R. Soc. London **195A**, 336 (1948).
[12]G. I. Ivanovski and M. I. Kaganov, Zh. Eksp. Teor. Fiz. **83**, 2320 (1982) [Sov. Phys. JETP **56**, 1345 (1982)].
[13]V. G. Peschanskiĭ, V. S. Lekhtsier, and A. Tada, Fiz. Met. Metalloved. **56**, 855 (1983).
[14]I. P. Ipatova, M. I. Kaganov, and A. V. Subashiev, Zh. Eksp. Teor. Fiz. **84**, 1830 (1983) [Sov. Phys. JETP **57**, 1066 (1983)].
[15]L. D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956) [Sov. Phys. JETP **3**, 920 (1956).
[16]I. M. Lifshits, M. Ya. Azbel', and M. I. Kaganov, Élektronnaya teoriya metallov (Electronic Theory of Metals), Nauka, Moscow, § 23, 1971.
[17]V. P. Silin, Zh. Eksp. Teor. Fiz. **35**, 1243 (1958) [Sov. Phys. JETP **8**, 870 (1959)].
[18]V. P. Silin, Zh. Eksp. Teor. Fiz. **33**, 495 (1957) [Sov. Phys. JETP **6**, 387 (1958)].
[19]E. M. Lifshitz and L. P. Pitaevskiĭ, Fizicheskaya kinetika, Vol. X, § 74, Nauka, Moscow, 1979 (Physical Kinetics, Pergamon Press, Oxford, 1981).
[20]D. Pines and P. Nozières, The Theory of Quantum Liquids, Vol. 1, Normal Fermi Liquids, Benjamin, New York (1966) (Russ. transl. Mir, Moscow, 1967).
[21]E. M. Lifshitz and L. P. Pitaevskiĭ, Statisticheskaya fizika, Vol. IX, Ch. 2, Nauka, Moscow, 1978, p. 32 (Statistical Physics).
[22]P. M. Platzman and P. A. Wolff, Waves and Interactions in Solid State Plasma, Suppl. 13 to Solid State Phys., Academic Press, New York (1973) (Russ. transl. Mir, Moscow, 1975).
[23]A. S. Kondrat'ev and A. E. Kuchma, Élektronnaya zhidkost' normal'nykh metallov (The Electron Liquid of Normal Metals), Izd-vo Leningr. un-ta, Leningrad, 1980.
[24]É. A. Kaner, L. V. Chebotarev, and A. V. Eremenko, Zh. Eksp. Teor. Fiz. **80**, 1058 (1981) [Sov. Phys. JETP **53**, 540 (1981)].
[25]V. M. Kontorovich, Zh. Eksp. Teor. Fiz. **45**, 1638 (1963) [Sov. Phys. JETP **18**, 1125 (1964)]; **59**, 2116 (1970) [Sov. Phys. JETP **32**, 1146 (1970)]; Usp. Fiz. Nauk **142**, 263 (1984) [Sov. Phys. Usp. **27**, 134 (1984)].
[26]V. P. Silin, Zh. Eksp. Teor. Fiz. **38**, 977 (1960) [Sov. Phys. JETP **11**, 703 (1960)].
[27]A. I. Akhiezer, M. I. Kaganov, and G. Ya. Lyubarskiĭ, Zh. Eksp. Teor. Fiz. **32**, 837 (1957) [Sov. Phys. JETP **5**, 685 (1957)].

Translated by Dave Parsons