Stability of inhomogeneous states in nonequilibrium superconductors

V. F. Elesin and V. A. Kashurnikov

Engineering Physics Institute, Moscow
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The stability of inhomogeneous distributions of the order parameter in a superconductor during optical pumping and tunnel injection is analyzed. Over a broad range of the pump power level, for the superconductor—normal-metal boundary conditions which are of interest for superconductors, there exist stable, steady-state, inhomogeneous distributions of the order parameter. This conclusion generalizes and extends the previous understanding of the stability, and it explains the experimentally observed onset of a resistance in a steady state.

INTRODUCTION

Inhomogeneous states in which the order parameter varies along the coordinates are known to arise in nonequilibrium superconductors.[1] The stability of such inhomogeneous states in an important question and one which has not been completely resolved.[1–3] Analogous problems arise in plasma theory,[4] in the theory of semiconductors with a negative differential conductivity,[5,6] and in other areas. The stability was studied in Refs. 3 and 5–7, by the approach proposed by Zel'dovich and Barenblatt.[8] According to Refs. 3, 6, and 7, only monotonic solutions are stable. Bass *et al.*[5] assert that the only stable solution is a singular layer solution which separates phases with identical free energies and which occurs at a certain pump power $\delta=\delta_0$.

In the present paper we show that the results of Refs. 3 and 5–7 do not apply to certain inhomogeneous solutions (which have a limiting point, as discussed below). In particular, we show that for the boundary conditions of primary interest in the case of superconductors, i.e., the boundary conditions corresponding to the interface between a superconductor and a normal metal, there may be stable inhomogeneous solutions over a broad range of the pump power.

This result yields an explanation for the experimentally observed[9–11] and previously unexplained onset of an intermediate resistance in nonequilibrium superconductors in a steady state.

1. EQUATION FOR THE ORDER PARAMETER

From Ref. 1 we have the following equation for the order parameter $\Delta(x,t)$ of a superconductor subjected to optical pumping or tunneling injection (wide sources):

$$\tau_\Delta\frac{\partial\Delta}{\partial t}=\frac{\partial^2\Delta}{\partial x^2}+\frac{\partial U}{\partial\Delta},\quad x\equiv\frac{x}{\xi_0},\tag{1}$$

$$U=-\Delta^2\left\{\frac{\delta}{2}-\varphi\frac{\Delta}{3}+\alpha\frac{\Delta^2}{4}\right\},\quad \Delta\equiv0.71\frac{\Delta}{\Delta_0},\tag{2}$$

where ξ_0 is the coherence length at the point of the phase transition with $\Delta=0$ and $T=0$, α is the numerical factor of order unity, the parameter φ is a measure of the deviation of the system from equilibrium, δ is the dimensionless extent to which the power exceeds the critical power [i.e., $\delta\sim(\beta-\beta_c)/\beta_c$], τ_Δ is the relaxation time of the order parameter, $-U$ is the energy of the nonequilibrium superconductor, and Δ_0 is the order parameter in the absence of a pump, with a temperature $T=0$ in the homogeneous and equilibrium case.

In the absence of a pump, the nonequilibrium parameter is $\varphi=0$, and Eq. (1) becomes the Ginzburg-Landau equation (as T approaches T_c, the critical temperature). The maximum value, $\varphi=1$, is reached at $T=0$ in thin films.[1] For simplicity we consider the one-dimensional case, also assuming $\xi_0\gg l$, where l is the diffusion length of the quasiparticles. Equations (1) and (2) are to be supplemented with the boundary conditions

$$\left.\frac{\partial\Delta}{\partial x}\right|_{\pm L/2}=\mp b_\pm\Delta|_{\pm L/2},\quad b_\pm\geqslant0.\tag{3}$$

We know[12,13] that we have $b=0$ for a semiconductor-insulator interface $b=0$ for the boundary conditions at an interface between a superconductor and a normal metal; in the latter case, b depends on the purity of the superconductor and on the reflection from the interface,[12] so that its value varies over a broad range: $0\leqslant b<\infty$.

To simplify the calculations we introduce some new variables:

$$\bar\Delta=\Delta/\varphi,\ \bar\delta=\delta/\varphi^2,\ \bar x=x\varphi,\qquad \tau_{\bar\Delta}=\tau_\Delta/\varphi^2,\ \bar b=b/\varphi.\tag{4}$$

In a steady state, Eq. (1) becomes

$$\frac{d^2\bar\Delta}{d\bar x^2}=-\frac{\partial\bar U}{\partial\bar\Delta},\quad \bar U=-\bar\Delta^2\left\{\frac{\bar\delta}{2}-\frac{\bar\Delta}{3}+\alpha\frac{\bar\Delta^2}{4}\right\},\quad \bar\delta<\frac{1}{4\alpha},\tag{5}$$

which is the equation of motion of a "particle" with a "coordinate" $\bar\Delta$ in the "field" $\bar U$. Figure 1 shows the "energy" $\bar U$ and the phase trajectories of Eq. (5).

2. INHOMOGENEOUS STEADY-STATE SOLUTIONS

A first integral of (5) is given by the equation

$$\frac{1}{2}\left(\frac{d\bar\Delta}{d\bar x}\right)^2=\bar C-\bar U(\bar\Delta),\quad \bar C=\bar U(\bar\Delta_2)=\bar U(\bar\Delta_3),\tag{6}$$

where $\bar C$ is a constant, and the $\bar\Delta_i$ $(i=1,2,3,4)$ are the roots of the equation $\bar C-\bar U(\bar\Delta_i)=0$.

Analysis of the trajectories shows that the following spatially inhomogeneous solutions $\bar\Delta(\bar x)$ are possible: 1) solutions which oscillate from $\bar\Delta_2$ to $\bar\Delta_3$, 2) solitary layers with $\bar\Delta\neq0$ or $\bar\Delta=0$ (soliton solutions), and 3) monotonic solutions.

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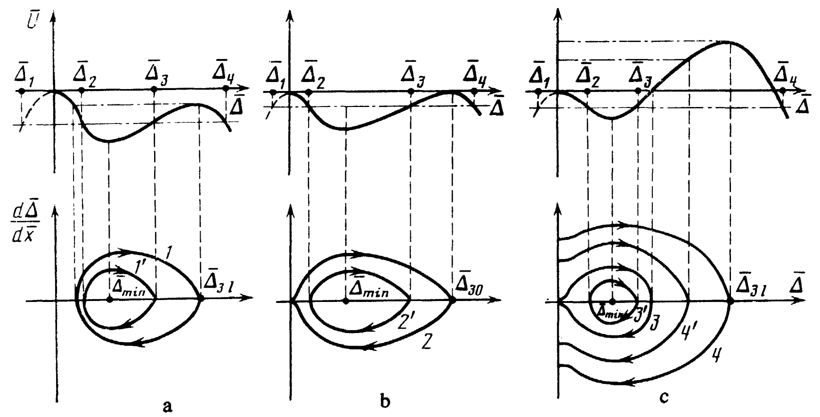

FIG. 1. The "energy" $\bar U$ and the phase trajectories of Eq. (5). a—$\bar\delta > \bar\delta_0$; b—$\bar\delta = \bar\delta_0$; c—$\bar\delta < \bar\delta_0$.

The solutions of Eq. (6) are expressed in terms of elliptic functions. Here we will write out only the solution which describe "limiting" trajectories 1, 2, 3, 4 (Fig. 1) and which can be expressed in terms of elementary functions:

a) trajectory 2 (a layer solution),

$$\Delta(\bar x)=\frac{\bar\Delta_{30}}{1+\exp\{-\delta_0^{1/2}(\bar x+C_1)\}},\quad \delta=\delta_0=\frac{2}{9\alpha}=\frac{\bar\Delta_{30}}{3},\tag{7}$$

where

$$(\bar\Delta_4-\bar\Delta_3)(\bar\Delta_2-\bar\Delta_1)=16\bar\Delta_{30}^2\exp\{-\delta_0^{1/2}\bar L\}\ll 1,\quad \bar L=L\varphi;\tag{8}$$

b) trajectories 1,4.

$$\bar\Delta(\bar x)=\bar\Delta_3-\frac{3(\bar\Delta_3-2\delta)}{|1-{}^3/_2\alpha\bar\Delta_3|^{1/2}Z(\delta,\bar x)+3\alpha\bar\Delta_3-1},\quad \delta\neq\delta_0,\tag{9}$$

where

$$Z(\bar\delta,\bar x)=\mathrm{sh}\{(\bar\Delta_3-2\bar\delta)^{1/2}(\bar x+C_2)\},\ \bar\delta<\bar\delta_0\ \text{(trajectory 4)},$$

$$Z(\bar\delta,\bar x)=\mathrm{ch}\{(\bar\Delta_3-2\bar\delta)^{1/2}(\bar x+C_3)\},\ \bar\delta>\bar\delta_0\ \text{(trajectory 1)},$$

$$\bar\Delta_4-\bar\Delta_3=24\frac{\bar\Delta_3-2\delta}{|1-{}^3/_2\alpha\bar\Delta_3|^{1/2}}\exp\{-(\bar\Delta_3-2\delta)^{1/2}\bar L\}\ll1;\tag{10}$$

c) trajectory 3,

$$\bar\Delta(\bar x)=3\bar\delta[1+(1-{}^9/_2\alpha\bar\delta)^{1/2}\,\mathrm{ch}\,(\bar\delta^{1/2}(\bar x+C_4))]^{-1},\quad 0<\bar\delta<\bar\delta_0,\tag{11}$$

where

$$\bar\Delta_2-\bar\Delta_1=24\bar\delta[1-{}^9/_2\alpha\bar\delta]^{-1/2}\exp\{-\bar\delta^{1/2}\bar L\}\ll1.\tag{12}$$

On the right side of (7)–(10),

$$\bar\Delta_3\equiv\bar\Delta_{3l}=\frac{1}{2\alpha}\{1+(1-4\alpha\delta)^{1/2}\}\tag{13}$$

is the limiting value of $\bar\Delta_3$, which corresponds to a maximum of the energy $\bar U(\bar\Delta)$ (the limiting point for trajectory 3 is $\bar\Delta = 0$), and the C_i ($i = 1, 2, 3, 4$) are constants to be determined from the boundary conditions. For these trajectories, the distance along $\bar x$ from $\bar\Delta_2$ to $\bar\Delta_3$ (the half-period) is $\bar L \gg 1$, and the roots of the equation $\bar C - \bar U(\bar\Delta) = 0$ coalesce at the accuracy specified in (8), (10), and (12).

3. STABILITY CONDITION FOR HOMOGENEOUS DISTRIBUTIONS

To analyze the stability we use Eq. (1). Setting

$$\Delta(\bar x, t)=\bar\Delta(\bar x)+\eta(\bar x)\exp(-\gamma t),\quad \eta(\bar x)\ll\bar\Delta(\bar x),$$

where $\bar\Delta(\bar x)$ is a solution of Eq. (5), we find from (1) an equation from which we can find γ:

$$\left\{-\frac{d^2}{d\bar x^2}-\frac{\partial^2\bar U}{\partial\bar\Delta^2}\right\}\eta(\bar x)=\bar\gamma\eta(\bar x),\quad \bar\gamma=\gamma\tau_\Delta.\tag{14}$$

Since the boundary conditions on $\eta(\bar x)$ are homogeneous, they are the same as the boundary conditions on $\bar\Delta(\bar x)$. Stability of the steady-state solutions $\Delta(x)$ with respect to spatially inhomogeneous perturbations results when we have $\bar\gamma\geqslant0$ for all $\bar\gamma$. If we have $\bar\gamma<0$ for at least one $\bar\gamma$, the solution $\bar\Delta(\bar x)$ is unstable.

The stability problem thus reduces to the solution of the Sturm-Liouville problem (14) with boundary conditions (3) for the eigenfunctions $\eta(\bar x)$ and the eigenvalue $\bar\gamma$; the steady-state solution $\bar\Delta(\bar x)$ determines the "potential energy"

$$V(\bar x)=-\partial^2\bar U/\partial\bar\Delta^2=\bar\delta-2\bar\Delta+3\alpha\bar\Delta^2.$$

The stability analysis in Refs. 3 and 5–7 was based on a representation of the general solution of (14) in the case $\bar\gamma = 0$ in the form

$$\eta(\bar x)=\bar C_1\frac{d\bar\Delta}{d\bar x}+\bar C_2\frac{d\bar\Delta}{d\bar x}\int^{\bar x}d\bar x'\left(\frac{d\bar\Delta}{d\bar x'}\right)^{-2}.\tag{15}$$

Using an oscillation theorem, Bass *et al.*[5] showed that all the solutions $\bar\Delta(\bar x)$ are unstable (except for the singular layer solution).

For the case of limiting trajectories 1, 3, and 4, however, expression (15) cannot be used for a stability analysis, since the second term diverges at the limiting point $\bar\Delta = \bar\Delta_l$. Using (6) we find the following result as $\bar\Delta\to\bar\Delta_l$:

$$\frac{d\bar\Delta}{d\bar x}\int^{\bar x}d\bar x'\left(\frac{d\bar\Delta}{d\bar x'}\right)^{-2}=\frac12\{\bar C-\bar U(\bar\Delta)\}^{1/2}\int^{\bar\Delta}\frac{d\bar\Delta'}{\{\bar C-\bar U(\bar\Delta')\}^{3/2}}$$
$$\sim(\bar\Delta-\bar\Delta_l)\int^{\bar\Delta}\frac{d\bar\Delta'}{(\bar\Delta'-\bar\Delta_l)^3}\sim(\bar\Delta-\bar\Delta_l)^{-1}.\tag{16}$$

We must therefore set $\bar C_2 = 0$. The solution which remains, however, does not satisfy the boundary conditions (3). The question of the stability of such solutions therefore remains open.

We now turn to an approximate analytic determination and a numerical determination of the decay rate $\bar{\gamma}$ for these solutions; we show that some of these solutions are stable.

4. STABILITY OF SOLUTIONS DESCRIBING TRAJECTORIES WITH A LIMITING POINT

Trajectory 4. We first consider the case of most interest, trajectory 4, described by expression (9) ($\bar{\delta} < \bar{\delta}_0$). We examine the stability of the half-period of this trajectory for the boundary conditions

$$\bar{\Delta}\left(-\frac{\bar{L}}{2}\right)=0, \quad \frac{d\bar{\Delta}}{d\bar{x}}\bigg|_{+\bar{L}/2}=0.$$

We write Eq. (14) in the form

$$\frac{d^2\eta}{d\bar{x}^2}+p^2(\bar{x})\eta(\bar{x})=0, \quad \eta\left(-\frac{\bar{L}}{2}\right)=0, \quad \frac{d\eta}{d\bar{x}}\bigg|_{+\bar{L}/2}=0, \tag{17}$$

where $p(\bar{x}) = \{\bar{\gamma} - V(\bar{x})\}^{1/2}$ is an analog of the classical momentum. In the limit $\bar{\delta} \to \bar{\delta}_0$, the conditions of the semiclassical approximation hold:

$$\frac{1}{p^2}\frac{dp}{d\bar{x}}\sim\left\{1-\frac{\bar{\delta}}{\bar{\delta}_0}\right\}^{1/2}\ll 1.$$

Using (10), we can write this condition more rigorously as

$$\frac{16}{3\alpha}\exp\{-\bar{\delta}_0^{1/2}\bar{L}\}\ll\left\{1-\frac{\bar{\delta}}{\bar{\delta}_0}\right\}^{1/2}\sim\frac{1}{p^2}\frac{dp}{d\bar{x}}\ll 1. \tag{18}$$

In the semiclassical approximation, the values of $\bar{\gamma}$ are found from the transcendental equation[14,15]

$$\operatorname{ctg}\omega=-{}^1/_2\exp\{-2S\}, \tag{19}$$

where

$$S=\int_{-\bar{L}/2}^{\bar{x}_1}|p(\bar{x}')|\,d\bar{x}', \quad \omega=\int_{\bar{x}_1}^{\bar{x}_2}p(\bar{x}')\,d\bar{x}', \tag{20}$$

and $\bar{x}_1$ and $\bar{x}_2$ are the roots of the equation $p(\bar{x}) = 0$. Substituting $\bar{\Delta}(\bar{x})$ into (20), and expanding ω and S in the small parameter $\chi = 1 - \bar{\delta}/\bar{\delta}_0$ (Ref. 16), we find from (19)

$$\begin{gathered}\bar{\gamma}_0=0.58\chi+o(\chi),\\ \bar{\gamma}_1=\frac{1}{6\alpha}+0.19\chi^{1/2}-0.65\chi\ln\chi-1.17\chi+o(\chi).\end{gathered} \tag{21}$$

We see that both values of $\bar{\gamma}$ are positive; i.e., this solution is stable. Numerical calculations show that this result also holds for values of $\bar{\delta}$ which are not close to $\bar{\delta}_0$. It should be noted that in the case we have the following estimate for $\bar{\gamma}_{\min}$:

$$\bar{\gamma}_{min}=\bar{\Delta}_3-2\bar{\delta}>0. \tag{22}$$

We now examine the total period of trajectory 4, which is described by (9) (with $x \to -|x|$) and the boundary conditions $\bar{\Delta}(\pm\bar{L}/2) = 0$.

In this case the potential $V(\bar{x})$ in the stability equation has two depressions separated by a large distance $\sim\bar{L}$. Expanding in the small parameter $\exp\{-\bar{\delta}_0^{1/2}\bar{L}\}$, we can easily show that the level splitting is weak and that the lower level, $\bar{\gamma}_0$, shifts downward by a small quantity of the same order, $\sim\exp\{-\bar{\delta}_0^{1/2}\bar{L}\}\ll 1$, as $\bar{\delta}\to\bar{\delta}_0$; using condition (18), we see that this shift does not change the sign of $\bar{\gamma}_0$. The solution which described the total period of trajectory 4 is therefore also stable.

The situation is different for trajectories which have 3/2 of a period or more. In such cases, the potential barriers in $V(\bar{x})$ are low, and the level shift is correspondingly significant and leads to negative values of $\bar{\gamma}$. Numerical calculations confirm these arguments.

Trajectory 3. For $0 < \bar{\delta} < \bar{\delta}_0$ we have yet another limiting trajectory: trajectory 3, described by (11). Let us find the decay rate $\bar{\gamma}$ for the half-period of this trajectory and for the boundary conditions

$$\frac{d\bar{\Delta}}{d\bar{x}}\bigg|_{\pm\bar{L}/2}=0.$$

Substituting (11) into (14) and into the boundary conditions, we find

$$\begin{gathered}\varphi(\xi)\frac{d^2\eta}{d\xi^2}+\frac{1}{2}\frac{d\varphi}{d\xi}\frac{d\eta}{d\xi}-\frac{1}{2}\frac{d^2\varphi}{d\xi^2}\eta=-\bar{\gamma}\eta,\\ \varphi^{1/2}(\xi)\frac{d\eta}{d\xi}\bigg|_{0;1}=0,\end{gathered} \tag{23}$$

where

$$\begin{gathered}\xi=\bar{\Delta}/\bar{\Delta}_2, \quad 0<\xi<1, \quad \bar{\Delta}_2=\frac{2}{3\alpha}\left\{1-\left(1-\frac{\bar{\delta}}{\bar{\delta}_0}\right)^{1/2}\right\},\\ \varphi(\xi)=\frac{\alpha}{2}\bar{\Delta}_2{}^2\xi^2(1-\xi)(\xi_1-\xi), \quad \xi_1=\bar{\delta}\left(\frac{\alpha}{2}\bar{\Delta}_2{}^2\right)^{-1},\end{gathered} \tag{24}$$

and

$$\begin{gathered}\xi_1=1+2(1-\bar{\delta}/\bar{\delta}_0)^{1/2}, \quad \bar{\delta}\to\bar{\delta}_0,\\ \xi_1={}^8/_9(\alpha\bar{\delta})^{-1}, \quad \bar{\delta}\to 0.\end{gathered} \tag{25}$$

Equation (23) is a Fuchs equation[17] with four singularities (poles), at 0, 1, ξ_1, and ∞.

In the limit $\bar{\delta} \to 0$ [condition (12) holds automatically in this case: $(24/\alpha)\bar{\delta}\exp\{-\bar{\delta}^{1/2}\bar{L}\}\ll 1$], the poles ξ_1 and $\xi = \infty$ coalesce, and Eq. (23) becomes the equation for Legendre polynomials [after the change of variable $t = (1-\xi)^{1/2}$]. Expanding in the small parameter ξ_1^{-1}, and retaining terms up to second order, we find

$$\begin{gathered}\bar{\gamma}_0=\bar{\delta}\left\{-1.25+\frac{3177}{1792}\xi_1{}^{-1}-\frac{129}{490}\xi_1{}^{-2}\right\},\\ \bar{\gamma}_1=\bar{\delta}\left\{0.75+\frac{12}{35}\xi_1{}^{-1}+\frac{129}{490}\xi_1{}^{-2}\right\}.\end{gathered} \tag{26}$$

Since $\bar{\gamma}_0$ is negative, the solution $\bar{\Delta}(\bar{x})$ is unstable.

In the other limiting case, $\bar{\delta} \to \bar{\delta}_0$, in which we have, according to (12),

$$\frac{16}{3\alpha}\exp\{-\bar{\delta}_0^{1/2}\bar{L}\}\ll\left(1-\frac{\bar{\delta}}{\bar{\delta}_0}\right)^{1/2}\ll 1,$$

the poles ξ_1 and $\xi = 1$ merge, since $\xi_1 \to 1$. With $\bar{\delta} = \bar{\delta}_0$, using a change of variables ($\xi_c = 1 - 2\xi$), we find from (23) the equation for Legendre polynomials, $\bar{\gamma}_0 = 0$, and $\bar{\gamma}_1 = 1/6\alpha$, Using the semiclassical approximation, as in the preceding case, we find the corrections to $\bar{\gamma}$ from the transcendental equation

$$\operatorname{ctg}\omega={}^1/_2\exp\{-2S\}, \tag{27}$$

where expressions (20) hold for ω and S. The integrals ω and

S can be written in terms of elliptic integrals.[16] Expanding them in the parameter $\chi = 1 - \delta/\delta_0$, we find

$$\bar{\gamma}_0 = -0.59\chi + o(\chi),$$
$$\bar{\gamma}_1 = \frac{1}{6\alpha} - 0.19\chi^{1/2} - 0.33\chi \ln \chi - 5.13\chi + o(\chi). \tag{28}$$

It follows from (28) that we have $\bar{\gamma}_0 < 0$, and the solution $\bar{\Delta}(\bar{x})$ describing the half-period of trajectory 3 is unstable. We can go through the same analysis for a half-period with the boundary conditions

$$\bar{\Delta}\left(-\frac{\bar{L}}{2}\right) = 0, \quad \left.\frac{d\bar{\Delta}}{d\bar{x}}\right|_{+\bar{L}/2} = 0 \ ;$$

we find that $\bar{\Delta}(\bar{x})$ is also unstable. This conclusion holds for the period of trajectory 3 and also for, longer periods; for a period with the boundary conditions $\bar{\Delta}(\pm \bar{L}/2) = 0$ the stability problem can be solved exactly: $\eta_1 = d\bar{\Delta}/d\bar{x}$ and $\bar{\gamma}_1 = 0$, $\bar{\gamma}_0 < 0$.

Trajectory 1. We now seek $\bar{\gamma}$ for the half-period of trajectory 1, described by (9) ($\bar{\delta}_0 < \bar{\delta} < 1/4\alpha$), and for the boundary conditions

$$\left.\frac{d\bar{\Delta}}{d\bar{x}}\right|_{\pm\bar{L}/2} = 0.$$

Changing variables in (14) by means of

$$\xi = (\bar{\Delta}_3 - \bar{\Delta}(\bar{x}))/(\bar{\Delta}_3 - \bar{\Delta}_{-\bar{L}/2}), \tag{29}$$

where

$$\bar{\Delta}_{-\bar{L}/2} = \frac{2}{3\alpha}\left\{1 - \frac{3\alpha}{2}\bar{\Delta}_3 + \left(1 - \frac{3\alpha}{2}\bar{\Delta}_3\right)^{1/2}\right\},$$

we find Eq. (23) to within the formal substitution

$$\bar{\delta} \to \bar{\Delta}_3 - 2\bar{\delta}, \quad \bar{\Delta}_2 \to \bar{\Delta}_3 - \bar{\Delta}_{-\bar{L}/2}; \tag{30}$$

here

$$\xi_1 = \frac{\bar{\Delta}_3 - 2\bar{\delta}}{(\alpha/2)(\bar{\Delta}_3 - \bar{\Delta}_{-\bar{L}/2})^2} = \begin{cases} 1 + 2\left(\dfrac{\delta}{\delta_0} - 1\right)^{1/2}, & \delta \to \delta_0, \\[2ex] \dfrac{4}{9}\left(1 - \dfrac{\delta}{\delta_i}\right)^{-1/2}, & \delta \to \delta_i = \dfrac{1}{4\alpha}. \end{cases} \tag{31}$$

The analysis of trajectory 3 above remains completely applicable here; using (30), we find

$$\bar{\gamma}_0 = (\bar{\Delta}_3 - 2\bar{\delta})\left\{-1.25 + \frac{3177}{1792}\xi_1^{-1} - \frac{129}{490}\xi_1^{-2}\right\},$$
$$\bar{\gamma}_1 = (\bar{\Delta}_3 - 2\bar{\delta})\left\{0.75 - \frac{12}{35}\xi_1^{-1} + \frac{129}{490}\xi_1^{-2}\right\}, \tag{32}$$

where $\bar{\delta} \to \bar{\delta}_i$. Condition (10) holds automatically:

$$\frac{24}{\alpha}\left(1 - \frac{\delta}{\delta_i}\right)^{1/4} \exp\left\{-\left(1 - \frac{\delta}{\delta_i}\right)^{1/4} \bar{L}\right\} \ll 1.$$

As $\bar{\delta} \to \bar{\delta}_0$ we have

$$\bar{\gamma}_0 = -0.58\chi_1 + o(\chi_1),$$
$$\bar{\gamma}_1 = \frac{1}{6\alpha} - 0{,}19\chi_1^{1/2} - 0.33\chi_1 \ln \chi_1 - 5.72\chi_1 + o(\chi_1), \tag{33}$$

where

$$\chi_1 = \bar{\delta}/\bar{\delta}_0 - 1 \ll 1.$$

More rigorously, we find from (10)

$$\frac{16}{3\alpha}\exp\{-\bar{\delta}_0^{1/2}\bar{L}\} \ll \left(\frac{\delta}{\delta} - 1\right)^{1/2} \ll 1,$$

As can be seen from (32) and (33), the solution describing the half-period of trajectory 1 is unstable. This result also applies to the complete period and more. A unique solution for this trajectory, which is stable, occurs only when it represents a part of the trajectory near the limiting point $\bar{\Delta}_3$ [see estimate (34) below].

For simplicity, expressions (21), (28), and (33) have been written for the particular value $\alpha = 1.9$.

Layer solution (trajectory 2.) For the layer solution described by (7), the stability problem can be solved exactly. Equation (14) can be reduced to the Legendre equation by the change of variables $\xi_c = 3\alpha\bar{\Delta}(\bar{x})$ and $\bar{\gamma}_0 = 0$, $\bar{\gamma}_1 = 1/6\alpha$. As was expected in accordance with the analysis of Refs. 3 and 5–7, we find $\bar{\gamma}_{\min} = 0$.

The results found above are shown in Fig. 2. Also shown here are the values of $\bar{\gamma}$ found by the numerical calculations. We see from this figure that the results of the numerical and analytic calculations agree completely.

For a numerical solution of the Sturm-Liouville problem we used the program of Ref. (18), modified for the boundary conditions. We studied the stability of all of the "limiting" solutions with various boundary conditions (with various values of $\bar{b}$), both analytically and numerically. For $\delta < \delta_0$, the stable solutions are the monotonic solutions which describe limiting trajectory 4 and the soliton solutions of the same trajectory with the limiting point $\bar{\Delta}_3$ at its center. The numerical calculations for $\delta > \delta_0$ for trajectory 1 lead to the same results if the solution describes this trajectory near $\bar{\Delta} = \bar{\Delta}_3$ and, correspondingly, if the values of $\bar{b}$ are in a small interval $0 \ll \bar{b}_{\pm} \ll \bar{b}_{cr}(\bar{\delta})$, where $\bar{b}_{cr}(\bar{\delta}_0) = \bar{\delta}_0^{1/2}$ and $\bar{b}_{cr}(\bar{\delta}_i) = 0$. This result is confirmed by the analytic calculations. If

$$(\bar{\Delta}_3 - \bar{\Delta}(\bar{x}))/\bar{\Delta}_3 \ll 1 \quad (\bar{b}_{\pm} \ll \delta_0^{1/2}),$$

then

$$\bar{b}_{cr}(\bar{\delta}) = (\bar{\Delta}_3 - 2\bar{\delta})^{1/2},$$

and

$$\bar{\gamma}_{min} = \bar{b}_{cr}^2(\bar{\delta}) - \max\{\bar{b}_{\pm}^2\}. \tag{34}$$

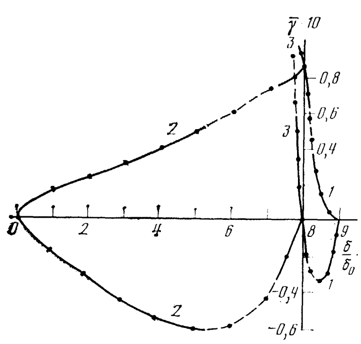

FIG. 2. The damping rate $\bar{\gamma}$ versus the relative source power $\bar{\delta}$ for $\alpha = 1.9$. 1—Half-period of trajectory 1; 2—half-period of trajectory 3; 3—half-period of trajectory 4. Solid lines) Analytic calculations; dashed lines) numerical calculations.

These stable solutions apparently correspond to a local minimum of the energy of the system [see (46) below].

5. STABLE INHOMOGENEOUS STEADY-STATE DISTRIBUTIONS

Here the explicit expression for the stable, inhomogeneous, steady-state distributions $\Delta(x)$, where $x \equiv x/\xi_0$ and $\delta < \delta_i$, are

$$\Delta(x) = \Delta_3 - \frac{3(\Delta_3\varphi - 2\delta)}{|{}^3/_2\alpha\varphi\Delta_3 - \varphi^2|^{1/2} Z(\delta, x) + 3\alpha\Delta_3 - \varphi}, \quad \delta \neq \delta_0,$$
$$\Delta(x) = \frac{\Delta_{30}}{1 + \exp\{-\delta_0^{1/2}(x+\theta_1)\}}, \quad \delta = \delta_0, \tag{35}$$

where

$$Z(\delta, x) = \mathrm{sh}\,\{(\varphi\Delta_3 - 2\delta)^{1/2}(x+\theta_2)\}, \quad \delta < \delta_0,$$
$$Z(\delta, x) = \mathrm{ch}\,\{(\varphi\Delta_3 - 2\delta)^{1/2}(x+\theta_3)\}, \quad \delta > \delta_0, \tag{36}$$

$$\Delta_3 = \frac{\varphi}{2\alpha}\left\{1 + \left(1 - \frac{4\alpha\delta}{\varphi^2}\right)^{1/2}\right\},$$
$$\delta_0 = \frac{2\varphi^2}{9\alpha}, \quad \Delta_{30} = \frac{2\varphi}{3\alpha}, \quad \delta_i = \frac{\varphi^2}{4\alpha}. \tag{37}$$

The constants θ_i ($i = 1, 2, 3$) are determined from the boundary conditions. Solutions (35) are shown (in dimensionless form, divided by φ) in Fig. 3 for various values of the constants b in the boundary conditions and for various values of the pump power.

The parameter φ is a measure of the deviation of the system from equilibrium.[1] In the equilibrium case (with $\varphi = 0$, $\delta < 0$), expression (35) becomes the Ginzburg-Landau solution ($T \to T_c$)

$$\Delta(x) = \left(\frac{|\delta|}{\alpha}\right)^{1/2} \mathrm{th}\left\{\left(\frac{|\delta|}{2}\right)^{1/2}(x+\theta_4)\right\}, \tag{38}$$

for which the damping factor γ_0 is positive and attains a maximum. Here $\xi_0|\delta|^{-1/2} = \xi(T)$ is the coherence length. In this case the stability problem can be solved analytically for arbitrary values of the constants θ_4 in (38) [for any boundary conditions of the type in (3)], and estimate (22) applies.

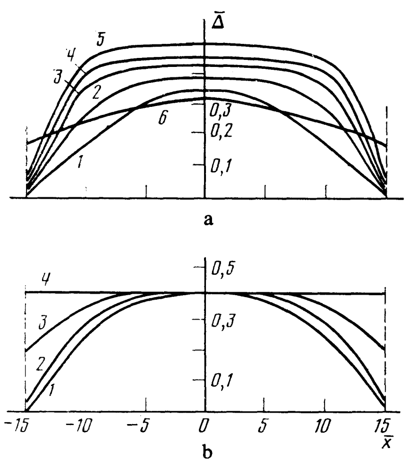

FIG. 3. Stable, nonequilibrium, inhomogeneous distributions of the order parameter in the one-dimensional case near the phase transition for $\bar{L} = 30$, $\bar{x} = x\varphi$, $\bar{\Delta} = \Delta/\varphi$, $(d\bar{\Delta}/d\bar{x}|_{\pm\bar{L}/2} = \pm\bar{b}\bar{\Delta}|_{\pm\bar{L}/2}$. a: $\bar{b} = 2.6$. 1—$\bar{\delta} = 0.95\bar{\delta}_0$, 2—$0.8\bar{\delta}_0$, 3—$0.6\bar{\delta}_0$, 4—$0.4\bar{\delta}_0$, 5—$0.2\bar{\delta}_0$, 6—$1.07\bar{\delta}_0$ ($\bar{b} = 0.09$). b: $\bar{\delta} = 0.7\bar{\delta}_0$. 1—$\bar{b} \gg 1$; 2—$\bar{b} = 2.6$; 3—$\bar{b} = 0.5$; 4—$\bar{b} = 0$.

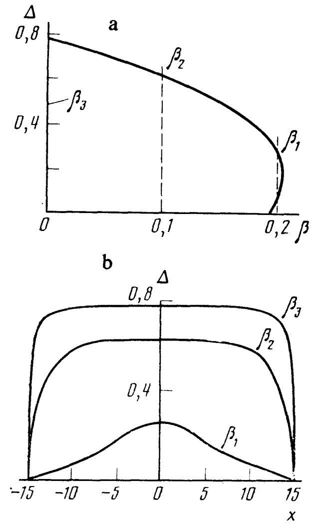

FIG. 4. Stable distributions of the order parameter versus the parameter φ, a measure of the deviation from equilibrium, in the limit $T \to T_c$ ($\varphi \sim \beta$). a—Homogeneous case ($b_\pm = 0$); b—inhomogeneous distributions ($b_\pm \gg 1$). $\beta_1 = 0.205$, $\beta_2 = 0.1$, $\beta_3 = 0$ (equilibrium).

Let us construct the φ dependence of $\Delta(x)$ in the limit $T \to T_c$, using the following results for thin films[1]:

$$\varphi = \beta\zeta_1, \quad \alpha = \frac{7\zeta(3)\Delta_0^2}{4\pi^2 T_c^2} \approx 0.66,$$
$$\delta = \frac{T - T_c}{T_c} + 2\beta\zeta_2, \quad \zeta_i \sim 1. \tag{39}$$

The results are shown in Fig. 4 ($T/T_c = 0.6$, $\zeta_i = 1$). Also shown in this figure, for clarity, is the β dependence of Δ in the homogeneous case (described by the equation $\partial U/\partial\Delta = 0$).

As can be seen from Figs. 3 and 4, the typical distance over which the order parameter changes, l_φ, increases substantially as the phase transition is approached ($\delta \to \delta_0$). Whereas with $\beta = 0$ and at equilibrium ($\varphi = 0$) we have $l_\varphi = \xi(T)$, in the limit $\beta \to \beta_0$ ($\delta \to \delta_0$) and at $b \gg 1$ we have

$$l_\varphi \approx \xi_0\delta_0^{-1/2} \ln\left\{2\left(1 - \frac{\delta}{\delta}\right)^{-1/2}\right\}. \tag{40}$$

For $T/T_c = 0.6$, $\alpha = 0.66$, and $\beta \approx \varphi = 0.205$ ($\delta/\delta_0 \approx 0.7$), for example, we have $l_\varphi \approx 5\xi(T)$ (see Figs. 4; $\beta = \beta_1$).

The change in the order parameter thus occurs over distances much greater than the coherence length.

We turn now to the resistance component which arises from the inhomogeneity of the order parameter in the steady state.

We consider a superconducting thin film under nonequilibrium conditions, through which a small direct current j is flowing. Taking the approach of Ref. 19, we find an equation for the electrostatic potential Φ, taking into account the fact that the order parameter varies slowly over the coherence length[1)]:

$$l_E^2 \frac{d^2\Phi}{dx^2} = \frac{\Delta(x)}{\Delta_3}\Phi, \quad x \equiv \frac{x}{\xi_0}, \tag{41}$$

$$l_E^2 = \frac{2}{7\pi\lambda}\left[\frac{\omega_D}{\varepsilon(T)}\right]^2 \frac{D}{\xi_0^2}\frac{1}{\overline{\overline{\Delta}}_3}, \tag{42}$$

where D is the quasiparticle diffusion coefficient, λ is the constant of the electron-phonon interaction, ω_D is the Debye frequency, $\overline{\overline{\Delta}}_3 = \Delta_3(\Delta_0/0.71)$ [see (2)], and $\varepsilon(T)$ is the energy scale over which the quasiparticle distribution function varies at the point of the phase transition ($\Delta = 0$):

$$\varepsilon(T) = T, \quad T \to T_c, \quad \varepsilon(T) = 0.6\Delta_0 \sim T_c, \quad T \to 0.$$

Equation (41) is the same as the corresponding equation in Ref. 19 in the limit $T \to T_c$. To solve (41), we substitute $\Delta(x)$ from (35) in the limit $\delta \to \delta_0$ in the form ($\delta < \delta_0$, $b \gg 1$)

$$\frac{\Delta(x)}{\Delta_3} = 1 - \left\{\left(1 - \frac{\delta}{\delta_0}\right)^{1/2} \operatorname{sh}\left[\delta_0^{1/2}\left(x + \frac{L}{2}\right)\right] + 1\right\}^{-1}. \tag{43}$$

To simplify the calculations below, we set $l_E = \delta_0^{-1/2}$.

Using $E = -d\Phi/dx$, we find, to lowest order in $1 - \delta/\delta_0$,

$$E(x) = E_N \begin{cases} 1, & x + \dfrac{L}{2} \ll l_E, \\ 1 + \left[\dfrac{1}{t(x)} - 1\right] \ln[1 - t(x)], & x + \dfrac{L}{2} \gg l_E, \end{cases} \tag{44}$$

where

$$t(x) = \left\{1 + \exp\left[\left(x + \frac{L}{2} - l_\varphi\right)\Big/ l_E\right]\right\}^{-1}, \quad E_N = j\sigma_N^{-1}.$$

Here σ_N is the conductivity in the normal state. The contribution of the superconducting film to the resistance is thus a linear function of l_φ:

$$R/R_N = (l_E + l_\varphi)/L, \tag{45}$$

where L is the length of the film.

The deviation from equilibrium thus leads to an increase in the resistance in the steady state. In principle, there could be a continuous transition in terms of the resistance to the normal state. This result apparently explains the resistance observed in the steady state in Refs. 9–11.

6. DISCUSSION OF RESULTS

The stability of inhomogenous distributions of the order parameter, studied above, is confirmed by energy considerations. Specifically, if we use the expression

$$I = \int_{-L/2}^{L/2} dx \left\{\frac{1}{2}\left(\frac{d\Delta}{dx}\right)^2 - U(\Delta)\right\} + \frac{b_-}{2}\Delta^2\left(-\frac{L}{2}\right) + \frac{b_+}{2}\Delta^2\left(\frac{L}{2}\right) \tag{46}$$

for the functional (a variation of this expression yields an equation for Δ), we can show that stable distributions (35) achieve an absolute minimum of the functional in the case $\delta < \delta_0$ or a local minimum in the case $\delta > \delta_0$ (we recall that if $\delta > \delta_0$ there is an absolute minimum, which is reached at $\Delta = 0$).

The quantity I is the difference between the free energies of the inhomogeneous superconducting state and the normal state (cf. the homogeneous case[20] and the equilibrium case[12,13]).

How does the behavior of the superconducting film depend on the boundary conditions? We recall that we are dealing with the static case, i.e., that in which the time scale (τ_δ) for a change in the pump power is much longer than the relaxation time (τ_Δ) of the order parameter.

If the film is homogeneous, i.e., if there is no nucleating region of a normal phase at $\beta > 0$, then we observe the following: When the pump is turned on slowly, a stable, inhomogeneous state (35) exists up to $\delta = \delta_0$. Under the condition $\max\{b_\pm\} \geqslant b_{cr}(\delta_0)$, there is transition to the normal state at $\delta = \delta_0$. If, on the other hand, we have $0 \leqslant b_\pm < b_{cr}(\delta_0)$, then at $\delta > \delta_0$ we again have a distribution as in (35) (in the metastable state now), and now the transition to the normal state occurs at $\delta = \delta_{cr}$ ($\delta_0 < \delta_{cr} \leqslant \delta_i$), which is the root of the equation $\max\{b_\pm\} = b_{cr}(\delta_{cr})$. In the latter case we observe hysteresis: As the pump power is lowered, the system remains at the absolute minimum of the energy ($\Delta = 0$) down to $\delta = \delta_0$ (which is lower than δ_{cr}), and at this point the inverse transition to the superconducting state (35) occurs.

It there is a nucleating region of the normal phase, a time-varying homogeneous state[1] will arise in which the layer solution separates super-conducting and normal phases. At $\delta > \delta_0$, the interface moves at a velocity proportional to $\delta - \delta_0$, and the film goes into the normal state. At $\delta < \delta_0$, the interface moves toward the normal phase (increasing the size of the super-conducting region), and the final state of the system is that described by inhomogeneous distribution (35).

[1)]We are assuming $\hbar + e + k_B = 1$.

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